CHAPTER 3
STEADY HEAT CONDUCTION

§ 3-6 Heat Transfer from **Finned** Surface

1 The Function of Fins (肋)

- **Increase heat transfer rate for a fixed surface temperature**

  –"Extending" A_s increases $\dot{Q}_s$ for a fixed T_s

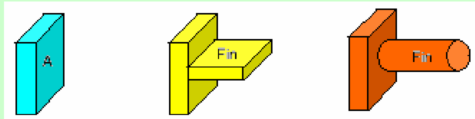
  Newton's law of cooling $\dot{Q}_s = h\, A_s\, (T_s - T_\infty)$

- **Or, Lower surface temperature for a fixed heat transfer rate**

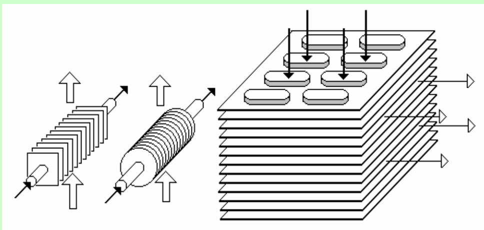
  –"Extending" A_s lowers T_s for a fixed $\dot{Q}_s$

$$T_s = T_\infty + \frac{\dot{Q}_s}{hA_s}$$

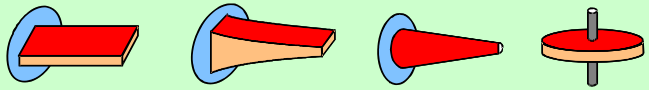
- **A surface is "extended" by adding fins**



- **Examples of fins:**
  - **Thin rods on the condenser in back of refrigerator.**
  - **Honeycomb surface of a car radiator.**
  - **Disks or plates attached to a baseboard radiator.**



2 Types of Fins



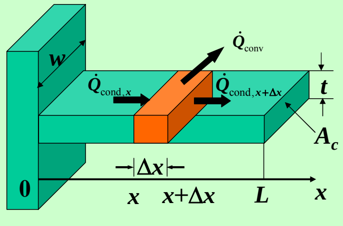
(a) constant area straight fin (b) variable area straight fin (c) pin fin (d) annular fin

- **Fin terminology and types**
  - **Fin *base***
  - **Fin *tip***
  - ***Straight* fin: (a) and (b).**
  - ***Variable cross-sectional area* fin: (b), (c) and (d).**
  - ***Spine* or a *pin* fin: (c).**
  - ***Annular* or *cylindrical*: (d).**

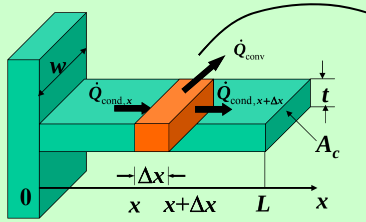
3 The Fin Heat Equation

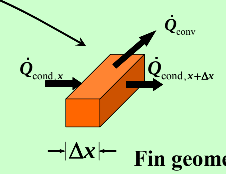
- Objective: Determine the **heat transfer rate** from a fin. Need the **temperature distribution**
- Select an origin and a coordinate axis x
- Procedure: Formulate the fin heat equation: Conservation of energy for a small element Δx.

h : heat transfer coefficient
T : fin temperature
T_∞: fluid temperature
A_c : fin cross sectional area
P : fin perimeter
k : fin thermal conductivity



a rectangular fin





Fin geometry: $A_s = P\Delta x$

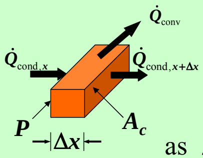
a rectangular fin

- Assume steady state and no heat generation
- Conservation of energy for the element :

$$\dot{E}_{in} = \dot{E}_{out}$$

$$\dot{Q}_{cond,x} = \dot{Q}_{cond,x+\Delta x} + \dot{Q}_{conv}$$

$$\dot{Q}_{cond,x+\Delta x} - \dot{Q}_{cond,x} + hP\Delta x(T - T_\infty) = 0$$



$$\frac{\dot{Q}_{cond,x+\Delta x} - \dot{Q}_{cond,x}}{\Delta x} + hP(T - T_\infty) = 0$$

as $\Delta x \to 0$

$$\frac{d\dot{Q}_{cond}}{dx} + hP(T - T_\infty) = 0$$

since $\dot{Q}_{cond} = -kA_c \dfrac{dT}{dx}$

$$\frac{d}{dx}(kA_c \frac{dT}{dx}) - hP(T - T_\infty) = 0$$

If define $\theta = T - T_\infty$

temperature excess

$$\frac{d}{dx}(kA_c \frac{d\theta}{dx}) - hP\theta = 0$$

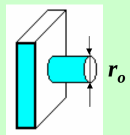
The general form of fin equation

Where $\begin{cases} A_c = A_c(x) & = \text{cross-sectional conduction area} \\ P = P(x) & = \text{circumference of the element} \end{cases}$

are determined from fin geometry.

$$\frac{d}{dx}(kA_c \frac{d\theta}{dx}) - hP\theta = 0$$

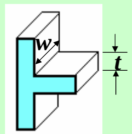
e.g.



For a circular fin of radius r_o

$$A_c = \pi r_o^2 \qquad p = 2\pi r_o$$

For a rectangulra bar of side w and t



$$A_c = wt \quad p = 2(w + t)$$

Assume: constant k and constant cross section of the fin

$$kA_c \frac{d^2\theta}{dx^2} - hP\theta = 0$$

or

$$\frac{d^2\theta}{dx^2} - a^2\theta = 0 \qquad \text{where} \quad a^2 = \frac{hP}{kA_c}$$

- This is a linear, homogenous, second-order differential equation with constant coefficients.
- The general solution of this equation is

$$\theta(x) = C_1 e^{ax} + C_2 e^{-ax}$$

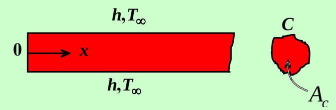
- **Two boundary conditions** are required to obtain C_1 and C_2.
- The temperature at the **fin base** (T_b) is usually known and is used as the **first boundary condition**.

$$T(x=0) = T_b \quad \text{or} \quad \theta(x=0) = \theta_b = T_b - T_\infty$$

4 Applications I: Constant Area Fins

Simplest fin problem: constant cross-sectional area A_c

A. Governing Equation



$$\frac{d^2\theta}{dx^2} - a^2\theta = 0$$

where $\quad \theta = T - T_\infty \qquad a^2 = \frac{hP}{kA_c}$

- **Equation is valid for:**
  - **(1) Steady state**
  - **(2) Constant k**
  - **(3) No heat generation**
  - **(4) $Bi \ll 1$**
  - **(5) Constant fin area**
  - **(6) Constant ambient temperature T_∞**

B. Solution

Assume: h is constant. Therefore a is constant.

Solution is

$$\theta(x) = A_1 \exp(ax) + A_2 \exp(-ax)$$

Or

$$\theta(x) = B_1 \sinh ax + B_2 \cosh ax$$

A_1 and A_2 or B_1 and B_2 are integration constants. They depend on:

- **Location of the origin**
- **Direction of coordinate axis x**
- **The two boundary conditions**

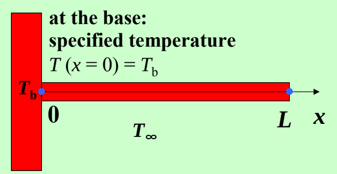
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

C. Special Cases

Consider 3 cases of constant area fins

- **Fin equation**
- **Temperature solution**
- **Objective: To determine**
  - **(1) The temperature distribution in the fin**
  - **(2) The heat transfer rate**



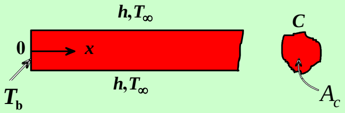
at the tip:

(i) Specified temperature
(ii) Convection
(iii) Insulated fin tip

Case (i): Infinite long fin

($L \to \infty$, $T(L) \to T_\infty$, specified temperature)

- **Solution:**



$$\theta(x) = A_1 \exp(ax) + A_2 \exp(-ax)$$

- **B.C. are:**

$$T(0) = T_b$$

$$T(L) = T_\infty$$

Introduce $\theta = T - T_\infty$

$$\theta(0) = \theta_o \qquad \text{(a)}$$

$$\theta(L) = 0 \qquad \text{(b)}$$

where θ_o **is**

$$\theta_o = T_b - T_\infty$$

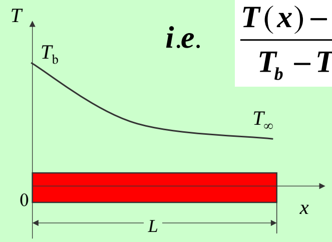
B.C. (b) $\quad 0 = A_1 \cdot \infty + A_2 \cdot 0, \quad \therefore \; A_1 = 0$

B.C. (a) $\quad A_2 = \theta_o$

so $$\frac{\theta(x)}{\theta_o} = \exp(-ax)$$

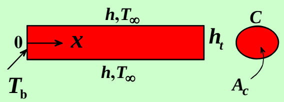
i.e. $$\frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-x\sqrt{hp/kA_c}}$$

Exponent distribution



Case (ii): Finite length fin with convection at tip

- **The base is at temperature** T_b
- **The tip exchanges heat by convection:** h_t, T_∞



- **Solution:** $\theta(x) = B_1 \sinh ax + B_2 \cosh ax$
- **B.C. are:**

$$T(0) = T_b$$

$$-k \left.\frac{dT}{dx}\right|_{x=L} = h_t[T(L) - T_\infty]$$

Introduce θ

$$\theta(0) = \theta_o \qquad \text{(c)}$$

$$-k \left.\frac{d\theta}{dx}\right|_{x=L} = h_t \theta(L) \qquad \text{(d)}$$

B.C. give B_2 **and** B_1:

$$B_2 = \theta_o$$

$$B_1 = -\theta_o \frac{[\cosh aL + (ka/h_t)\sinh aL]}{[\sinh aL + (ka/h_t)\cosh aL]}$$

Solution becomes

$$\frac{\theta(x)}{\theta_o} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh a(L-x) + (h_t/ak)\sinh a(L-x)}{\cosh aL + (h_t/ak)\sinh aL}$$

Case (iii): Finite length fin with insulated tip

Same as Case (ii) except the tip is insulated. B.C. (d) becomes

$$\left.\frac{d\theta}{dx}\right|_{x=L} = 0 \qquad \text{(e)}$$

- Two B.C. give B_1 and B_2
- Simpler approach: Set $h_t = 0$ in solutions of case (ii)

$$\frac{\theta(x)}{\theta_o} = \frac{T(x)-T_\infty}{T_b - T_\infty} = \frac{\cosh a\,(L\text{-}x)}{\cosh aL}$$

Hyperbolic distribution

Corrected Length L_c for fins with *convection at the tip*

- Fins with *convection* at the tip

$$\frac{\theta(x)}{\theta_o} = \frac{T(x)-T_\infty}{T_b - T_\infty} = \frac{\cosh\ a(L\text{-}x)+(h_t/ak)\sinh a(L\text{-}x)}{\cosh\ aL+(h_t/ak)\sinh aL}$$

- Fins with *insulated* tips—— *Solutions are simpler !*

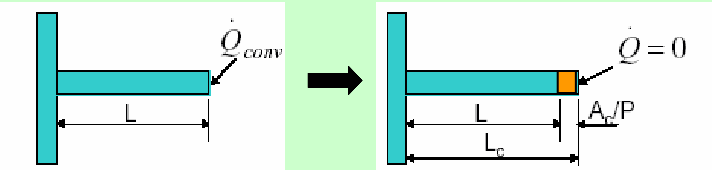
$$\frac{\theta(x)}{\theta_o} = \frac{T(x)-T_\infty}{T_b - T_\infty} = \frac{\cosh a\,(L\text{-}x)}{\cosh aL}$$

- Simplified model for fins with *convection* at the tip : *assume insulated tip* and introduce corrected length L_c

$$\frac{\theta(x)}{\theta_o} = \frac{T(x)-T_\infty}{T_b - T_\infty} = \frac{\cosh a\,(L_c\text{-}x)}{\cosh aL_c}$$

- The *corrected length* L_c is $L_c = L + \Delta L_c$

Insulation assumption is compensate by increasing the length by ΔL_c



Correction increment : $\Delta L_c = \frac{A_c}{p}$

Increase in surface area = Tip area

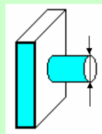
i.e. $\Delta L_c \cdot p = A_c$

e.g. Fin with convection from fin tip

Temperature distribution is

$$\frac{\theta(x)}{\theta_o} = \frac{T(x)-T_\infty}{T_b - T_\infty} = \frac{\cosh a\,(L_c\text{-}x)}{\cosh aL_c} \qquad \Delta L_c \cdot p = A_c$$

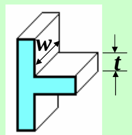
1) For a circular fin of radius r_o



$\pi\ r_o^2 = 2\pi\, r_o\, \Delta L_c \qquad \Delta L_c = r_o / 2$

$L_c = L + r_0 / 2$

2) For a rectangular bar of side w and t



$wt = 2(w+t)\Delta L_c \qquad \Delta L_c = \frac{t}{2(1+t/w)} \approx t/2$

$L_c = L + t/2$

3) For a square bar of side t $\quad \Delta L_c = t/4$

$L_c = L + t/4$

D. Determination of Fin Heat Transfer Rate $\dot{Q}_{\text{fin}}$

- Conservation of energy applied to a fin at steady state:

$$\dot{Q}_{\text{fin}} = \text{conduction at the base} = \text{convection at the surface}$$

- Two methods to determine $\dot{Q}_{\text{fin}}$:

(1) Convection at the fin surface: Newton's law

$$\dot{Q}_{\text{fin}} = \int_{A_{\text{fin}}} h[T(x) - T_\infty]\, dA_{\text{fin}} = \int_0^L h[T(x) - T_\infty]\, p\, dx$$

(2) Conduction at the base: Fourier's law

$$\dot{Q}_{\text{fin}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0}$$

Case(i) Infinite long fin or specified tip temperature

- Fin heat transfer $\dot{Q}_{\text{fin}}$

$$\frac{\theta(x)}{\theta_o} = \exp(-ax)$$

$$\dot{Q}_{\text{fin}} = -kA_c \left.\frac{d\theta}{dx}\right|_{x=0} = kA_c a\theta_o$$

$$\dot{Q}_{\text{fin}} = \sqrt{hpkA_c}\,(T_b - T_\infty)$$

Case(ii) fin with convection at tip

$$\dot{Q}_{\text{fin}} = \sqrt{hpkA_c}\, \frac{(T_b - T_\infty)[\sinh aL + (h_t/ak)\cosh aL]}{\cosh aL + (h_t/ak)\sinh aL}$$

Case(iii) fin with insulated tip

$$\frac{\theta(x)}{\theta_o} = \frac{\cosh a(L_c - x)}{\cosh aL_c}$$

$$\dot{Q}_{\text{fin}} = \sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL$$

5 Fin Efficiency η_{fin} and Fin Effectiveness ε_{fin}

Fin performance is described by two parameters:

1) *Fin Efficiency* η_{fin}

2) *Fin Effectiveness* ε_{fin}: Measures heat transfer enhancement due to fin addition.

Fin *Efficiency* η_{fin}

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin,max}}} = \frac{\text{Actual heat transfer rate from the fin}}{\text{Ideal/maximum heat transfer rate from the fin if the entire fin were at base temperature}}$$

$\dot{Q}_{\text{fin,max}}$ = heat transfer from fin if its entire surface is at the base temperature

$$\dot{Q}_{\text{fin,max}} = hA_{\text{fin}}(T_b - T_\infty)$$

A_{fin} = total surface area $\left(\text{Constant area fins } A_{\text{fin}} = pL\right)$

$$\therefore\ \eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{hA_{\text{fin}}(T_b - T_\infty)}$$

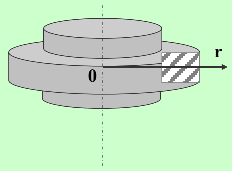
Case(i) Infinite long fin or specified tip temperature

$$\eta_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin,max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_{\text{fin}}(T_b - T_\infty)} = \frac{1}{aL}$$

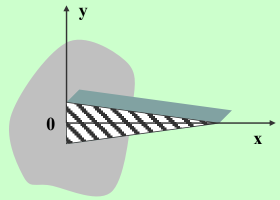
Case(ii) fin with convection at tip
Case(iii) fin with insulated tip

$$\eta_{\substack{\text{insulated}\\ \text{tip}}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin,max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL}{hA_{\text{fin}}(T_b - T_\infty)} = \frac{\tanh aL}{aL}$$

6 Applications II: Variable Area Fins



Annular fins



Triangular fins

$$\dot{Q}_{\text{fin}} = \eta_{\text{fin}}\dot{Q}_{\text{fin,max}} = \eta_{\text{fin}} hA_{\text{fin}}(T_b - T_\infty)$$

Fin efficiency of annular fins of length *L* and constant thickness *t*.

Figure 3-29

Fin efficiency circular, rectangular, and triangular fins on a plain surface of width *w*.

Figure 3-30