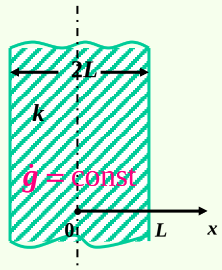
CHAPTER 2
HEAT CONDUCTION EQUATION

§ 2-6 Heat Conduction with Heat Generation in a solid

1 Plane Wall with Uniform Heat Generation

$\dot g \neq 0,$ *thermal resistance is inapplicable !!*

- **Assume:**
  - **(a) Steady state**
  - **(b) One-dimensional**
  - **(c) Constant *k***
  - **(d) Thermal symmetry**
- **Find:**
  - **(1) Temperature distribution** $T(x)$
  - **(2) Heat transfer rate** $\dot Q_x$



1) The Heat Conduction Equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\cancelto{0}{\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)}+\cancelto{0}{\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)}+\dot g=\rho c_p\cancelto{0}{\frac{\partial T}{\partial t}}$$

It becomes
$$\boxed{\frac{d^2T}{dx^2}+\frac{\dot g}{k}=0}$$

2) General Solution

Rearrange
$$\frac{d^2T}{dx^2}=-\frac{\dot g}{k}$$

Integrate
$$\frac{dT}{dx}=-\frac{\dot g}{k}x+C_1$$

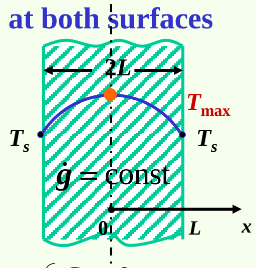
Integrate again
$$\boxed{T(x)=-\frac{\dot g}{2k}x^2+C_1x+C_2}$$

- **Temperature distribution is parabolic, not linear.**

3) Application to Special Cases

- **Case (i): Specified temperatures at both surfaces**

B.C.
$$\begin{cases}\left.\dfrac{dT}{dx}\right|_{x=0}=0\\ T(L)=T_s\end{cases}$$



Solution $T(x)=-\dfrac{\dot g}{2k}x^2+C_1x+C_2$

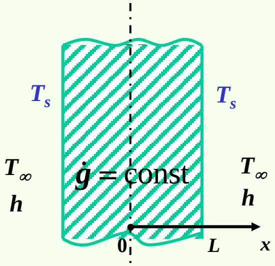
Applying B.C.

$$\begin{cases}0=-\dfrac{\dot g}{k}\cdot 0+C_1\\ T_s=-\dfrac{\dot g}{2k}\cdot L^2+C_1\cdot L+C_2=-\dfrac{\dot g}{2k}\cdot L^2+C_2\end{cases}\Rightarrow\begin{cases}C_1=0\\ C_2=T_s+\dfrac{\dot g}{2k}\cdot L^2\end{cases}$$

Solution becomes $$T(x)=T_s+\frac{\dot g}{2k}(L^2-x^2)$$

- **Case (ii): Convection at both surfaces**

B.C. $\begin{cases} \left.\dfrac{dT}{dx}\right|_{x=0} = 0 \\ -k\left.\dfrac{dT}{dx}\right|_{x=L} = h(T_s - T_\infty) \end{cases}$



Solution

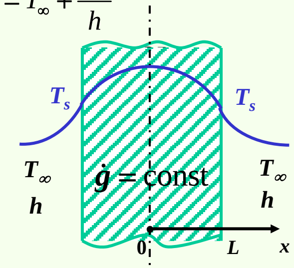
$$T(x) = -\frac{\dot{g}}{2k}x^2 + C_1 x + C_2$$

Applying B.C.

$$\begin{cases} 0 = -\dfrac{\dot{g}}{k}\cdot 0 + C_1 \\ -k(-\dfrac{\dot{g}}{k}L + C_1) = h(T_s - T_\infty) \end{cases} \Rightarrow \begin{cases} C_1 = 0 \\ T_s = T_\infty + \dfrac{\dot{g}L}{h} \end{cases}$$

$$T_s = T(x)\big|_{x=L} = -\frac{\dot{g}}{2k}L^2 + C_1 L + C_2 = T_\infty + \frac{\dot{g}L}{h}$$

$$\therefore \quad C_2 = T_\infty + \frac{\dot{g}L}{h} + \frac{\dot{g}}{2k}L^2$$



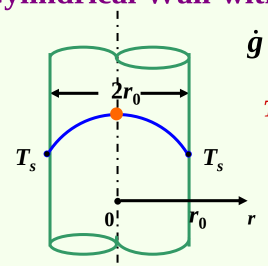
Solution becomes

$$T(x) = T_\infty + \frac{\dot{g}L}{h} + \frac{\dot{g}}{2k}(L^2 - x^2)$$

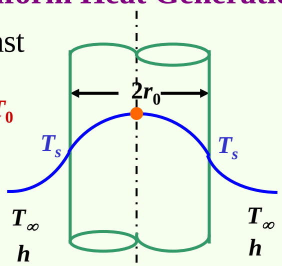
$$= T_s + \frac{\dot{g}}{2k}(L^2 - x^2)$$

2 Cylindrical Wall with Uniform Heat Generation

$\dot{g}$ = const



$T_{max} = T_0$



Case (i) Specified Temp. B.C.

- $T(x) = T_s + \dfrac{\dot{g}}{4k}(r_0^2 - r^2)$
- $\dot{q}(r) = -k\dfrac{dT(r)}{dr} = -k\cdot(-\dfrac{\dot{g}}{2k}r) = \dfrac{\dot{g}}{2}r$;

  $\dot{Q}(r) = \dot{q}(r)\cdot A(r) = \dfrac{\dot{g}}{2}r\cdot(2\pi rL) = \pi g r^2 L$

Case (ii) Conv. B.C.

- $\Delta T_{max} = T_0 - T_s = \dfrac{\dot{g}}{4k}r_0^2$

  $T_s = T_\infty + \dfrac{\dot{g}r_0}{2h}$

Note:

- **For steady state, with heat generation** $\dot{Q}(x) \neq \text{const}$
  while without heat generation $\dot{Q}(x) = \text{const}$
- **For convection B.C., another way to get**
  Surface Temperature T_s :

Energy balance $\quad \dot{G}_{gen} = \dot{Q}_{conv}$ (Steady state)

$$\dot{g}V = hA(T_s - T_\infty)$$

$$\therefore \quad T_s = T_\infty + \frac{\dot{g}V}{hA}$$

❖ Single Layer Wall with Uniform Heat Generation

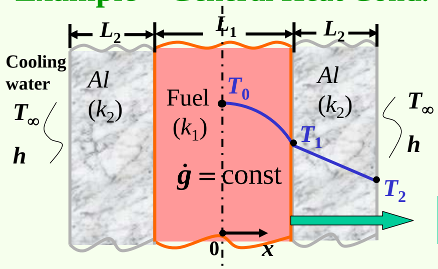
$$T(x) = T_s + \frac{\dot{g}}{2nk}(S^2 - x^2) \qquad T_s = T_\infty + \dot{g}S/(nh)$$

$$\Delta T_{max} = T_0 - T_s = \dot{g}S/(2nk) \qquad \dot{q}(x) = \dot{g}x/n$$

S — Characteristic Length, *m*
n — Shape factor

	S	n	$T(x)$ or $T(r)$	ΔT_{max}	T_s	$\dot{q}(x)$
Plane wall (2L)	L	1	$T_s + \frac{\dot{g}}{2k}(L^2 - x^2)$	$\frac{\dot{g}L^2}{2k}$	$T_\infty + \frac{\dot{g}L}{h}$	$\dot{g}x$
Cylindrical wall (r_0)	r_0	2	$T_s + \frac{\dot{g}}{4k}(r_0^2 - r^2)$	$\frac{\dot{g}r_0^2}{4k}$	$T_\infty + \frac{\dot{g}r_0}{2h}$	$\frac{\dot{g}r_0}{2}$
Spherical wall (r_0)	r_0	3	$T_s + \frac{\dot{g}}{6k}(r_0^2 - r^2)$	$\frac{\dot{g}r_0^2}{6k}$	$T_\infty + \frac{\dot{g}r_0}{3h}$	$\frac{\dot{g}r_0}{3}$

Example General Heat Cond. : Find T_0, T_1 and T_2



Thermal symmetry:
1/2 system
V=A(L_1/2)

Energy balance

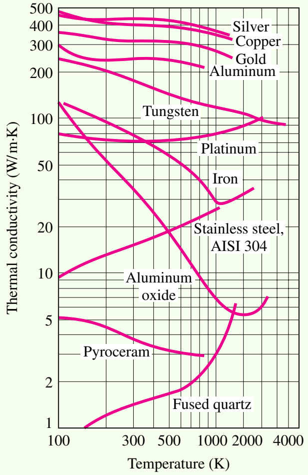
$$\dot{G}_{\text{gen,fuel}} = \dot{Q}_{\text{cond, }Al} = \dot{Q}_{\text{conv}}$$

1) $\dot{G}_{\text{gen,fuel}} = \dot{Q}_{\text{conv}} \qquad \dot{g}V = hA(T_2 - T_\infty) \quad \therefore T_2 = T_\infty + \frac{\dot{g}V}{hA} = T_\infty + \frac{\dot{g}(L_1/2)}{h}$

2) $\dot{G}_{\text{gen,fuel}} = \dot{Q}_{\text{cond, }Al} \qquad \dot{g}V = \frac{T_1 - T_2}{L_2/(kA)} \quad \therefore T_1 = T_2 + \frac{\dot{g}L_1L_2}{2k_2}$

3) $\Delta T_{\text{max, plane wall}} = T_0 - T_1 = \frac{\dot{g}(L_1/2)^2}{2k_1} \quad \therefore T_0 = T_1 + \frac{\dot{g}(L_1/2)^2}{2k_1}$

§ 2-7 Variable Thermal Conductivity, *k* (*T*)



- If thermal conductivity is a function of temperature, ***k=k(T)***, we will use an *average* value over the temperature range we are dealing with

$$k_{ave} = \frac{\int_{T_1}^{T_2} k(T)dT}{T_2 - T_1}$$

- Example: ***k = k₀(1+ βT)***

where k_0 — the thermal conductivity at zero temperature
β — the temperature coefficient of thermal conductivity.

$$k_{ave} = \frac{\int_{T_1}^{T_2} k_0(1+\beta T)dT}{T_2 - T_1} = k_0\left(1 + \beta\frac{T_2 + T_1}{2}\right) = k_0(1 + \beta T_{ave}) = k(T_{ave})$$

❑ Can use k_{ave} instead of constant k, and use heat transfer rate equations for constant thermal conductivity.

$$\dot{Q}_{\text{plane wall}} = k_{ave} A \frac{T_1 - T_2}{L}$$

$$\dot{Q}_{\text{cylinder}} = 2\pi k_{ave} L \frac{T_1 - T_2}{\ln(r_2 / r_1)}$$

$$\dot{Q}_{\text{sphere}} = 4\pi k_{ave} r_1 r_2 \frac{T_1 - T_2}{r_2 - r_1}$$

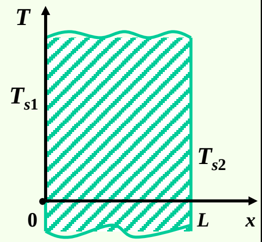
Example 2-8 Variation of Temperature in a Wall with $k(T)$

$$k = k_0(1 + \beta T)$$

Find:

(1) Heat transfer rate $\dot{Q}_x$

(2) Temperature distribution $T(x)$



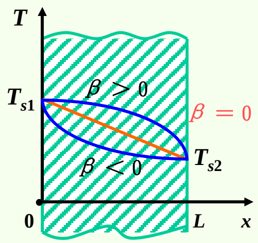
Solution:

(1) Determine $\dot{Q}_x$ **:**

$$k_{ave} = k(T_{ave}) = k_0\left(1 + \beta \frac{T_2 + T_1}{2}\right)$$

$$\dot{Q} = k_{ave} A \frac{T_1 - T_2}{L}$$

(2) Determine $T(x)$:



Apply Fourier's law $\dot{Q} = -k(T) A \dfrac{dT}{dx}$

Integrate $\int_0^x \dot{Q}\,dx = -A \int_{T_1}^{T} k(T)\,dT$

Substitute $\begin{cases} \dot{Q} = k_{ave} A \dfrac{T_1 - T_2}{L} = \text{const} \\ k(T) = k_0(1 + \beta T) \end{cases}$

Get $$T^2 + \frac{2}{\beta} T + \frac{2 k_{ave} x}{\beta k_0 L}(T_1 - T_2) - T_1^2 - \frac{2}{\beta} T_1 = 0$$

$$\therefore \quad T(x) = -\frac{1}{\beta} \pm \sqrt{\frac{1}{\beta^2} - \frac{2 k_{ave} x}{\beta k_0 L}(T_1 - T_2) + T_1^2 + \frac{2}{\beta} T_1}$$