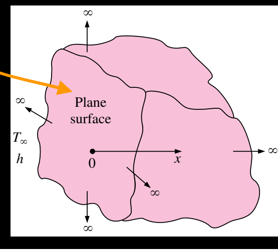
§ 4-3

Transient heat cond. in semi-infinite solids

Concept of semi-infinite solids

- **An idealized body**
- **It has a *single plane surface* and extends to infinity in all direction**

e.g. earth, furnace/building base

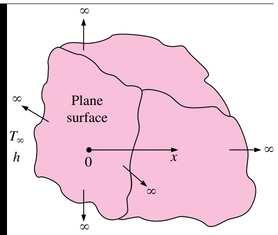


$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}$$

$$t = 0, T(x,0) = T_i$$

$$t > 0, x = 0, h[T_\infty - T(0,t)] = -k\frac{\partial T(0,t)}{\partial x}$$

$$x \to \infty, T(\infty,t) = T_i$$



$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty}$$

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial x^2}$$

$$t = 0, \theta(x,0) = 0$$

$$t > 0, x = 0, h[1 - \theta(0,t)] = -k\frac{\partial \theta(0,t)}{\partial x}$$

$$x \to \infty, \theta(\infty,t) = 0$$

Solution:

$$1 - \theta(x,t) = \frac{T(x,t) - T_i}{T_\infty - T_i}$$

$$= \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right)\left[\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)\right]$$

Fig.4-13

Where complementary error function is defined as

$$\operatorname{erfc}(\xi) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-u^2} du$$

Table 4-3

$$\operatorname{erfc}'(\xi) = -\frac{2}{\sqrt{\pi}} e^{-\xi^2} \cdot \xi'$$

❑ **Specified temperature B.C. ($h \to \infty$, $T_s \to T_\infty$)**

$$1-\theta(x,t)=\frac{T(x,t)-T_i}{T_s-T_i}=\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

- **Local transient heat flux:**

$$\dot{q}(x,t)=-k\frac{\partial T(x,t)}{\partial x}=-k(T_s-T_i)\frac{\partial}{\partial x}[\mathrm{erfc}(\frac{x}{2\sqrt{at}})]$$

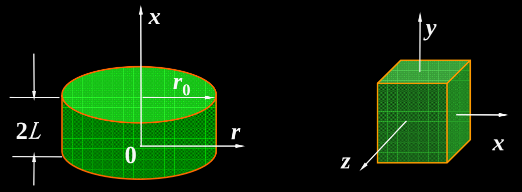
$$=k\frac{T_s-T_i}{\sqrt{\pi a t}}e^{-x^2/(4at)}$$

$$\mathrm{erfc}'(\xi)=-\frac{2}{\sqrt{\pi}}e^{-\xi^2}\cdot\xi'$$

- **Surface transient heat flux (x = 0) :**

$$\dot{q}_s(t)=\frac{k(T_s-T_i)}{\sqrt{\pi a t}}$$

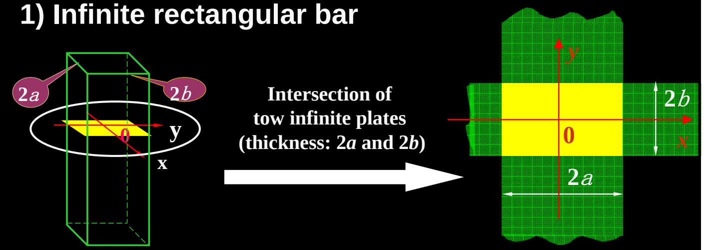
§ 4-4
Transient heat cond. in multidimensional system



Product solutions

- **The solutions for a regular multi-dimensional solid is the product of the solutions for 1-D geometries whose intersection is the multidimensional body.**

1) Infinite rectangular bar



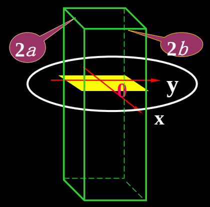
Heat conduction equation is :

$$\frac{\partial\theta}{\partial t}=a\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}\right) \qquad \theta=\frac{T-T_\infty}{T_i-T_\infty}$$

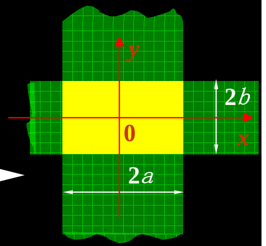
Assume $\theta(x,y,t)=\theta_x(x,t)\cdot\theta_y(y,t)$ **so**

$$\theta_y\left(\frac{\partial\theta_x}{\partial t}-a\frac{\partial^2\theta_x}{\partial x^2}\right)=\theta_x\left(\frac{\partial\theta_y}{\partial t}-a\frac{\partial^2\theta_y}{\partial y^2}\right)$$

1) Infinite rectangular bar



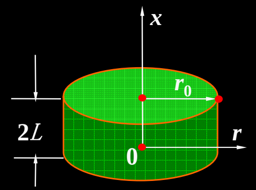
Intersection of
tow infinite plates
(thickness: 2*a* and 2*b*)

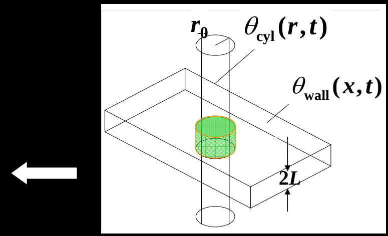


$$\theta_{\substack{\text{rectangular}\\\text{bar}}}(x,y,t)=\theta_{\text{wall}1}(x,t)\cdot\theta_{\text{wall}2}(y,t)$$

$$\left(\frac{T(x,y\ t)-T_\infty}{T_i-T_\infty}\right)_{\substack{\text{rectangular}\\\text{bar}}}=\left(\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right)_{\text{wall}1}\left(\frac{T(y,t)-T_\infty}{T_i-T_\infty}\right)_{\text{wall}2}$$

2) Short cylinder





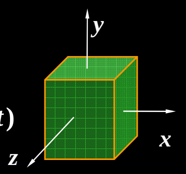
an infinite cylinder and
an infinite plane wall

$$\theta_{\substack{\text{short}\\\text{cyl}}}(x,r,t)=\theta_{\text{cyl}}(r,t)\cdot\theta_{\text{wall}}(x,t)$$

3) Rectangular parallel-piped

three infinite plane walls

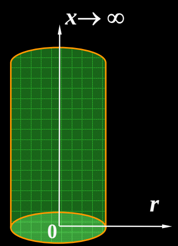
$$\theta(x,y,z,t)=\theta_{\text{wall}1}(x,t)\cdot\theta_{\text{wall}2}(y,t)\cdot\theta_{\text{wall}3}(z,t)$$



4) Semi-infinite Cylinder

an infinite cylinder and a semi-infinite solids

$$\theta(x,r,t)=\theta_{\text{cyl}}(r,t)\cdot\theta_{\text{semi-inf}}(x,t)$$



Product solutions

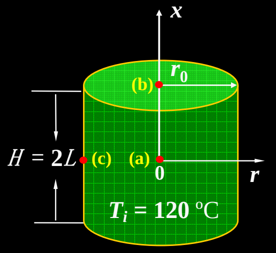
- The solutions for a regular multi-dimensional solid is **the product of the solutions** for 1-D geometries whose intersection is the multidimensional body.

So $\theta_{2D}=\theta_{1D}\cdot\theta_{1D}$

$\theta_{3D}=\theta_{1D}\cdot\theta_{1D}\cdot\theta_{1D}$

Example 4-5 A short brass cylinder (D = 10 cm , H = 12 cm) t = 15 min

(a) $T(0,0,t) = ?$ (b) $T(L,0,t) = ?$ (c) $T(0, r_0, t) = ?$



h = 60 W/m² · °C
T_∞ = 25 °C

Solution :

(a)

$$\left(\frac{T(0,0,t)-T_\infty}{T_i-T_\infty}\right)_{\text{short cyl}} = \theta_{\text{wall}}(0,t)\,\theta_{\text{cyl}}(0,t)$$

(b)

$$\left(\frac{T(L,0,t)-T_\infty}{T_i-T_\infty}\right)_{\text{short cyl}} = \theta_{\text{wall}}(L,t)\,\theta_{\text{cyl}}(0,t)$$

(c)

$$\left(\frac{T(0,r_0,t)-T_\infty}{T_i-T_\infty}\right)_{\text{short cyl}} = \theta_{\text{wall}}(0,t)\,\theta_{\text{cyl}}(r_0,t)$$

Transient heat transfer

$$\left(\frac{Q}{Q_{\max}}\right)_{\text{total,2D}} = \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2\left[1-\left(\frac{Q}{Q_{\max}}\right)_1\right]$$

$$\left(\frac{Q}{Q_{\max}}\right)_{\text{total,3D}} = \left(\frac{Q}{Q_{\max}}\right)_1 + \left(\frac{Q}{Q_{\max}}\right)_2\left[1-\left(\frac{Q}{Q_{\max}}\right)_1\right] + \left(\frac{Q}{Q_{\max}}\right)_3\left[1-\left(\frac{Q}{Q_{\max}}\right)_1\right]\left[1-\left(\frac{Q}{Q_{\max}}\right)_2\right]$$

❖ Applicability of Product solutions

- No heat generation and constant thermal conductivity ($\dot{g} = 0, \quad k = \text{const}$)
- $T\big|_{t=0} = T_i \quad (uniform)$
- **Convection B.C. and** $h, T_\infty = \text{const}$

  $(h \to \infty, T_s \to T_\infty$, **specified temperature B.C.**)