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A GLOBULAR-CLUSTER MODEL WITH VARIABLE MEAN MASS OF A SINGLE STAR

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SUMMARY: The application of the basic hydrodynamical equations to globular clusters is reconsidered. As usually, these stellar systems are assumed to be spherically symmetric and in a steady state with an isotropic velocity distribution. A system of differential equations with four independent functions describing a globular-cluster model is developed. The obtained results show that it is possible to find a solution with two input empirical functions describing the spatial distribution of stars and the variation of the mean mass of a single star.

1. INTRODUCTION

As well known, the globular clusters have been usually considered as spherically symmetric, self-consistent, stellar systems in a steady state. A global isotropy in the velocity space has been also assumed (e.g. Lightman and Shapiro, 1978). In such a case, as well known, a usual approach has been to form a model with three descriptive functions: density, potential and pressure. In the case of existing of an additional relation connecting the pressure with the density the number of independent descriptive functions is reduced to two. Since there are only two equations (Euler's and Poisson's), it is possible to solve the system purely theoretically, i. e. without introducing any function based empirically. A good example is the well-known polytrope model (e.g. Ogorodnikov, 1958 - p. 460).

However, the stellar statistics for globular clusters has demonstrated that the spatial distribution of stars within them does not agree with the polytrope models though in the beginning of this century the Plummer-Schuster model (polytrope of index 5) was usually assumed for globular clusters. Instead of it a new empirical formula for description of stars spatial distribution was offered (King, 1962). The same author extended his concept including also the velocity distribution (King, 1965, 1966) finally forming in this way the well-known King model for globular clusters. In later years King's formulae have been often used in modelling both real and imaginary globular clusters with some modifications, such as: introducing the time varying of the King parameters, rotation of a globular cluster and small deviations from the spherical symmetry and velocity isotropy, dividing the total star populations into several subpopulations with different masses, etc. (e.g. Pryor *et al.* 1986; Meylan and Mayor, 1986; Chernoff *et al.* 1986; Meylan, 1988). In principle, the isotropy in the velocity space has been practically always assumed and the deviations have concerned only some parts of a globular cluster (for example, the periphery due to the tidal field). This assumption is based on the fact that the globualr clusters (at least those belonging to the Milky Way) are old enough to have suffered a significant relaxation. However, another, well-known and expected, consequence of the relaxation is the mass segregation. In the existing models this problem has been solved in some cases by introducing various mass classes, or several subpopulations as said above, where stars within a given class have the same velocity distribution (e.g. Davoust, 1977). However, it is clear that the mass distribution is a continuous function as it is the velocity one so that, evidently, a mass-velocity correlation arises. Something similar was applied in King's (1965) paper but seems to have been abandoned later on (King, 1966).

Bearing in mind such a state of the problem the present author's intention is to present his own point of view where the crucial moment is the introducing of a generalised stellar distribution function involving the mass as well. In such a case the set of usual equations can be rewritten in a somewhat different way where one can look for a solution. It should be noted that some special phenomena, such as the core collapse, the tidal forces, the deviations from the spherical symmetry towards the more general axial symmetry, etc, are not considered in the present paper because their intensity is, nevertheless, low.

2. EQUATIONS

The first step is to introduce a new distribution function (number density within phase space). This function corresponds to a generalised phase space involving the mass, as well. Therefore, the following relations are valid – for the ordinary phase-space density

$$\Psi = \int \Psi_m \mathrm{d}m \ ,$$

for the number density within a mass interval (m, $m + \mathrm{d}m$)

$$\mathcal{F}_m = \int \Psi_m \mathrm{d}^3 V \ ;$$

the former integral is taken over the entire mass space, the latter one over the entire velocity space. Finally, the number density of stars (of all masses) will be given as

$$n = \int \mathcal{F}_m \mathrm{d}m = \int \Psi_m \mathrm{d}^3 V \mathrm{d}m \ .$$

The idea of including the mass among the phase coordinates is not a new one. It can be found, for example, in von der Pahlen's (1947 - p. 43-44) textbook. Since throughout the rest of the present paper the spherical symmetry and the isotropy in the velocity space will be assumed, the dependence on the spatial coordinates will be reduced to the radius (r) only and the volume element in the velocity space will be equal to $4\pi v^2 \mathrm{d}v$ (v is the modulus of the residual velocity, equal to the total velocity of a star with respect to the cluster centre because the assumed spherical symmetry implicates no centroid velocity).

The number of the independent descriptive functions will be enlarged to four now because the mean mass of a single star - $\overline{m}(r)$ - is no longer constant. It is defined as

$$\overline{m} = \frac{\int \mathcal{F}_m m \mathrm{d}m}{n}$$

and it is for obvious reasons equal to the ratio of the mass density, ρ, (further on density) and the number one, n. Therefore, as for the Poisson equation there will be no changes. On the other hand the Euler one will be transformed into the following form

$$\rho \frac{\mathrm{d}\Pi}{\mathrm{d}r} = \frac{1}{3} \frac{\mathrm{d}}{\mathrm{d}r} (n \ \overline{mv^2}) \ , \qquad (1)$$

where Π is the potential. The essence of the change is in the appearance of the mean double kinetic energy of a single star – $\overline{mv^2}$ – instead of the specific one (per unit mass) because now there is the correlation between the mass and velocity. In view of the Poisson equation relation (1) becomes

$$\frac{-G}{4\pi} \frac{\mathrm{d}\mathcal{M}(r)}{\mathrm{d}r} \frac{\mathcal{M}(r)}{r^4} = \frac{\mathrm{d}n}{\mathrm{d}r} \frac{\overline{mv^2}}{3} + \frac{n}{3} \frac{\mathrm{d}(\overline{mv^2})}{\mathrm{d}r} \ , \qquad (2)$$

where G is the universal gravitation constant and $\mathcal{M}(r)$ is the cluster mass within the radius r.

As seen, in equation (2) there are three unknown functions of r. This is not surprising with regard that the model description is performed with four independent functions and there are only two equations. Clearly, the possibilities are either to introduce additional equations or to assume suitable (say, of empirical origin) forms for two functions from this set. This problematics will be discussed in more details in the next section.

3. SOME EXAMPLES

In this section some examples concerning equation (2) are given assumning some particular forms for two arbitrary functions of the given set.

Example No 1: the potential follows the Plummer-Schuster law, i. e.

$$\Pi = \frac{G\mathcal{M}}{(r^2 + b^2)^{1/2}} \ , \ \mathcal{M} = \mathrm{const} \ , \ b = \mathrm{const} \ .$$

As well known, then for the density is valid

$$\rho \propto (r^2 + b^2)^{-5/2}$$

yielding for the pressure

$$p \propto (r^2 + b^2)^{-3} \ .$$

With regard that the pressure is equal to

$$p = \frac{1}{3} n \ \overline{mv^2} \ ,$$

one can look for various solutions concerning n and $\overline{mv^2}$. For example any solution of the type $\overline{mv^2} =$ const is not acceptable because it yields an increasing function of the radius for $\overline{m}$. The classical case, treated in the polytrope theory corresponds to a constant mean mass of a single star. Of course, monotonously decreasing functions of the type $\overline{m} \propto (r^2 + b^2)^{-q/2}$, where $q > 0$, are also possible and, as easy to see, they yield $\overline{mv^2} \propto (r^2 + b^2)^{-(q+1)/2}$.

Example No 2: the density is given as

$$\rho = \rho(0)(1 + x^2)^{-1} \ , \ x = \frac{r}{r_c} \ , \ r_c = \text{const} \ .$$

As easy to see, the corresponding expression for the mass inside an arbitrary radius r will be

$$\mathcal{M}(r) = 4\pi\rho(0)r_c^3(x - \text{arctan}x) \ .$$

Among the various possibilities it will be considered $\overline{mv^2} =$ const. It yields

$$n = C - 12\pi G\frac{\rho^2(0){r_c}^2}{\overline{mv^2}}(\frac{1}{2}\text{arc}^2\text{tan}x + \frac{\text{arctan}x}{x}) \ ,$$
$$C = \text{const}$$

This solution yields a decreasing function $\overline{m}$.

Example No 3: $\rho \propto r^{-2}$ (the classical isothermal distribution).
This time nothing new is obtained, i. e. both $\overline{m}$ and $\overline{mv^2}$ are constant.

Example No 4: the density follows a power law, i. e. $\rho \propto r^{-3/2}$.
(As easy to see, $\mathcal{M}(r)$ will follow also a power law, but the exponent is equal to $+3/2$. If the mean double kinetic energy of a single star were constant, the number density would follow a power law (r^{-1}), as well as the mean mass of a single star, however with a different exponent ($r^{-1/2}$).

All these examples indicate that it is possible to obtain physically based solutions. They all offer analytical solutions, but this is due to their simplicity which, on the other hand, appears as a significant limitation to their applicability to real globular star clusters. Thus in order to obtain a better fit to the real situation, models using numerical solutions seem inevitable.

4. DISCUSSION AND CONCLUSIONS

In the present author's opinion the basic advantage of the procedure proposed above is in introducing a generalised distribution function adding the mass to the usual phase coordinates (components of radius and velocity vectors). The reason is, firstly, that the star distribution is a continuous mass function and, secondly, that the number of coordinates in such a generalised phase space, on which the generalised distribution function in this case depends, is only three due to the assumptions of spherical symmetry and velocity isotropy. Therefore, any complication arising in connexion with the phase-space generalisation is small compared to those appearing in the case of many functions describing the distribution in the velocity space (also in the ordinary one) for different mass classes when one attempts to explain the mass distribution (basically continuous) by applying discontinuity approximmations.

It may be concluded that in the process of solving equation (1) (more precisely the system formed by it and the Poisson equation) the best way, if achieving of a reasonable fit to the observations is required, is to use some empirical functions as the input ones. No doubt, the best known empirical function used in the case of globular clusters is that describing the spatial distribution, proposed by King (1962, formula (27)). Though nominally used for representing the spatial mass density, it is not clear whether it, in fact, yields the mass density or the number density because as an empirical formula it was obtained by interpreting star counts. On the other hand, due to the observational selection the information based on star counts is severely restricted for some categories of star masses. For example, very low-mass stars are always problematic because of their low luminosities. In the case of globular clusters (at least when those of the Milky Way are concerned) very massive stars are also problematic because their evolution was finished long ago by converting them into dark bodies. Therefore, obtaining of a reliable information concerning the variation of the mean mass of a single star with radius within a globular cluster is connected with difficulties.

On the other hand, a suitable approach may be to use two functions of the King type (formula (27) or (14) - King, 1962) with different r_c parameters where one of them represents the number density and the other one the mass one (as for the other two parameters of formulae (14) and (27) of King's paper, K must be different because ρ and n have different dimensions, whereas r_t should be equal because both densities are expected to vanish at the same distance). The ratio of r_c parameters for the two densities would correspond to the dependence of the mean mass of a single star on the radius.

However, it should be aware of a disadvantage, from the mathematical point of view, concerning the application of King's formula. Namely, it yields no analytical solution for the potential (Poisson's equation). Hence any solving of the equation system ((1) + Poisson) with input functions of the King type should be numerical.

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МОДЕЛ ЗА ЗБИЈЕНА ЗВЕЗДАНА ЈАТА СА ПРОМЕНЉИВОМ СРЕДЊОМ МАСОМ ЈЕДНЕ ЗВЕЗДЕ

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Примена основних хидродинамичких једначина на збијена звездана јата је поново размотрена. Као и обично, за ове звездане системе се усвајају сферна симетрија, стационарно стање и изотропна расподела брзина. Развијен је један систем диференцијалних једначина са четири независне функције који описује модел једног збијеног звезданог јата. Добијени резултати показују да је могуће наћи решење са две улазне емпиријске функције које описују просторну расподелу звезда и промену средње масе једне звезде.