CHAPTER 4

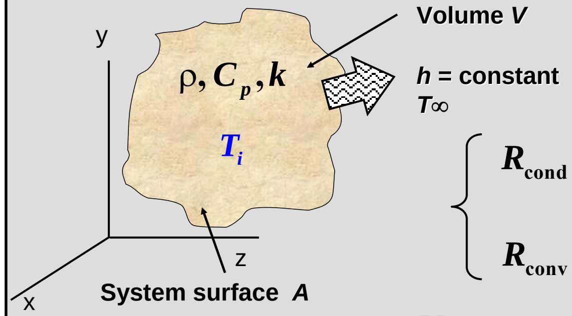
TRANSIENT HEAT CONDUCTION

$$\frac{\partial T}{\partial t} \neq 0$$

$$T = f(x, y, z, t)$$

4.1 Lumped System Analysis

Consider a body which is exchanging heat with the ambient by convection.



$$R_{cond} = \frac{L_c}{kA}$$

$$R_{conv} = \frac{1}{hA}$$

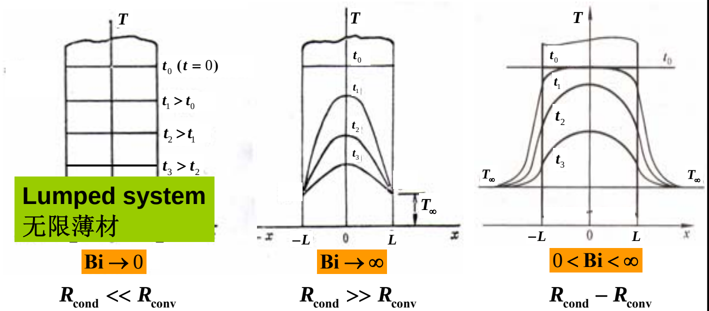
Characteristic length $L_c = \frac{V}{A}$

e.g. plane wall, sphere

Biot number

$$Bi_V = \frac{L_c / k}{1/h} = \frac{hL_c}{k}$$

$$= \frac{R_{cond}\ \text{(internal, within the body)}}{R_{conv}\ \text{(external, at the surface)}} = \frac{\dot{q}_{conv}}{\dot{q}_{cond}}$$

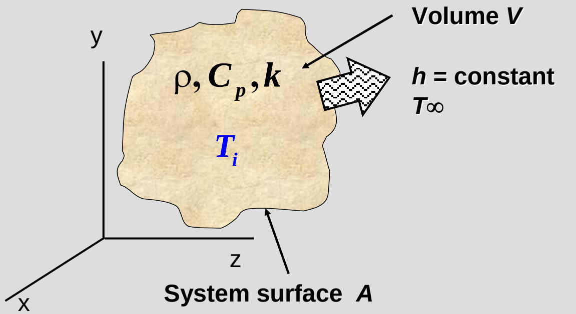


Criterion for *lumped-system*

$$Bi_V = \frac{h L_c}{k} \leq 0.1$$ (small body with high k)

A special case of transient heat transfer in which temperature does not change with the space, only change with the time.

$$T = f(x, y, z, t) \xrightarrow[Bi_V \leq 0.1]{\text{lumped-system}} T = f(t)$$



- Objective: Determine the transient temperature and amount of heat transfer

  Assume: Bi ≤ 0.1, then $T = T(t)$

  Assume no heat generation and heat is removed from the body

Heat balance during time interval dt:

$$Q_{in} - Q_{out} = \Delta U$$

$$-hA(T - T_\infty)dt = mC_p dT$$

Separate variables

$$\frac{d(T - T_\infty)}{T - T_\infty} = -\frac{hA}{\rho V C_p} dt$$

- This is the *lumped-system governing* equation for *all* bodies exchanging heat by convection. Valid for $Bi \le 0.1$
- Initial condition: $T(0) = T_i$

- Solution:

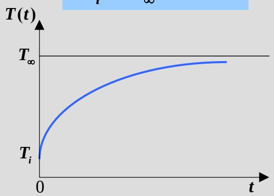
  Assume constant c_p, h and T_∞. Integrate governing eq.

$$\int_{T_i}^{T} \frac{d(T - T_\infty)}{T - T_\infty} = -\int_0^t \frac{hA}{\rho V C_p} dt$$

$$\ln \frac{T(t) - T_\infty}{T_i - T_\infty} = -\frac{hA}{\rho V C_p} t \qquad \text{let } b = \frac{hA}{\rho V C_p} \quad (1/s)$$

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt}$$

(Exponential approach)



❖ Discussion of *b*

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt}$$

$$b = \frac{hA}{\rho V C_p} = \frac{\left[\frac{W}{m^2 K}\right] \cdot \left[m^2\right]}{\left[\frac{kg}{m^3}\right] \cdot [m^3] \cdot \left[\frac{J}{kg \cdot K}\right]} = \frac{W}{J} = \frac{1}{s}$$

- $1/b$ (s) is called the time constant of lumped system. It's a measure of how fast temperature change in a system.
- $1/b \downarrow$, $t \downarrow$, the faster temperature change

1) Transient heat transfer rate

$$\dot{Q}(t) = hA[T(t) - T_\infty] = hA(T_i - T_\infty)e^{-bt} \quad \text{(W)}$$

2) Total amount of heat transfer (0 ~ t)

$$Q = \int_0^t \dot{Q}(t)dt = hA(T_i - T_\infty)\int_0^t e^{-bt}dt$$

$$= \frac{hA(T_\infty - T_i)}{b}(1 - e^{-bt}) = \rho V C_p (T_\infty - T_i)(1 - e^{-bt}) \quad \text{(kJ)}$$

or basing on energy conservation

$$Q = \rho V C_p [T(t) - T_i] \quad \text{(kJ)}$$

3) Total heat transfer from the system to the ambient

$t \to \infty$

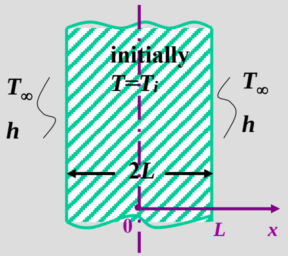
$$Q_{max} = \rho V C_p (T_\infty - T_i) \quad \text{(kJ)}$$

4.2 One-dimensional Transient Heat Conduction

- For $Bi_V > 0.1$, lumped system analysis is not applicable
- Spatial temperature variations must be accounted for
- 1-D conditions:
  1) large plane wall $T = f(x, t)$
  2) long cylinder $T = f(r, t)$
  3) sphere $T = f(r, t)$

4.2.1 1-D transient cond. in an infinite large plate

- no energy generation and constant conductivity



$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

B.C.

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T(L,t) - T_\infty]$$

I.C. $T(x,0) = T_i$

Solution:

Introduce temperature excess

$$\theta(x,t) = T(x,t) - T_\infty(x,t)$$

eq. $$\alpha\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t}$$

B.C. $$\left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0$$

$$-k\left.\frac{\partial \theta}{\partial x}\right|_{x=L} = h\theta(L,t)$$

I.C. $\theta(x,0) = T_i - T_\infty$

Use separation of variables. Assume a product solution

$$\theta(x,t) = X(x)\tau(t)$$

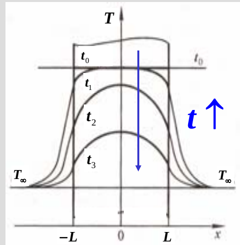
function of only t function of only x

eq. turns to $\frac{1}{a\tau}\frac{d\tau}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}$

It must have $\frac{1}{a\tau}\frac{d\tau}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \text{D (const)}$

let $D = -\beta^2$ $\Rightarrow$ $\begin{cases} \frac{1}{a\tau}\frac{d\tau}{dt} = -\beta^2 \\ \frac{1}{X}\frac{d^2X}{dx^2} = -\beta^2 \end{cases}$

❖ Sign of D is determined from the physical concept.



$$\frac{1}{a\tau}\frac{d\tau}{dt} = D$$

Integrate

$$\tau = C_1 e^{aDt}$$

$t \to \infty$, $T(x,t) \to T_\infty$ (reach steady state) ,

So D has to be **negative**. $D = -\beta^2$

Otherwise, $T(x,t) \to \infty$ (be physically meaningless !)

$$\frac{1}{a\tau}\frac{d\tau}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\beta^2 \Rightarrow \begin{cases} \frac{1}{a\tau}\frac{d\tau}{dt} = -\beta^2 \\ \frac{1}{X}\frac{d^2X}{dx^2} = -\beta^2 \end{cases}$$

- General solutions of the above 2 eqs. are

$$\tau(t) = C_1 e^{-a\beta^2 t}$$

$$X(x) = C_2\cos(\beta x) + C_3\sin(\beta x)$$

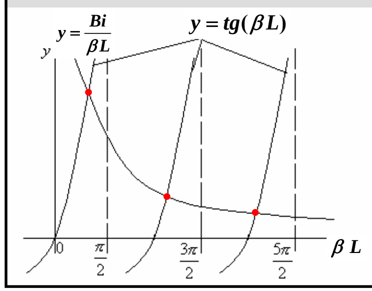
So get

$$\theta(x,\tau) = X(x)\,\tau(t) = e^{-a\beta^2 t}[A\cos(\beta x) + B\sin(\beta x)]$$

- **Determine A, B and β by B.C.s and I.C.**

B.C. $\begin{cases} \left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0 \\ -k\left.\frac{\partial\theta}{\partial x}\right|_{x=L} = h\theta(L,t) \end{cases}$ $\Rightarrow$ $\begin{cases} B = 0 \\ tg(\beta L) = \frac{h}{k\beta} \end{cases}$

characteristic equation



$$y = \frac{h}{k\beta} = \frac{hL}{k}\frac{1}{\beta L} = \frac{Bi}{\beta L}$$

characteristic values

$$\beta = f(\text{Bi})$$

$$\beta_1\ \beta_2\ \beta_3 \cdots$$

Get infinite solutions:

$$\theta_1(x,t) = e^{-a\beta_1^2 t}[A_1\cos(\beta_1 x)]$$

$$\theta_2(x,t) = e^{-a\beta_2^2 t}[A_2\cos(\beta_2 x)]$$

....

$$\theta_n(x,t) = e^{-a\beta_n^2 t}[A_n\cos(\beta_n x)]$$

Sum all solutions: $\theta(x,t) = \sum_{n=1}^{\infty} e^{-a\beta_n^2 t}[A_n\cos(\beta_n x)]$

I.C.

$\theta(x,0) = T_i - T_\infty$ ⇨ $A_n = (T_i - T_\infty)\dfrac{2\sin(\beta_n L)}{\beta_n L + \sin(\beta_n L)\cos(\beta_n L)}$

- Final solution (infinite series):

$$\frac{T(x,t)-T_\infty}{T_i - T_\infty} = \sum_{n=1}^{\infty}\frac{2\sin(\beta_n L)}{\beta_n L + \sin(\beta_n L)\cos(\beta_n L)} e^{-a\lambda_n^2 t}\cos(\beta_n x)$$

- Rearrange the solution, let $\beta_n L = \lambda_n$

$$\frac{T(x,t)-T_\infty}{T_i - T_\infty} = \sum_{n=1}^{\infty}\frac{2\sin(\lambda_n)}{\lambda_n + \sin\lambda_n\cos\lambda_n} e^{-\lambda_n^2 \frac{at}{L^2}}\cos(\lambda_n \frac{x}{L})$$

dimensionless temperature $\theta(x,t) = \dfrac{T(x,t)-T_\infty}{T_i - T_\infty}$

dimensionless distance $X = x/L$

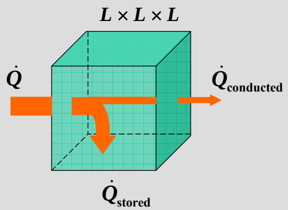
dimensionless time $\tau = at/L^2$ **Fourier number (Fo)**

$$\theta(x,t) = \sum_{n=1}^{\infty}\frac{2\sin(\lambda_n)}{\lambda_n + \sin\lambda_n\cos\lambda_n} e^{-\lambda_n^2\tau}\cos(\lambda_n X)$$

Fourier number (Fo)

$$\tau = \frac{\alpha t}{L^2} = \frac{kL^2(1/L)\,\Delta T}{\rho C_p L^3/t\ \Delta T} = \frac{kL^2\dfrac{\Delta T}{L}}{\rho C_p L^3\dfrac{\Delta T}{t}} = \frac{\dot{Q}_{\text{conducted}}}{\dot{Q}_{\text{stored}}}$$

$= \dfrac{\text{The rate at which heat is conducted across } L \text{ of a body of volume } L^3}{\text{The rate at which heat is stored in a body of volume } L^3}$



$$\theta(x,t) = \sum_{n=1}^{\infty}\frac{2\sin(\lambda_n)}{\lambda_n + \sin\lambda_n\cos\lambda_n} e^{-\lambda_n^2\tau}\cos(\lambda_n X)$$

Dimensionless quantities:

$\theta(x,t) = \dfrac{T(x,t)-T_\infty}{T_i - T_\infty}$ $X = x/L$

$\tau = at/L^2$ $\text{Bi} = hk/L$ $(\lambda_n - \text{Bi})$

$\therefore \quad \theta(x,t) = f(\text{Bi}, \tau, X)$

- $\tau > 0.2$, one term approximation : (error <2%)

$$\theta(x,t)_{\text{wall}} = \frac{T(x,t)-T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2\tau}\cos(\lambda_1 x/L), \quad \tau > 0.2$$

- Center of plane wall ($x = 0$) $\theta_{\text{o,wall}} = \dfrac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2\tau}$

$\therefore \quad \dfrac{\theta(x,t)_{\text{wall}}}{\theta_{\text{o,wall}}} = \cos(\lambda_1 x/L)$ Table 4-1: A_1, λ_1

1) Local transient heat transfer rate

$$\dot{Q}(x,t) = -kA\frac{\partial T}{\partial x} = -kA(T_i - T_\infty)A_1 e^{-\lambda_1 \tau}\frac{\lambda_1}{L}[-\sin(\lambda_1 x/L)]$$

$$= k\frac{A}{L}(T_i - T_\infty)A_1\lambda_1 e^{-\lambda_1 \tau}\sin(\lambda_1 x/L) \quad \text{(W)}$$

2) Total heat transfer from the system to the ambient

$Q_{max} = \rho V C_p (T_\infty - T_i)$ (kJ) energy conservation $(t \to \infty)$

3) Total amount of heat transfer up to time t

$$Q(t) = \rho C_p A\int_{-L}^{L}[T(x,t) - T_i]dx \qquad (V = 2AL)$$

$$\frac{Q(t)}{Q_{max}} = \frac{1}{2L}\int_{-L}^{L}\frac{T(x,t)-T_i}{T_\infty - T_i}dx = 1 - \frac{1}{2L}\int_{-L}^{L}\frac{T(x,t)-T_\infty}{T_i - T_\infty}dx$$

$$= 1 - \frac{1}{2L}\int_{-L}^{L}A_1 e^{-\lambda_1^2\tau}\cos\left(\frac{\lambda_1 x}{L}\right)dx = 1 - \frac{\theta_{o,wall}}{2\lambda_1}\sin\left(\frac{\lambda_1 x}{L}\right)\Big|_{-L}^{L} = 1 - \theta_{o,wall}\frac{\sin\lambda_1}{\lambda_1}$$

❖ Graphical Representation of Solutions: Non-dimensional Form (Heisler's Charts)

- **Express the result in non-dimensional form.**

$$\theta(x,t) = \sum_{n=1}^{\infty}\frac{2\sin(\lambda_n)}{\lambda_n + \sin\lambda_n\cos\lambda_n}e^{-\lambda_n^2\tau}\cos(\lambda_n X)$$

Dimensionless quantities **Characteristic length : L**

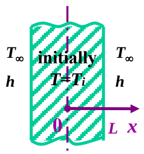
$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty} \qquad X = x/L \qquad \tau = \alpha t/L^2 \qquad \text{Bi} = hk/L$$

$$\therefore \qquad \theta(x,t) = f(\text{Bi}, \ \tau, \ X)$$

- **Heisler charts: Dimensionless quantities are used to construct charts to determine transient temperature in plates**

Fig. 4-7 (a) Midplane transient temperature of a plate of thickness 2*L*

Fig. 4-7 (b) Temperature distribution in a plate of thickness 2*L*



$$\frac{T(x)-T_\infty}{T_i - T_\infty} = \frac{T(x,t)-T_\infty}{T_o - T_\infty}\cdot\frac{T_o - T_\infty}{T_i - T_\infty}; \qquad f\left(\text{Bi}, \frac{x}{L}\right)\cdot f(\text{Bi}, \ \tau)$$

$$\theta(x,t) = \theta_m \cdot \theta_0$$

$(\text{Bi}, \frac{x}{L}) \Rightarrow \theta_m = \dfrac{T(x,t) - T_\infty}{T_o - T_\infty}$ Fig. 4-7 (b)

$(\text{Bi}, \ \tau) \Rightarrow \theta_0 = \dfrac{T_o - T_\infty}{T_i - T_\infty}$ Fig. 4-7 (a)

Problem a) $t \to T$?

Bi, τ, x/L ⟹ Fig. 4-7 (a) θ_o, Fig. 4-7 (b) θ_m ⟹ calculate $\theta = \theta_m \cdot \theta_o = \dfrac{T - T_\infty}{T_i - T_\infty} \Rightarrow T$

Problem b) $T \to t$?

Bi, x/L ⟹ Fig. 4-7 (b) θ_m ⟹ calculate $\theta_o = \theta / \theta_m$

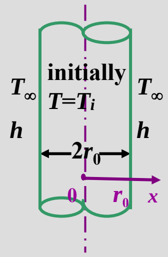
$\tau = \alpha t/L^2 \Rightarrow t$ ⟸ Fig. 4-7 (a) τ

$$Q_{max} = \rho V C_p (T_\infty - T_i) \quad \text{(kJ)} \Rightarrow Q = \left(\frac{Q}{Q_{max}}\right) \cdot Q_{max}$$

Fig. 4-7 (c) Heat transfer rate in a plate of thickness 2L

4.2.2 1-D transient Cond. in an infinite long Cyl.

- no energy generation and constant conductivity



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

B.C.

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0$$

$$-k\left.\frac{\partial T}{\partial r}\right|_{r=r_0} = h\left[T(r_0,t) - T_\infty\right]$$

I.C. $\quad T(r,0) = T_i$

Non-dimensionalization

- Dimensionless temperature:

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty}$$

- Dimensionless distance:

$$R = r/r_0$$

- Dimensionless time : (Fourier number)

$$\tau = at/r_0^2$$

- Biot number

$$\mathrm{Bi} = hr_0/k$$

Characteristic length : r_0

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \theta}{\partial R}\right) = \frac{\partial \theta}{\partial \tau}$$

B.C.

$$\left.\frac{\partial \theta}{\partial R}\right|_{R=0} = 0$$

$$\left.\frac{\partial \theta}{\partial R}\right|_{R=1} = -\mathrm{Bi}\cdot\theta(1,\tau)$$

I.C.

$$\theta(R,0) = 1$$

Solution and Heisler's Charts for Cylinders

- Temperature distribution — Fig. 4-8 (b)

$$\theta(r,t) = f(\mathrm{Bi}, \ \tau, \ R)$$

Zero-order Bessel function of the first kind

$$\theta(r,t)_{cyl} = \frac{T(r,t) - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau} J_0(\lambda_1 R), \quad \tau > 0.2$$

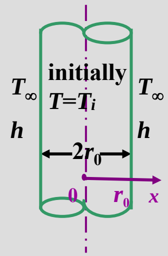
- Midpoint temperature — Fig. 4-8 (a)

$$\theta_{o,cyl} = \frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau}$$

- Heat transfer rate — Fig. 4-8 (c)

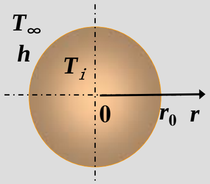
$$\left(\frac{Q}{Q_{max}}\right)_{cyl} = 1 - 2\theta_{0,cyl}\frac{J_1(\lambda_1)}{\lambda_1}$$

$$Q_{max} = \rho V C_p (T_\infty - T_i) \quad \text{(kJ)}$$



4.2.3 1-D transient conduction in spheres

- no energy generation and constant conductivity



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

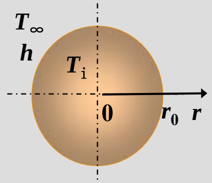
B.C.

$$\left.\frac{\partial T}{\partial r}\right|_{r=0}=0$$

$$-k\left.\frac{\partial T}{\partial r}\right|_{r=r_0}=h\left[T(r_0,t)-T_\infty\right]$$

I.C. $\quad T(r,0)=T_i$

Solution and Heisler's Charts for Spheres



- Temperature distribution — Fig. 4-9 (b)

$$\theta(r,t)_{\text{sph}}=\frac{T(r,t)-T_\infty}{T_i-T_\infty}=A_1e^{-\lambda_1^2\tau}\frac{\sin(\lambda_1 R)}{\lambda_1 R},\quad \tau>0.2$$

- Midpoint temperature — Fig. 4-9 (a)

$$\theta_{0,\text{sph}}=\frac{T_0-T_\infty}{T_i-T_\infty}=A_1e^{-\lambda_1^2\tau}$$

- Heat transfer rate — Fig. 4-9 (c)

$$\left(\frac{Q}{Q_{\max}}\right)_{\text{sph}}=1-3\theta_{0,\text{sph}}\frac{\sin\lambda_1-\lambda_1\cos\lambda_1}{\lambda_1^3}$$

Characteristic length : r_0

❖ Applicability

- no energy generation and constant conductivity ($\dot{g}=0,\ k=\text{const}$)
- $T(x,0)=T_i$ (*uniform*)
- $h, T_\infty=\text{const}$ ($h\to\infty, T_s\to T_\infty$, specified temperature B.C.)

❖ Note

- $\text{Bi}_V\le 0.1$, alternative lumped system analysis.

	$\text{Bi}_V=\frac{h\,(V/A)}{k}$	$\text{Bi}=\frac{h\,L_c}{k}$	
Plane wall ($2L$)	$\frac{V}{A}=\frac{A\cdot 2L}{2A}=L$	Bi=hL/k	$\text{Bi}_V=\text{Bi}$
Cylinder (r_0)	$\frac{V}{A}=\frac{\pi r_0^2H}{2\pi r_0H}=\frac{r_0}{2}$	Bi=hr_0/k	$\text{Bi}_V=\frac{\text{Bi}}{2}$
Sphere(r_0)	$\frac{V}{A}=\frac{4\pi r_0^3/3}{4\pi r_0^2}=\frac{r_0}{3}$	Bi=hr_0/k	$\text{Bi}_V=\frac{\text{Bi}}{3}$