CHAPTER 2

§ 2-2 § 2-3

HEAT CONDUCTION EQUATION

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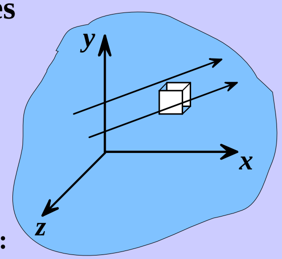
2.2 The General Heat Conduction Equation in Rectangular Coordinates



General case:

- 3-D
- Unsteady state
- Energy generation, $\dot{g}$:Examples:
  - Nuclear element
  - Electric energy dissipation in devices
  - Metabolic heat production in tissue

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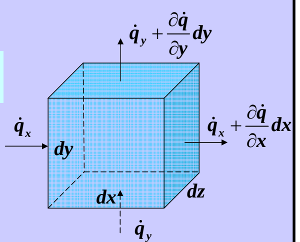
Apply conservation of energy to element $dV_{element}=dxdydz$ during time dt :

Heat added- Heat removed + heat generation = *Energy change within element*

$$Q_{in} - Q_{out} + G_{element} = \Delta E_{element}$$

Express in terms of *T*.



$dV_{element}=dxdydz$

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- Heat in by conduction, Q_{in}:

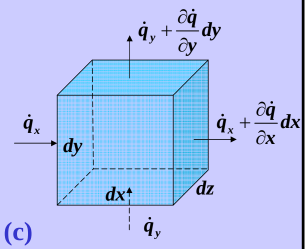
$$Q_{in} = \dot{q}_x \; dydzdt \; + \; \dot{q}_y \; dxdzdt + \; \dot{q}_z dxdydt \qquad \text{(a)}$$

- Heat generation, $G_{element}$:

$$G_{element} = \dot{g}dxdydzdt \qquad \text{(b)}$$

- Heat out by conduction, Q_{out}:

$$Q_{out} = (\dot{q}_x + \frac{\partial \dot{q}_x}{\partial x} dx) dydzdt$$
$$+ (\dot{q}_y + \frac{\partial \dot{q}_y}{\partial y} dy) dxdzdt$$
$$+ (\dot{q}_z + \frac{\partial \dot{q}_z}{\partial z} dz) dxdydt \qquad \text{(c)}$$



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- Energy change within the element ΔE

Expressing E in terms of T. Neglecting changes in kinetic and potential energy

$$\Delta E = \Delta U = \rho(dxdydz)C_p\Delta T$$

$$\Delta T = \frac{\partial T}{\partial t}dt$$

$$\Delta E = \rho C_p \frac{\partial T}{\partial t} dxdydzdt \qquad \text{(d)}$$

Substitute (a), (b), (c) and (d) into energy conservation equation and dividing through by $dx\, dy\, dz\, dt$

$$-\frac{\partial \dot{q}_x}{\partial x} - \frac{\partial \dot{q}_y}{\partial y} - \frac{\partial \dot{q}_z}{\partial z} + \dot{g} = \rho c_p \frac{\partial T}{\partial t} \qquad \text{(e)}$$

Apply Fourier's law of conduction

$$\dot{q}_x = -k\frac{\partial T}{\partial x}, \quad \dot{q}_y = -k\frac{\partial T}{\partial y}, \quad \dot{q}_z = -k\frac{\partial T}{\partial z}$$

Substituting into (e)

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{g} = \rho c_p \frac{\partial T}{\partial t}$$

- It is the ***heat conduction differential equation***

- Assume: constant k

Diffusive term, Source term, Transient term:

$$\alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{\dot{g}}{\rho c_p} = \frac{\partial T}{\partial t}$$

or

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Where $\alpha = k / \rho C_p$ (m^2/s) is the thermal diffusivity.

- NOTE:

(1) It is the differential formulation of the principle of conservation of energy. Valid at every point in the material

(2) Limited to isotropic and constant k

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

➢ Simplifications for special cases:

- Steady state: set $\frac{\partial T}{\partial t} = 0$
- One-dimensional: set $\frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial z^2} = 0$
- No energy generation: set $\dot{g} = 0$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

unsteady

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ Fourier-Biot eq.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ Diffusion eq.

Steady

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = 0$$ Possion eq.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$ Laplace eq.

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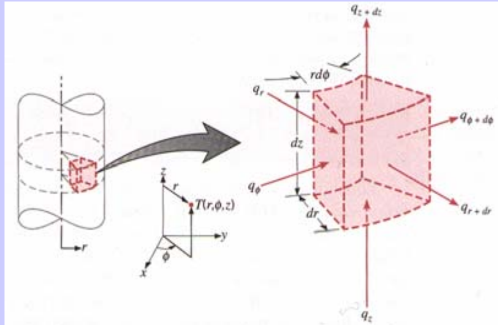
2.3 Heat Cond. Eq. in Cyl. and Sph. Coordinates

- Cylindrical coordinates r, ϕ, z

$$\alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{\dot{g}}{\rho\, c_p} = \frac{\partial T}{\partial t}$$



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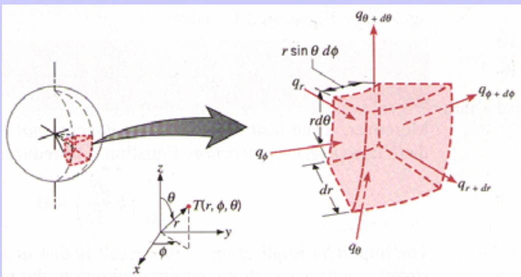
- Spherical coordinates r, θ, ϕ

$$\alpha\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)\right] + \frac{\dot{g}}{\rho c_p} = \frac{\partial T}{\partial t}$$



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2.4 Boundary and Initial Conditions

1 Initial Conditions

I.C. are mathematical expression for the temperature distribution of the medium initially.

Unsteady: $t = 0, \quad T(x, y, z, 0) = f(x, y, z)$

uniform—— $T(x, y, z, 0) = T_i$

2 Boundary Conditions

B.C. are mathematical equations describing what takes place physically at a boundary.

To write boundary conditions we must:

- Select an origin
- Select coordinate axes
- Identify the physical conditions at the boundaries

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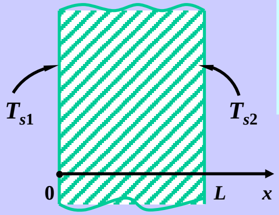
(1) Specified temperature. (*the first kind* B.C.)



$$T_s = \begin{cases} \text{const} \\ T_{si} \pm Ct \\ \cdots \end{cases} \quad \left(\frac{\partial T_s}{\partial t} = \pm C \right)$$

e.g. $$\begin{cases} T(0,t) = T_{s1} \\ T(L,t) = T_{s2} \end{cases}$$

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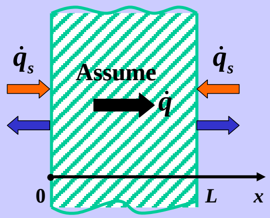
(2) Specified heat flux. (*the Second kind* B.C.)

Using Fourier's law $\dot{q}_s = -k\left(\frac{\partial T}{\partial n}\right)_s = \begin{cases} \text{const} \\ 0 \end{cases}$ Insulated B.C.



e.g. $|\dot{q}_s| = 50\ \text{W/m}^2$

Heating $$\begin{cases} -k\frac{\partial T(0,t)}{\partial x} = 50 \\ -k\frac{\partial T(L,t)}{\partial x} = -50 \end{cases}$$

Cooling $$\begin{cases} -k\frac{\partial T(0,t)}{\partial x} = -50 \\ -k\frac{\partial T(L,t)}{\partial x} = 50 \end{cases}$$

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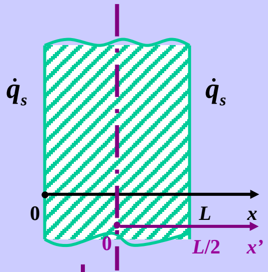
❖ Special case



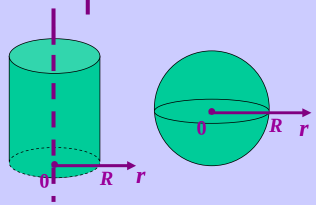
(a) Insulated B.C. $\dot{q}_s = 0$

$$\begin{cases} k\frac{\partial T(0,t)}{\partial x} = 0 \\ k\frac{\partial T(L,t)}{\partial x} = 0 \end{cases}$$

(b) Thermal symmetry

$$x' = 0, \quad \frac{\partial T(0,t)}{\partial x'} = 0$$

e.g.



$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0$$

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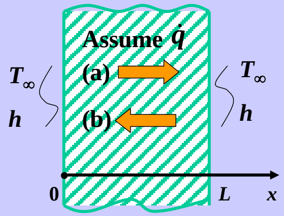
(3) Convection.

Fluid temperature T_∞ and convection heat transfer coefficient h are known. T_s is unknown.

• Equating Newton's law with Fourier's law: $-k\left(\frac{\partial T}{\partial n}\right)_s = h(T_s - T_\infty)$



▪ Left boundary ($x = 0$)

$(a) \quad -k\frac{\partial T(0,t)}{\partial x} = h(T_\infty - T_s)$

or $(b) \quad k\frac{\partial T(0,t)}{\partial x} = h(T_s - T_\infty)$

▪ Right boundary ($x = L$)

$(a) \quad -k\frac{\partial T(L,t)}{\partial x} = h(T_s - T_\infty)$

or $(b) \quad k\frac{\partial T(0,t)}{\partial x} = h(T_\infty - T_s)$

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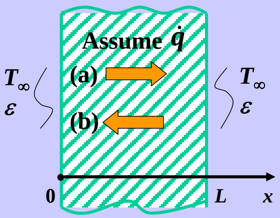
(4) Radiation.

T_∞ and ε of material surface are known. T_s is unknown.

- **Equating Stefan-Boltzmann law with Fourier's law:**



$$-k\left(\frac{\partial T}{\partial n}\right)_s = \varepsilon\sigma(T_s^4 - T_\infty^4)$$

e.g. Left boundary ($x = 0$)

$$(a)\quad -k\frac{\partial T(0,t)}{\partial x} = \varepsilon\sigma(T_\infty^4 - T_s^4)$$

or

$$(b)\quad k\frac{\partial T(0,t)}{\partial x} = \varepsilon\sigma(T_s^4 - T_\infty^4)$$

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- **Note:** $$-k\left(\frac{\partial T}{\partial n}\right)_s = \varepsilon\sigma(T_s^4 - T_\infty^4)$$

Or express as $$-k\left(\frac{\partial T}{\partial n}\right)_s = h_{rad}(T_s - T_\infty)$$

Where h_{rad} is equivalent heat transfer coefficient.

$$h_{rad}(T_s - T_\infty) = \varepsilon\sigma(T_s^4 - T_\infty^4) \Rightarrow h_{rad} = \frac{\varepsilon\sigma(T_s^4 - T_\infty^4)}{(T_s - T_\infty)}$$

- **Combined heat transfer coefficient:** $h_{combined} = h_{rad} + h_{conv}$

- ***The Third kind* B.C.**
  - (3) Convection
  - (4) Radiation

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(5) Interface.

Two different materials with a perfect interface contact.

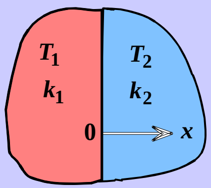
Two B.C.:

(i) Equality of temperature:

$$T_1(0, y) = T_2(0, y)$$

(ii) Equality of heat flux:

$$-k_1\frac{\partial T_1}{\partial x}\bigg|_{x=0} = -k_2\frac{\partial T_2}{\partial x}\bigg|_{x=0}$$



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