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Causal Temperature Profiles in Horizon-free Collapse

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Abstract. We investigate the causal temperature profiles in a recent model of a radiating star undergoing dissipative gravitational collapse without the formation of a horizon. It is shown that this simple exact model provides physically reasonable behaviour for the temperature profile within the framework of extended irreversible thermodynamics.

Key words. Horizon-free collapse—thermodynamics.

1. Introduction

The Cosmic Censorship Conjecture occupies center stage within the realms of relativistic astrophysics. The final outcome of the gravitational collapse of a star is still very much open to debate with the discovery of models admitting naked singularities (Harada *et al.* 1998; Kudoh *et al.* 2000). Various scenarios of gravitational collapse have been considered in which the energy momentum tensor is taken to be either a perfect fluid or an imperfect fluid with heat flux and anisotropic pressure (Bonnor *et al.* 1989; Herrera & Santos 1997a; Naidu *et al.* 2006). It has been shown that shearing effects delay the formation of the apparent horizon by making the final stages of collapse incoherent thus leading to the generation of naked singularities (Joshi *et al.* 2002). In this paper we revisit a radiating stellar model proposed by Banerjee *et al.* (Banerjee *et al.* 2002), (hereafter referred to as the *BCD* model) in which the horizon is never encountered. The interior matter distribution is that of an imperfect fluid with heat flux and the exterior spacetime is described by the radiating Vaidya metric (Vaidya 1951). The junction conditions required for the smooth matching of the interior and exterior spacetimes across a four-dimensional time-like hypersurface are solved exactly.

In this paper we investigate the physical viability of the *BCD* model. In particular, we analyse the relaxational effects on the temperature profiles within the framework of extended irreversible thermodynamics. We are in a position to obtain exact solutions to the causal heat transport equation for both the special case of constant collision time as well as variable collision time. Our results are in agreement with earlier thermodynamical investigations of radiating stellar models. We find that relaxational effects enhance the temperature at each interior point of the stellar configuration. Our investigations show that the *BCD* model displays physically reasonable temperature profiles throughout the evolution of the star.

2. The *BCD* radiating model revisited

In the *BCD* model the following form of the metric for the interior spacetime is assumed

$$ds^2 = -A^2(r,t)dt^2 + B^2(r,t)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2], \tag{1}$$

in which the metric functions A and B are yet to be determined. The energy momentum tensor for the interior matter distribution is given by

$$T^{\mu\nu} = (\rho + p)\, v^\mu v^\nu + p g^{\mu\nu} + q^\mu v^\nu + q^\nu v^\mu. \tag{2}$$

The heat flow vector q^μ is orthogonal to the velocity vector so that $q^\mu v_\mu = 0$. In order to generate an exact model of radiative gravitational collapse, the following ansatz was adopted for the metric functions in (equation 1)

$$A = a(r), \tag{3}$$

$$B = b(r)R(t), \tag{4}$$

which reduces the Einstein field equations for the interior matter distribution to

$$\rho = \frac{1}{R^2}\left[\frac{3}{a^2}\dot{R}^2 - \frac{1}{b^2}\left(\frac{2b''}{b} - \frac{b'^2}{b^2} + \frac{4b'}{rb}\right)\right], \tag{5}$$

$$p = \frac{1}{R^2}\left[-\frac{1}{a^2}(2R\ddot{R} + \dot{R}^2) + \frac{1}{b^2}\left(\frac{b'^2}{b^2} + \frac{2a'b'}{ab} + \frac{2}{r}\left(\frac{a'}{a} + \frac{b'}{b}\right)\right)\right], \tag{6}$$

$$q^1 = -\frac{2a'\dot{R}}{R^3 a^2 b^2}, \tag{7}$$

where '·' and '′' indicate derivatives with respect to time and the radial coordinate respectively. The condition of pressure isotropy yields

$$\frac{a''}{a} + \frac{b''}{b} - 2\frac{b'^2}{b^2} - 2\frac{a'b'}{ab} - \frac{a'}{ra} - \frac{b'}{rb} = 0. \tag{8}$$

Since the star is radiating energy the exterior spacetime is described by the Vaidya metric given explicitly in the form

$$ds^2 = -\left(1 - \frac{2M(v)}{\bar{r}}\right)dv^2 - 2d\bar{r}dv + \bar{r}^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{9}$$

where v is the retarded time and $M(v)$ is the exterior Vaidya mass. The junction conditions required for the smooth matching of the interior metric (equation 1) and the exterior Vaidya metric (equation 9) across a time-like hypersurface Σ are given by

$$(rB)_\Sigma = \bar{r}_\Sigma, \tag{10}$$

$$p_\Sigma = (q^1 B)_\Sigma, \tag{11}$$

$$m_\Sigma = \left[\frac{r^3 B \dot{B}^2}{2A^2} - r^2 B' - \frac{r^3 B'^2}{2B}\right]_\Sigma, \tag{12}$$

where m_Σ represents the total mass of the stellar configuration of radius r inside Σ. Utilising equations (6) and (7) in the boundary condition (equation 11) yields

$$2R\ddot{R} + \dot{R}^2 + m\dot{R} = n, \tag{13}$$

where m and n are constants. A simple particular solution of equation (13) is

$$R(t) = -Ct, \tag{14}$$

where $C > 0$ is a constant of integration. As pointed in (Banerjee *et al.* 2002) the mass-to-radius ratio, $m_\Sigma/\bar{r}_\Sigma$, is independent of time. A simple calculation yields

$$\frac{2m_\Sigma}{\bar{r}_\Sigma} = \frac{2m_\Sigma}{(rB)_\Sigma} = 2\left[\frac{C^2 r_0^2 b_0^2}{2a_0^2} - \frac{r_0 b_0'}{b_0} - \frac{r_0^2 b_0'^2}{2b_0^2}\right], \tag{15}$$

where $b(r_0) = b_0$ and r_0 defines the boundary of the stellar configuration. It is interesting to note that the parameters in equation (15) may be chosen so that $2m_\Sigma/\bar{r}_\Sigma < 1$ in order to avoid the appearance of horizon at the boundary.

3. Causal temperature profiles

In this section we consider the physical viability of the *BCD* model. In order to satisfy the condition of pressure isotropy (equation 8), the *BCD* model assumes $b(r) = 1$ and

$$A = a(r) = 1 + \xi_0 r^2. \tag{16}$$

The fluid volume collapse rate is

$$\Theta = \frac{3}{A}\frac{\dot{B}}{B} = \frac{3}{(1+\xi_0 r^2)t}, \tag{17}$$

and in the absence of shear is the same in both the radial and tangential directions. The proper stellar radius is given by

$$r_p(t) = \int_0^{b_0} B dr = -Ctb_0. \tag{18}$$

Since the star is collapsing we require that C be positive which corresponds to $-\infty < t < 0$. We further have

$$C^2 < 4\xi_0(1 + \xi_0 r_0^2). \tag{19}$$

The Einstein field equations (5)–(7) reduce to

$$\rho = \frac{3}{t^2(1+\xi_0 r^2)^2}, \tag{20}$$

$$p = \frac{1}{t^2(1+\xi_0 r^2)^2}\left[\frac{4\xi_0}{C^2}(1+\xi_0 r^2) - 1\right], \tag{21}$$

$$q^1 = -\frac{4\xi_0 r}{(1+\xi_0 r^2)^2}\frac{1}{C^2 t^3}. \tag{22}$$

We note that all the above thermodynamical quantities diverge as $t \to 0$. The regularity conditions $\rho > 0,\ p > 0$ and $\rho' < 0$, and $p' < 0$ together with the dominant energy condition, $(\rho - p) > 0$ and the more stringent requirement $(\rho + p) > 2|q|$ are all satisfied when

$$\left[1 - \frac{2\xi_0 r}{C}\right]^2 > -\frac{2\xi_0}{C^2}(1 - \xi_0 r^2). \tag{23}$$

We can now write

$$1 - \frac{2m_\Sigma}{\bar{r}_\Sigma} = \left[1 - \frac{C^2 r_0^2}{(1+\xi_0 r_0^2)^2}\right]. \tag{24}$$

We note that when

$$C^2 < \frac{1}{r_0^2} + \xi_0^2 r_0^2 + 2\xi_0, \tag{25}$$

the boundary surface can never reach the horizon (Banerjee *et al.* 2002). Furthermore, the surface redshift is given by

$$1 + z_\Sigma = \left(1 + r_0\frac{b_0'}{b_0} + r_0\dot{b}_0\right)^{-1}, \tag{26}$$

which diverges for an observer at infinity at the time of the appearance of the horizon. For the *BCD* model, equation (26) reduces to

$$1 + z_\Sigma = (1 - Cr_0)^{-1}, \tag{27}$$

which diverges when $C = 1/r_0$. In order to avoid the divergence of the surface redshift we must have

$$\frac{1}{r_0^2} < C^2 < \frac{1}{r_0^2} + \xi_0^2 r_0^2 + 2\xi_0, \tag{28}$$

where we have taken equation (25) into account. The luminosity of the star as perceived by an observer at infinity is given by

$$L = -\frac{dm}{dv} = \frac{c^3 r^3(1 + \xi_0 r^2 - rC)}{(1+\xi_0 r^2)^4}, \tag{29}$$

which is independent of time. We now turn our attention to the evolution of the temperature profiles of the *BCD* model. To this end we employ the causal transport equation for the heat flux, which in the absence of rotation and viscous stress is given by

$$\tau h_a^b \dot{q}_b + q_a = -\kappa(D_a T + T\dot{u}^a)\,, \tag{30}$$

where τ is the relaxation time for the thermal signals. Setting $\tau = 0$ in the above, we regain the so-called Eckart transport equations which predict infinite propagation velocities for the dissipative fluxes. For the line element (equation 1) the causal transport equation (equation 30) reduces to

$$\tau(qB)_{,t} + A(qB) = -\kappa\frac{(AT)_{,r}}{B}, \tag{31}$$

which governs the behaviour of the temperature. Setting $\tau = 0$ in equation (31) we obtain the familiar Fourier heat transport equation

$$A(qB) = -\kappa\frac{(AT)_{,r}}{B}, \tag{32}$$

which predicts reasonable temperatures when the fluid is close to quasi-stationary equilibrium. In order to study the evolution of the temperature in the *BCD* model, we employ the thermodynamic coefficients for radiative transfer as outlined in Maharaj & Govender (2005). The thermal conductivity takes the form

$$\kappa = \gamma T^3 \tau_c, \tag{33}$$

where γ (≥ 0) is a constant and τ_c is the mean collision time between the massless and massive particles. We further adopt the generalised power-law behaviour for τ_c

$$\tau_c = \left(\frac{\alpha}{\gamma}\right) T^{-\omega}, \tag{34}$$

where α (≥ 0) and ω (≥ 0) are constants. The velocity of thermal dissipative signals is assumed to be comparable to the adiabatic sound speed, which is satisfied if the relaxation time is proportional to the collision time:

$$\tau = \left(\frac{\beta\gamma}{\alpha}\right) \tau_c, \tag{35}$$

where τ (≥ 0) is a constant. The constant β is a measure of the strength of relaxational effects, with $\beta = 0$ giving the noncausal case. Using the above definitions for τ and κ, equation (31) takes the form

$$\beta(qB)_{,t} T^{-\omega} + A(qB) = -\alpha\frac{T^{3-\omega}(AT)_{,r}}{B}. \tag{36}$$

The Eckart temperature is readily obtained by setting $\beta = 0$ in equation (36). We are in a position to integrate equation (36) for the special case $\omega = 0$ which corresponds to constant collision time and more interestingly, the case $\omega = 4$ which gives a variable collision time. For constant collision time, the causal temperature profile is given by

$$T^4(r,t) = \left(\frac{L}{4\pi\delta}\right)\frac{1}{r_0^2c^2t^2}\left(\frac{1+\xi_0 r_0^2}{1+\xi_0 r^2}\right)^4 + \frac{8\beta\xi_0\left[2(r_0^2-r^2)+\xi_0(r_0^4-r^4)\right]}{\alpha t^2(1+\xi_0 r^2)^4} + \frac{8\xi_0\left[3(r^2-r_0^2)+3\xi_0(r^4-r_0^4)+\xi_0^2(r^6-r_0^6)\right]}{3\alpha t(1+\xi_0 r^2)^4}. \tag{37}$$

For $\omega = 4$, the causal temperature is given by

$$\begin{aligned} T^4(r,t) = {} & \frac{8\beta\xi_0}{\alpha t^2(1+\xi_0 r^2)^3}\left[\left(\frac{1+\xi_0 r_0^2}{1+\xi_0 r^2}\right)[8+\alpha t(1+\xi_0 r_0^2)]\mathrm{e}^{\frac{8\xi_0}{\alpha t}\left(\frac{r^2-r_0^2}{(1+\xi_0 r^2)(1+\xi_0 r_0^2)}\right)}\right] \\ & - \frac{8\beta\xi_0}{\alpha t^2(1+\xi_0 r^2)^3}[8+\alpha t(1+\xi_0 r^2)] \\ & + \frac{512\beta\xi_0 \mathrm{e}^{-\left(\frac{8}{\alpha t(1+\xi_0 r^2)}\right)}}{\alpha^2 t^3(1+\xi_0 r^2)^4}\left[\text{ExpIntegral } Ei\left(\frac{8}{\alpha t(1+\xi_0 r^2)}\right)\right] \\ & - \frac{512\beta\xi_0 \mathrm{e}^{-\left(\frac{8}{\alpha t(1+\xi_0 r^2)}\right)}}{\alpha^2 t^3(1+\xi_0 r^2)^4}\text{ ExpIntegral } Ei\left(\frac{8}{\alpha t(1+\xi_0 r_0^2)}\right) \\ & + \left[\frac{1+\xi_0 r_0^2}{1+\xi_0 r^2}\right]^4\frac{L}{(4\pi\delta)r_0^2c^2t^2}\mathrm{e}^{\frac{8\xi_0}{\alpha t}\left[\frac{r^2-r_0^2}{(1+\xi_0 r^2)(1+\xi_0 r_0^2)}\right]}, \end{aligned} \tag{38}$$

where L is given by equation (29) and δ is a constant. We note that the noncausal temperature ($\beta = 0$) and causal temperature are equal at the boundary ($r = r_0$). Figure 1 shows that the relaxational effects are dominant when the stellar fluid is far from equilibrium (large values of β). In the case of variable collision time, Fig. 2, we see that the causal temperature is everywhere greater than the corresponding noncausal temperature within the stellar interior. Furthermore, Figs. 1 and 2 indicate that the causal temperatures at late times (large values of β) decrease more rapidly than the causal temperatures when the star is close to quasi-static equilibrium. This is in agreement with the perturbative results of Herrera & Santos (1997b) as well as the acceleration-free model studied in Govender *et al.* (1998).

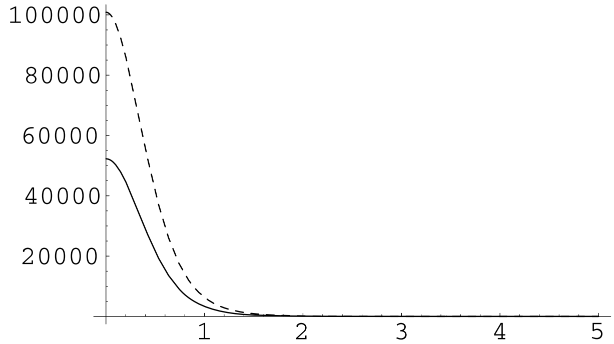

Figure 1. Temperature profiles for constant collision time (close to equilibrium – solid line), (far from equilibrium – dashed line) *versus* r.

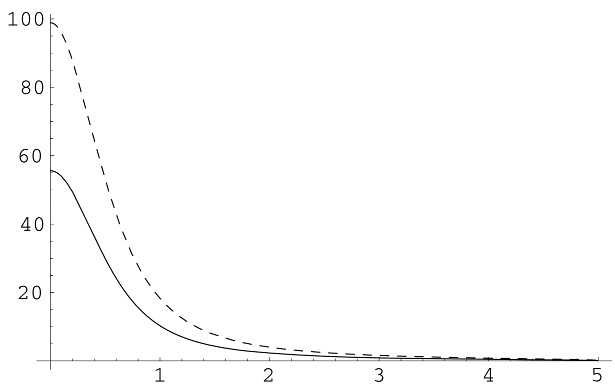

Figure 2. Temperature profiles for variable collision time (close to equilibrium – solid line), (far from equilibrium – dashed line) *versus* r.

4. Concluding remarks

We have investigated the physical viability of the *BCD* model within the framework of extended irreversible thermodynamics. We have shown that this simple model allows us greater insight into the evolution of the temperature for different collision times. More importantly, we were able to confirm earlier findings that the causal temperature dominates the Eckart temperature within the stellar core, even for variable collision time. As pointed out in earlier treatments, the constant collision time approximation is only valid for a limited period of the stellar evolution (Naidu *et al.* 2006). One expects that the collision time between the particles making up the stellar fluid will change with temperature. Such effects on the evolution of the temperature profiles were clearly demonstrated with the variable collision time solution. It must be pointed out that the truncation of the transport equations leads naturally to an implicitly defined temperature law (Govender & Govinder 2001). Such a temperature law may only be valid for a limited period of collapse. What remains is to investigate the behaviour of the temperature by employing the full transport equation for the heat flux as well as to include the effects of shear. The general framework for such an investigation has recently been provided in Herrera & Santos (2004).

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