Bull. Astron. Belgrade № 153 (l996), 23 – 27

UDC 521.11
Original scientific paper

THE LINDBLAD APPROXIMATION AS THE LINEAR ONE

S. Ninković

Astronomical Observatory, Volgina 7, 11050 Belgrade, Yugoslavia

(Received: February 22, 1996)

SUMMARY: Low-eccentricity orbits for the case of a spherically symmetric potential are studied. The Lindblad approximation concerning nearly circular orbits is treated. It is shown that this approximation is a linear one, i. e. it coincides with the exact formulae when the eccentricity terms exceeding the first order are neglected. For a special case it is found that the eccentricities up to almost 0.3-0.4 allow these orbits to be treated as low-eccentricity ones.

1. INTRODUCTION

The study of nearly planar and nearly circular orbits is important bearing in mind the motions of stars in stellar discs. It is well known that sufficiently long ago a method for treating such kind of orbits was proposed by B. Lindblad (e. g. Lindblad, 1959). This method is applicable for any particular form of the potential, but only provided that the motion is nearly circular. There is also another possibility in studying nearly circular orbits. It consists of adapting a general formula for the potential to the case of a sufficiently small region in a stellar disc (e. g. Ollongren, 1967). The advantage of Lindblad's method is that it results in a very simple differential equation which yields a sinusoidal dependence of the radius variation on time. The only question remaining is if such a solution (evidently approximative) is sufficiently correct.

Therefore, in the present paper this approach is discussed in more details. The attention is concentrated on the purely planar motions and hence, as more suitable for analysing, the motion in the spherically symmetric force field is examined. A good example is found where after expanding the basic formula in the Taylor series and preserving the first term only (linear in eccentricity), the Lindblad formula describing the distance dependence on time is obtained.

2. THEORETICAL BASE

First one should define the orbital eccentricity considering that for it various definitions are possible (e. g. Kuzmin and Malasidze, 1970; Kutuzov, 1985). Here the following one is adopted

$$e = \frac{r_a - r_p}{r_a + r_p} \ , \qquad (1)$$

where r_p and r_a are the distances of pericentre and apocentre, respectively. The mean distance to the centre is also introduced as follows

$$r_m = \frac{r_a + r_p}{2} \ .$$

As well known in Lindblad's classical approach (e. g. Ogorodnikov, 1958, p. 324) there are three important steps. In the first one it is assumed that the angular momentum of a test particle (here the modulus of the total angular momentum - spherical symmetry!) is the same as for the corresponding circular orbit (orbit of equal mean distance to the centre r_m). However, it is well known that for an orbit of nonzero eccentricity the angular-momentum modulus is also related to the orbital eccentricity (e. g. Ninkovich, 1986). It will be shown now that the approximation used in this step, i. e. the neglecting of the orbital eccentricity, is justified due to the preserving of the linear term only.

Based on the energy integral we have

$$J^2 = \frac{(1-e^2)^2}{2e} r_m^2 \Delta\Pi$$

where J is the modulus of the angular momentum per unit mass, e is the orbital eccentricity and $\Delta\Pi$,

$$\Delta\Pi = \Pi(r_p) - \Pi(r_a) \ ,$$

is the potential difference at the distances of pericentre and of apocentre. This difference can be successfully represented by expanding the potential at either of the two distances in the Taylor series about the mean distance r_m, i. e.

$$\Pi(r_k) = \Pi(r_m) + \sum_{i=1}^{\infty} \frac{(r_k - r_m)^i}{i!} \left(\frac{\mathrm{d}^i\Pi}{\mathrm{d}r^i}\right)_{r_m} \ ,$$

$$r_k = r_p \text{ or } r_k = r_a$$

.
The i-th derivative of the potential can be written in the following way

$$\left(\frac{\mathrm{d}^i\Pi}{\mathrm{d}r^i}\right)_{r_m} = \gamma_i(r_m){\omega_c}^2(r_m) r_m^{2-i} \ ,$$

where γ_i are dimensionless coefficients and ω_c is the cyclical frequency of the circular motion; note that there is valid $\gamma_1 = -1$ regardless of the particular form of the potential. If the mass interior of an arbitrary radius depends on this radius according to a power law, then the coefficients γ_i are constant, i. e. distance independent. Finally due to the symmetry in the Taylor series with respect to the apocentre and pericentre one obtains the following expression for the angular-momentum modulus

$$J^2 = {J_c}^2(1-e^2)^2 \left(1 - \sum_{i=1}^{\infty} \frac{\gamma_{2i+1}}{(2i+1)!} e^{2i}\right) \qquad (2)$$

where J_c,

$$J_c = r_m^2 \omega_c(r_m)$$

is the angular momentum of the corresponding circular orbit.

It is seen from (2) that if the eccentricity powers higher than 1 are neglected, the factor multiplying ${J_c}^2$ in (2) becomes equal to one, i. e. the angular momentum for an eccentric orbit does not differ from that of the corresponding circular one as assumed in the Lindblad approximation. Thus this step is justified.

The further steps are well known so that finally the following differential equation is obtained

$$\delta\ddot{r} = -(3 - \gamma_2)\omega_c^2 \delta r \ , \qquad (3)$$

where

$$\delta r = r - r_m \ .$$

It is often said that any realistic spherically symmetric potential is expected to be "something between" the two limiting cases - that of homogeneous sphere and the one of point mass. Since in both of them the mass within an arbitrary radius follows a power-law dependence, the coefficient γ_2 in (3) is constant, being equal to -1 (homogeneous sphere), or to 2 (point mass), respectively. Therefore, in a general case of spherical symmetry one can expect the coefficient γ_2 in (3) to be, though radius dependent, always within the limits -1 - 2. It is not difficult to see that then the following solution of (3) is obtained

$$r(t) = r_m(1 + e \sin \kappa t) \ , \qquad (4)$$

$$\kappa = (3 - \gamma_2)^{1/2} \omega_c \ , \qquad (5)$$

the last quantity usually known as the cyclical frequency of the so-called epicyclic motions.

It is well known that in describing a finite motion for the case of spherical symmetry one can define two important periods, i. e. the corresponding cyclical frequencies. They are the anomalistic period (time elapsing between two successive peri- or apocentric passages) and the sidereal one (time corresponding to the difference of the position angle in the orbital plane equal to 2π - e. g. Kuzmin and Malasidze (1970). It is easy to see that the anomalistic cyclical frequency is equal to that of epicyclic motions. As for the relationship between the sidereal cyclical frequency and that of circular motion, it is somewhat more complicated. Namely, in view of the angular-momentum integral and equations (4)-(5) and also taking into account that small orbital eccentricities are treated here (which allows certain self-evident approximations), one obtains the following expression for the dependence $\psi(t)$ where ψ is the position angle in the orbital plane

$$\psi(t) = \psi(0) - \frac{2e}{(3-\gamma_2)^{1/2}} + \omega_c t + \frac{2e}{(3-\gamma_2)^{1/2}} \cos \kappa t \ .$$

From here one easily obtains the following relation which connects the sidereal cyclical frequency ω_s with that of circular motion

$$2\pi\frac{\omega_c}{\omega_s} + \frac{2e}{(3-\gamma_2)^{1/2}}[\cos 2\pi(3-\gamma_2)^{1/2}\frac{\omega_c}{\omega_s} - 1] = 2\pi \ . \qquad (6)$$

This is obviously a transcendental equation allowing the ratio ω_c/ω_s to be equal to 1 only if the square root from (5) yields an integer (provided that eccentricity is not exactly zero - trivial case!). In the interval [-1, 2] admitted for γ_2 this is only possible just for the two limiting values. This fact is not surprising because it is well known that only for the homogeneous sphere, i. e. point mass the orbits are closed and the sidereal cyclical frequency is eccentricity independent. It is interesting to note that although equation (6) is obtained for the case of low eccentricities, this conclusion is valid for an arbitrary eccentricity.

It should be noted that in the more general case of axial symmetry γ_2 can exceed the value of 2, even that of 3, in the periphery for flattened models of finite size so that solutions like (4) are no longer possible. It is interesting to mention that the case $\gamma_2 = 3$ also yielding an integer for the ratio in (5) (zero), like the well-known cases of homogeneous sphere and point mass, also admits an analytical solution for the orbit as in the cases of the latter two (e. g. Kuzmin and Malasidze, 1970).

3. A CONCRETE CASE

For the purpose of making the present analysis as clear as possible a concrete case will be treated. The homogeneous-sphere potential is chosen. The main reason is that in this case it is not only possible to solve the exact differential equation of motion analytically, but also to represent the dependence $r(t)$ in an explicit form. In addition, the cyclical frequency ω_c is constant for this model. As well known, the final equation of motion for the given case is

$$r(t) = r_m(1 + e^2 + 2e\sin\kappa t)^{1/2} \qquad (7) \ .$$

From the said above it is clear that the other cyclical frequency κ is also constant in this case being equal to $2\omega_c$. The expansion of the expression within the parentheses in (7) in the Taylor series yields

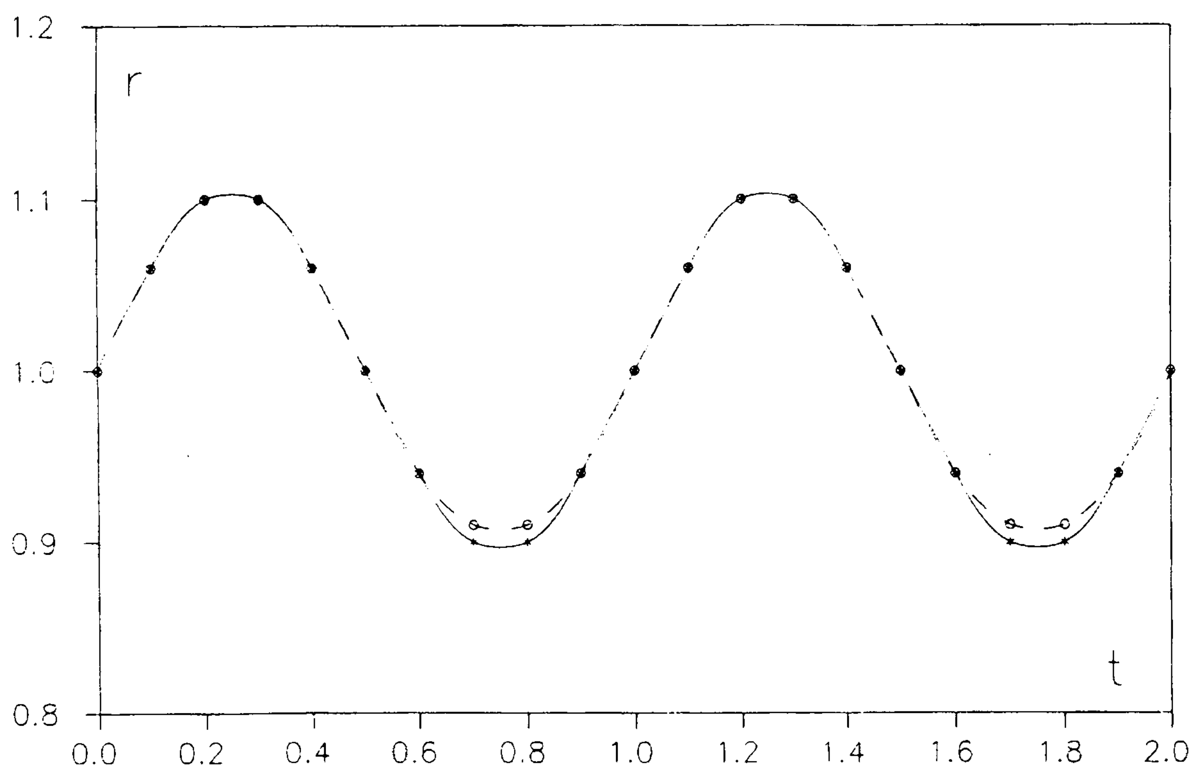

Fig. 1 The dependence $r(t)$ for eccentricity 0.1 - solid line according to (4), dashed one according to (7); r is expressed in units of r_m, t in those of anomalistic period.

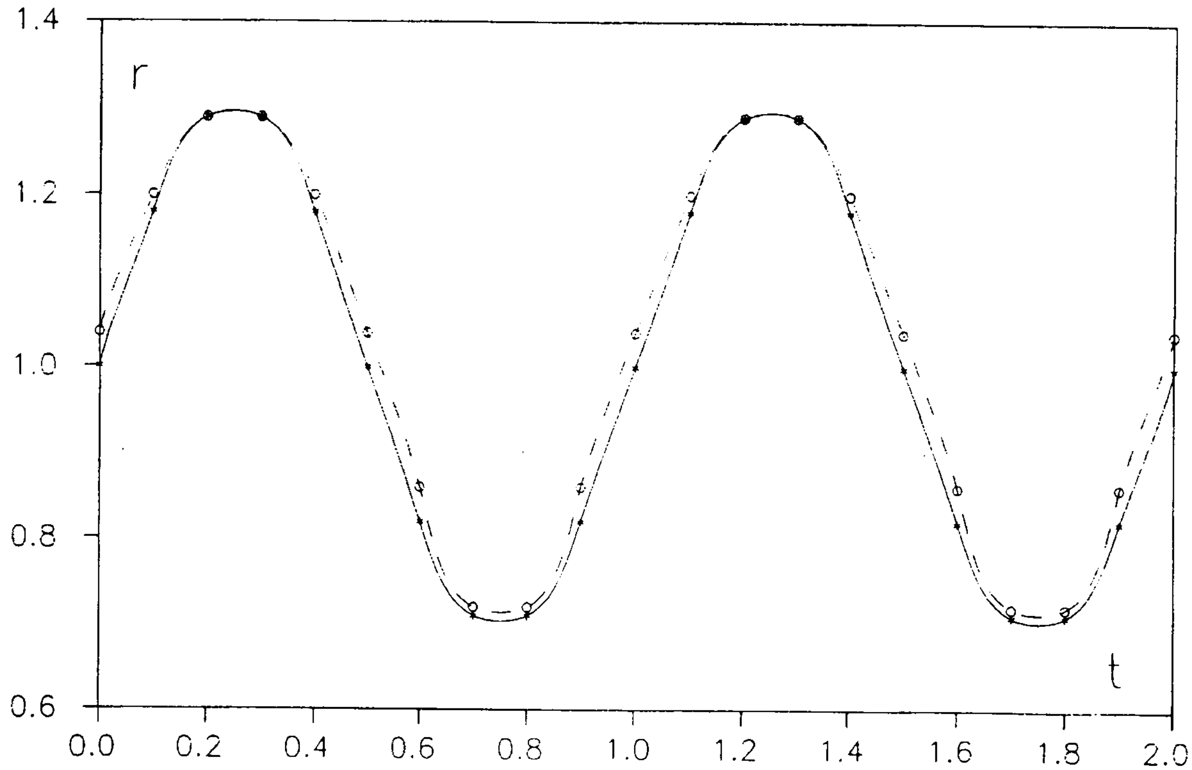

Fig. 2 Same as in Fig. 1, but $e = 0.3$.

$$B^{1/2} + \frac{1}{2}B^{-1/2}x - \frac{1}{8}B^{-3/2}x^2 + \frac{3}{48}B^{-5/2}x^3 + ... ,$$

$$B = 1 + e^2 \quad , \quad x = 2e \sin \kappa t .$$

As easy to see, if only those terms containing no powers of e higher than 1, which is also valid for the constant B, are retained, then (7) becomes

$$r(t) = r_m(1 + e \sin \kappa t) ,$$

i. e. it is reduced to Lindblad's approximate formula.

The agreement (or disagreement) between (4) and (7) is presented in Figs. 1 - 2. The main difference between the two formulae is that due to the ellipse curvature (as well known the true orbit for this potential is an ellipse) the time interval between $r = r_p$ and $r = r_m$ is shorter than the corresponding one between $r = r_m$ and $r = r_a$ which is not the case when the calculations are carried out by using the approximate formula, where these time intervals are mutually equal. Therefore the values for the instantaneous distance yielded by the two formulae and corresponding to the same argument, that of $\sin \kappa t = 0$, are also mutually different. The higher is the eccentricity, the more prominent is this difference. For example for $e = 0.1$ its value is 0.005 only (in units of r_m), for $e = 0.2$ the corresponding value is already 2% etc. This difference attains 10% for the eccentricity value of about 0.46. At the pericentre, i. e. apocentre ($\sin \kappa t = \pm 1$) both formulae yield the same values for the instantaneous distances. It is clear that by adding the higher-order terms in the expansion the agreement is improved.

This agreement can be examined in another way. This is the ratio of the time intervals between $r = r_m$ and $r = r_a$ and that between $r = r_p$ and $r = r_m$. As easily seen, according to the approximate formula it is equal to 1. The exact formula yields approximately, for example, 1.066 for $e = 0.1$, 1.1 for $e = 0.15$, 1.136 for $e = 0.2$ etc. Since due to the angular-momentum integral the time interval is proportional to the angle, one can express the results in the angle terms. Either of the time intervals mentioned above, i. e. of the corresponding angles, will be different from $\pi/2$, obtained by applying the approximate formula, by a small angle. This small angle compared to $\pi/2$ attains about 10% for $e = 0.3$. Thus as a final conclusion concerning this special case, one may say that the linear approximation, introduced by B. Lindblad, is applicable for orbital eccentricities as high as 0.3-0.4.

Of course, one should not forget that the case considered above is too special. Otherwise we cannot expect something similar in real stellar systems. A question concerning the eccentricity limits for such cases arises. Since an arbitrary spherically symmetric potential appears, as already noted above, as an intermediate case between the homogeneous-sphere potential and that of the point mass, something may be inferred by examining the situation for the latter one. However, an explicit exact formula like (7) does not exist for that case (the difficulties with the well-known Kepler equation). Therefore, the only possibility is a numerical study including the angles (true anomalies). It may be said that already for orbital eccentricities of about 0.2 the linear approximation fails in its applicability. This is not difficult to understand because due to the eccentric position of the gravity centre (at a focus instead of the ellipse centre) the ratios of the time intervals mentioned above are significantly higher than in the case of the homogeneous-sphere potential for the same orbital eccentricity. A preliminary conclusion may be that for the case of an arbitrary spherically symmetric potential the applicability domain for the linear approximation most likely does not exceed the eccentricity value of about 0.25. This conclusion, certainly, deserves to be subjected to further examinations. However, it could be of interest in the case of the Milky Way and other similar galaxies where the orbits of the disc stars are nearly planar so that their motion in the main plane can be studied as independent of that perpendicular to the plane. Then an additional question arises, namely if their orbital eccentricities are always low enough to justify the application of the linear approximation.

4. CONCLUSIONS

The main purpose of the present paper is to throw more light on the case of planar motion with angular-momentum integral. The obtained results show a prospective way in further research consisting of studying the dependence of the radius, i. e. of the position angle on time. The circumstance that here only one of several kinds of orbital eccentricity is studied is of no importance because for an orbit of low eccentricity in the sense of (1) the eccentricity defined in any other way will be also low due to the general concept of orbital eccentricity.

Acknowledgements – The author is grateful to Prof. Dr S. A. Kutuzov and to Dr L. P. Osipkov for valuable suggestions and advices.

This work is a part of the project "Astrometrical, Astrodynamical and Astrophysical Researches", supported by Ministry of Science and Technology of Serbia.

REFERENCES

Kutuzov, S. A.: 1985, in *Astronomo-geodezicheskie issledovaniya*, ed. K. A. Barkhatova, Ural'skij gos. universitet, Sverdlovsk, p. 56.

Kuzmin, G. G. , Malasidze, G. A.: 1970, *Publ. Tartuskoj astrof. obs. im. Struve*, **38**, 181.

Lindblad, B.: 1959, in *Encyclopaedia of Physics, vol. LIII Sternsysteme*, ed. S. Flügge, Springer-Verlag, Berlin, p. 21.

Ninkovich, S.: 1986, *Astrofizika*, **24**, 411.

Ogorodnikov, K. F.: 1958, *Dinamika zvezdnykh sistem*, Gos. izd. fiz.-mat. lit., Moskva.

Ollongren, A.: 1967, *Astron. J.*, **72**, 474.

ЛИНДБЛАДОВА АПРОКСИМАЦИЈА КАО ЛИНЕАРНА

Слободан Нинковић

Астрономска опсерваторија, Волгина 7, 11050 Београд, Југославија

УДК 521.11
Оригинални научни рад

Путање мале ексцентричности су проучаване за случај сферно симетричног потенцијала. Обрађује се Линдбладова апроксимација за скоро кружне путање. Показује се да је то једна линеарна апроксимација, тј., она се подудара са тачним формулама када се занемаре чланови који садрже степене ексцентричности више од првог. За један посебан случај је нађено да се за путање са ексцентричношћу до скоро 0.3-0.4 може рећи да су мало ексцентричне.