# Are heliospheric flows magnetic line- or flux-conserving?

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**Abstract.** This article discusses and tests the validity of the frozen in magnetic field paradigm (or 'ideal magnetohydro-dynamics (MHD) constraint') which is usually adopted by many authors dealing with heliospheric physics.

To show the problem of using ideal MHD in such a counterflow configuration like the heliosphere, we first recapitulate the basic concepts of freezing-in of magnetic fields, respectively magnetic topology conservation and its violation (= magnetic reconnection) in 3-D, already done by other authors with different methods with respect to derivations and interpretations. Then we analyse different heliospheric plasma environments. As a model of the stagnation region/stagnation point in front of the heliospheric nose, we present and discuss the general solution of the ideal MHD Ohm's law in the vicinity of a 2-D stagnation point, which was found by us.

We show that ideal MHD either leads necessarily to a diverging magnetic field strength in the vicinity of such a stagnation point, or to a vanishing mass density on the heliopause boundaries. In the case that components of the electric field parallel to the magnetic field do not exist due to the chosen form of the non-ideal Ohm's law, it is always possible to formulate the transport equation of the magnetic field as a modified ideal Ohm's law.

We find that the form of the Ohm's law which is often used in heliospheric physics (see e.g. Baranov and Fahr, 2003), is not able to change magnetic topology and thus cannot lead to magnetic reconnection, which necessarily has to occur at the stagnation point. The diverging magnetic field, for instance, implies the breakdown of the flux freezing paradigm for the heliosphere. Its application, especially at the heliospheric nose, is therefore rather doubtful. We conclude that it is necessary to search for an Ohm's law which is able to violate magnetic topology conservation.

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## 1 Introduction

An important question with respect to heliospheric physics is, whether the magnetic field is frozen into the plasma flow, or if topology changes of magnetic field can take place, e.g. in the vicinity of the heliopause. The topology change would imply, that the heliosphere could become leaky due to magnetic reconnection processes. Whether this happens or not depends on the shape of Ohm's law for the complex plasma interface region between the outer heliosphere and the Very Local Interstellar Medium (VLISM).

The question which velocity fields do transport magnetic flux and how this is connected with the velocity fields of plasma components was discussed, e.g. by Newcomb (1958). Stern (1966) (and references given therein) gives a good review about the topology conservation of general vector fields. Here we will apply these concepts to the heliosphere to shed light on the question, how leaky such an astrospheric boundary can be due to non-ideal plasma processes.

This paper is organized in the following way: First, in Sect. 2 and partially in Sect. 4, a necessary review of the theorems respectively the equations which are important for the derivation of the transport equation for the flux of a vector field with a vanishing divergence is given. To demonstrate the connection between flux conservation and topology (conservation) of such a vector field the partial differential equation (PDE) for a velocity is derived in Sect. 3. This PDE guarantees the connectivity of the field lines during the time dependent or stationary evolution of the electromagnetic field. In Sect. 4 a reasonable connection between magnetic reconnection and the possibility of defining flux transporting velocity fields of non-ideal MHD flows, starting with a generalized non-ideal term on the right hand side of Ohm's law, is given. Then, in Sect. 5 it is shown, how the results of Sect. 2 and Sect. 3 work in the vicinity of the magnetopause of the Earth, where a simplified version of the two fluid MHD, the so called Hall MHD (with neglected elec-
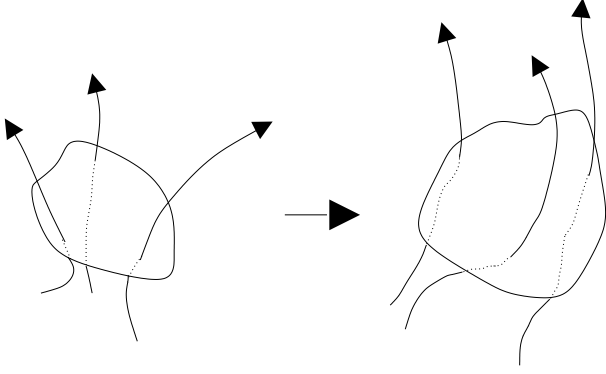
**Fig. 1.** Flux conservation: The magnetic flux of a magnetic field, intersecting a closed fluid line, does not change during the movement of the closed fluid line.

tron inertia) is given. Section 6 presents an application to an Ohm's law which is valid within the heliosphere: Here it is shown how flux conservation works in a multifluid plasma. A discussion is started about the problem that a flux conserving velocity field should have a connection to a plasma species velocity or any other velocity related to the plasma (components). In Sect. 7 we discuss the 'frozen in' or ideal MHD paradigm of heliospheric physics and the problem of validity in the vicinity of a stagnation point (here: the heliospheric nose). In the last section we summarize the results of our discussion and draw conclusions, leading us to a series of important questions, which remain open and could only be answered by future investigations.

## 2 The derivation of the flux conserving velocity field

We write down a generalized Ohm's law

$$E + v_p \times B = R,\tag{1}$$

where $E$ is the electric field and $B$ is the magnetic field. Here we introduce the term $R$, a generalized non-ideal term (see Priest and Forbes, 2000, and references therein, p. 41), $v_p$ is a velocity which is not necessarily the ion velocity. The velocity may be an averaged velocity of (all) existing plasma species or their weighted mean, or the velocity of a certain plasma species. An example that $v_p$ is dependent on the multifluid character of the plasma and the corresponding Ohm's law will be given in Sect. 5.

Strictly speaking, the ideal Ohm's law is a 'relic' of the momentum equation of the electron in the two-fluid theory, containing e.g. 'friction' respectively resistive terms, so called inertial terms, additional isotropic or anisotropic pressure terms, Hall term and so on (see Priest and Forbes, 2000). In the case of multifluid MHD, especially for partially ionized plasmas, or plasmas with different ion species, the terms on the right hand side of Ohm's law could look even much

more complicated (see Schlüter, 1958; Kulikovskii and Lyubimov, 1965).

If we now take also Faraday's law

$$\nabla \times E = -\frac{\partial B}{\partial t}\tag{2}$$

into consideration we get

$$\frac{\partial B}{\partial t} = \nabla \times (v_p \times B) - \nabla \times R.\tag{3}$$

The criterion for magnetic flux conservation is given by the following condition: The magnetic flux $\Phi_m$ intersecting the area which is enclosed by a closed fluid line with the surface $F$, does not change during the movement of the closed fluid line (see Fig. 1). Using Ohm's law, Faradays law and the Gauss-Stokes theorem we can recapitulate the condition for flux conservation in the form of the vanishing convective derivative of the magnetic flux $\Phi_m$, see e.g. the derivation in Priest and Forbes (2000):

$$\frac{d\Phi_m}{dt} = 0 \equiv \frac{d}{dt} \int_F B \cdot df\tag{4}$$

$$= \int_F \frac{\partial B}{\partial t} \cdot df + \int_{\partial F} B \cdot (v_p \times ds)\tag{5}$$

$$= \int_F \frac{\partial B}{\partial t} \cdot df - \int_{\partial F} (v_p \times B) \cdot ds,\tag{6}$$

yielding

$$\frac{d\Phi_m}{dt} = 0\tag{7}$$

$$= \int_F \frac{\partial B}{\partial t} \cdot df - \int_F \nabla \times (v_p \times B) \cdot df\tag{8}$$

$$\Rightarrow \qquad 0 = \int_F (-\nabla \times R) \cdot df\tag{9}$$

$$\Rightarrow \quad \nabla \times R = 0,\tag{10}$$

where $F$ is the comoving surface enclosed by the fluid line, and $\partial F$ the closed fluid line itself, $ds$ the line element of the closed fluid line, and $df$ the corresponding surface element. This implies, that it is a necessary and sufficient criterion for flux conservation that the non-ideal term can be written as a gradient, if Ohm's law is given by Eq. (1). The physical interpretation is, that any moving closed fluid line in this ideal plasma (ideal conductor), which is intersected by the magnetic field, encloses the same (integrated) magnetic flux at every moment, as illustrated in Fig. 1.

In general the above shown procedure can also be interpreted in the following way: for a given $E(x, t)$ and $B(x, t)$ a velocity field $\mathbf{v}(x, t)$ which represents the transport of the magnetic field, especially the magnetic flux, must be found such that $E^\star$ defined by

$$E^\star := E + \mathbf{v} \times B\tag{11}$$

can be written as a gradient. The velocity field $\mathbf{v} \neq \boldsymbol{v}_p$ is not necessarily a plasma (species) velocity. The equation to solve is then

$$\nabla \times (\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}) = \mathbf{0}$$
$$\Leftrightarrow \frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{B}) = \mathbf{0}, \tag{12}$$

with $\mathbf{v}$ as an 'abstract' velocity field which describes the movement of the magnetic field. Thus the vector field $\mathbf{v}$ is nothing else than the transport velocity of the magnetic flux. Therefore the relation Eq. (12) also holds, if a kinetic description of the plasma has to be taken into account. The relation above and also the field line conservation critereon (see below Eq. (28)) do only depend on the structure of the magnetic field and not on the question if a kinetic description or a fluid theory is used. The Eq. (12) is therefore an exact definition for the flux conserving velocity field and is equivalent to Newcomb's result, see the system Eq. (31) and Eq. (32). With additional appropriate boundary conditions, this linear partial differential equation will have a unique solution for given respectively known magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$. For reasonable physical boundary conditions it may happen that there is no solution, which is a criterion for magnetic reconnection (see Sect. 4).

An example is given for an ideal MHD plasma, which has only localized nonideal regions, see Sect. 4. In Sect. 4.1 and also Sect. 4.2 it can also be seen that there is a difference between $\boldsymbol{v}_p$ and $\mathbf{v}$. In the case of a magnetic neutral sheet the flux conserving velocity field is a field aligned flow, in contrast to the plasma flow, which may have perpendicular components if e.g. a resistive term is present in this region around the current sheet, as discussed in Sect. 4.2.

## 3 Line conservation

The condition for topology, respectively line conservation for vector fields was derived in different ways, e.g. by Newcomb (1958) and Stern (1966). We will now derive this condition for line conserving flows in a different way and show the connection to flux conservation with a vivid respectively visual method: The sketch in Fig. 2 shows how plasma elements are 'mapped' within time from one field line to another. In an ideal conductor, but in general also in a non-ideal but magnetic topology conserving plasma flow, two plasma elements which are lying on one field line, connected by the arc length or line element $\delta l$ at a certain time $t$ are also connected by one field line after each time step $dt$, i.e. are also lying on one field line at a later time $t + dt$. The plasma elements are then connected by the line element $\delta l + d(\delta l)$, and the corresponding tangential vectors to the magnetic field lines are given by

$$\delta \boldsymbol{l} = |\delta l| \, \boldsymbol{B}/|\boldsymbol{B}| \equiv \delta l \, \boldsymbol{B}/|\boldsymbol{B} \tag{13}$$

and by analogy

$$\delta \boldsymbol{l} + d(\delta \boldsymbol{l}) = (\delta l + d(\delta l)) \, \boldsymbol{B}/|\boldsymbol{B}|. \tag{14}$$
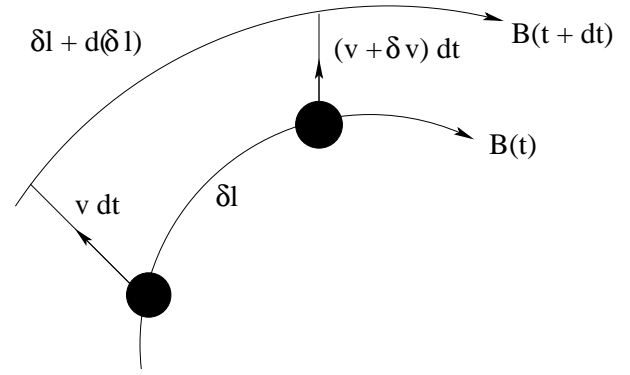


**Fig. 2.** This sketch shows how plasma elements are 'mapped' within time from one field line to another. In an ideal conductor two plasma elements which are lying on one field line at a certain time $t$ are also connected by one field line after each time step $dt$, i.e. are also lying on one field line at a later time $t + dt$. This defines magnetic connection (magnetic topology) and enables to find topology conserving velocity fields (conserving the line connection).

The above described behaviour defines magnetic connection, respectively magnetic topology (conservation) and enables to find corresponding velocity fields with the help of the definitions in Eq. (13), which we will show now: Using the Taylor expansion for the velocity field $\mathbf{v}$ (for a fixed time) we get

$$\delta \mathbf{v} = (\delta \boldsymbol{l} \cdot \nabla)\mathbf{v} \tag{15}$$

and with the help of Fig. 2

$$\delta \boldsymbol{l} + d(\delta \boldsymbol{l}) = \delta \boldsymbol{l} + (\mathbf{v} + \delta \mathbf{v})dt - \mathbf{v}\,dt \tag{16}$$

it follows

$$\frac{d\delta \boldsymbol{l}}{dt} = \delta \mathbf{v} = (\delta \boldsymbol{l} \cdot \nabla)\mathbf{v}. \tag{17}$$

With

$$\delta \boldsymbol{l} \times \boldsymbol{B} = \mathbf{0} \tag{18}$$

as initial condition, it is necessary for line conservation (conservation of line connection), i.e. for magnetic topology conservation to demand that the time derivative of the initial condition Eq. (18) vanishes, i.e.

$$\frac{d(\delta \boldsymbol{l} \times \boldsymbol{B})}{dt} = \mathbf{0}. \tag{19}$$

The left hand side of Eq. (19) results in

$$\frac{d(\delta \boldsymbol{l} \times \boldsymbol{B})}{dt} = \frac{d(\delta \boldsymbol{l})}{dt} \times \boldsymbol{B} + \delta \boldsymbol{l} \times \frac{d\boldsymbol{B}}{dt} \tag{20}$$
$$= [(\delta \boldsymbol{l} \cdot \nabla)\mathbf{v}] \times \boldsymbol{B}$$
$$\quad + \delta \boldsymbol{l} \times [(\boldsymbol{B} \cdot \nabla)\mathbf{v} - \boldsymbol{B}(\nabla \cdot \mathbf{v}) - \nabla \times \boldsymbol{E}^{\star}]. \tag{21}$$

With the help of the convective derivative of $\boldsymbol{B}$, the 'convective' electric field $\boldsymbol{E}^{\star}$ in Eq. (11), and Faraday's law (Eq. (2))

the operator on the left hand side of Eq. (12) can be written as:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} = \nabla \times (-\boldsymbol{E}^{\star} + \mathbf{v} \times \boldsymbol{B}) \tag{22}$$

$$= (\boldsymbol{B} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\boldsymbol{B} - (\nabla \cdot \mathbf{v})\boldsymbol{B} - \nabla \times \boldsymbol{E}^{\star}. \tag{23}$$

With Eq. (13) we see that

$$(\delta \boldsymbol{l} \cdot \nabla)\mathbf{v} \times \boldsymbol{B} = -\delta \boldsymbol{l} \times (\boldsymbol{B} \cdot \nabla)\mathbf{v} \tag{24}$$

is valid. Thus, the first term and the first term in brackets of the line conserving constraint Eq. (21) cancel each other and it follows

$$0 = \delta \boldsymbol{l} \times [-\boldsymbol{B}(\nabla \cdot \mathbf{v}) - \nabla \times \boldsymbol{E}^{\star}] \tag{25}$$

$$= -\delta \boldsymbol{l} \times (\nabla \times \boldsymbol{E}^{\star}). \tag{26}$$

As $\delta \boldsymbol{l} \parallel \boldsymbol{B}$[1] we can therefore conclude that

$$\boldsymbol{B} \times (\nabla \times \boldsymbol{E}^{\star}) = \boldsymbol{0} \tag{27}$$

$$\Leftrightarrow \quad \frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{B}) = \lambda \boldsymbol{B}. \tag{28}$$

This equation is similar to the well known induction equation of ideal MHD and is a differential equation for calculating the line respectively topology conserving velocity field **v** for a given magnetic field. The difference to the usual induction equation is the term $\lambda \boldsymbol{B}$ which corresponds to the freedom of the motion of the flux-/line conserving velocity field in the direction of the magnetic field. For vanishing $\lambda$ we conclude that **v** conserves magnetic flux, for non-vanishing $\lambda$ magnetic lines are conserved (magnetic flux is transported along the fieldlines). For ergodic vector fields $\boldsymbol{B}$, i.e. for magnetic fields for which no first integrals exist, $\lambda$ must be a constant. This implies that the general solution of Eq. (28) has a 'continous spectrum' and the form of an eigenvalue problem.

With this it is shown:

– Flux conservation implies line conservation, but not vice versa (Stern, 1966)!

– A non-ideal term is necessary for violation of line or flux conservation, but not sufficient.

## 4 Searching for 'redefined' velocity fields

The chosen plasma velocity field $v_p$ in generalized Ohm's law $\boldsymbol{E} + \boldsymbol{v}_p \times \boldsymbol{B} = \boldsymbol{R}$, where $\boldsymbol{R}$ is a given non-idealness, is only flux conserving, if $\boldsymbol{R}$ can be written as a gradient. If $\boldsymbol{R}$ cannot be written as a gradient it is possible to find a 'substitute' or 'redefined' velocity field which is flux conserving/transporting instead. In the first subsection we will show how this works for the general case $\boldsymbol{B} \neq \boldsymbol{0}$, and in the second subsection the problem will be briefly discussed for the special case $\boldsymbol{B} = \boldsymbol{0}$ locally.

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[1] Due to the initial condition $\delta \boldsymbol{l} \times \boldsymbol{B} = \boldsymbol{0}$.

### 4.1 The case: $\boldsymbol{B} \neq \boldsymbol{0}$ everywhere

To ensure the ideal transport of the magnetic flux and to ensure topology conservation it is necessary to find a function $X$ and a vector field **v** such that for known $\boldsymbol{E}$ and $\boldsymbol{B}$, and an Ohm's law given by $\boldsymbol{E} + \boldsymbol{v}_p \times \boldsymbol{B} = \boldsymbol{R}$,

$$\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B} = \nabla X \quad \Leftrightarrow \quad \boldsymbol{E} + \boldsymbol{v}_p \times \boldsymbol{B} = \boldsymbol{R}, \tag{29}$$

where $\boldsymbol{v}_p$ is a plasma velocity and $\boldsymbol{R}$ a given non-idealness. By solving for **v** the magnetic flux is frozen-in with respect to velocity fields

$$\mathbf{v} = \boldsymbol{v}_p - \frac{\boldsymbol{B} \times (\boldsymbol{R} - \nabla X)}{|\boldsymbol{B}|^2} + \nu \boldsymbol{B} \tag{30}$$

or, equivalently

$$\mathbf{v} = \frac{(\boldsymbol{E} - \nabla X) \times \boldsymbol{B}}{|\boldsymbol{B}|^2} + \tilde{\nu} \boldsymbol{B} \tag{31}$$

which was found first by Newcomb (1958). Here $\nu$, respectively $\tilde{\nu}$ are functions in space and $X$ is the solution of the PDE

$$\boldsymbol{B} \cdot \nabla X = \boldsymbol{R} \cdot \boldsymbol{B} \equiv \boldsymbol{E} \cdot \boldsymbol{B}. \tag{32}$$

That this equation needs appropriate boundary conditions and/or auxiliary conditions had not been noted by Newcomb (1958). The fulfillment of Eq. (32) is a necessary condition for the fulfillment of Eq. (31). The system Eq. (32) and Eq. (31) is equivalent to Eq. (12). Note: The whole procedure, discussed in this section, is only necessary and reasonable if $\boldsymbol{R}$ is not itself a gradient.

The parallel transport of the magnetic field, described by the term $\nu \boldsymbol{B}$ respectively $\tilde{\nu} \boldsymbol{B}$ does not influence the flux-freezing properties of the flow with respect to the magnetic field. Nevertheless, we briefly discuss the meaning of the the function $\nu$ respectively $\tilde{\nu}$: these functions are connected with the parallel Alfvén Mach number of the flow $M_{A\parallel}$ by

$$\tilde{\nu} := \frac{\pm M_{A\parallel}}{\sqrt{\mu_0 \rho}}. \tag{33}$$

Since $M_{A\parallel}$ is defined by

$$|\mathbf{v}_\parallel|^2 = \frac{M_{A\parallel}^2}{\mu_0 \rho} |\boldsymbol{B}|^2, \tag{34}$$

it follows that $\nu$ respectively $\tilde{\nu}$ is connected with the magnetic field. Here the mass density is $\rho$. The magnetic field aligned component of the velocity field is $\mathbf{v}_\parallel$.

To find solutions of Eq. (32), appropriate constraints and/or appropriate boundary conditions are needed. In most cases in astrophysical plasmas, the $\boldsymbol{E} \times \boldsymbol{B}$–drift velocity ($\boldsymbol{v}_\perp := \boldsymbol{E} \times \boldsymbol{B}/|\boldsymbol{B}|^2$) is the perpendicular component of the plasma velocity $\boldsymbol{v}_p$ and is a good approximation for the flux transporting velocity **v** almost everywhere (see Priest and Forbes, 2000). Thus only in a small subset of the domain

the perpendicular component of the flux transporting velocity is different from the velocity of the $E \times B$-drift. This can happen e.g. due to MHD, multifluid-MHD- or electrostatic turbulence, leading to non negligible non-ideal terms or resistive terms (current driven) on the right hand side of Ohm's law. The deviations from the drift velocity $E \times B$ are often strictly localized in space, which has to be taken into account for the boundary conditions of the potential $X$. Thus the flux conserving velocity field inside this non-ideal region $D_R$ must converge to the usual flux conserving velocity field between the boundary of the ideal and the non-ideal region, i.e. on the boundary $\partial D_R$ of the non-ideal domain $D_R$. We conclude here: To enable the convergence $\mathbf{v}_\perp \to \mathbf{v}_\perp$ outside the non-ideal domain, it is necessary to assume that $\nabla X \times B = \mathbf{0}$ outside $D_R$ in Eq. (31). This leads us to

$$\nabla X = \mu B \,, \tag{35}$$

where $\mu$ is a scalar function. Taking the curl of Eq. (35) we get

$$\nabla \mu \times B + \mu \, \nabla \times B = \mathbf{0} \,, \tag{36}$$

yielding

$$(\nabla \times B) \cdot B = 0 \,. \tag{37}$$

This equation implies either only perpendicular currents, writing

$$j \cdot B = 0 \tag{38}$$

with respect to the magnetic field, or with $\mu = \text{const}$ it follows

$$\nabla \times B = \mathbf{0} \,, \tag{39}$$

implying that the magnetic field must be potential. This result allows only a very special class of magnetic fields to be transported by the $E \times B$-drift velocity. In addition the influence of the $E_\parallel := \frac{(E \cdot B)}{|B|^2} B$ should be minimized. Strong electric field components will accelerate each of the charged particle species in a different way, e.g. with respect to the different masses of ions and electrons. This will lead to strong charge separation, induce additional collisions between the particles and thus destroy the quasi-neutrality of the plasma and the idealness of the plasma. Due to this reason it makes sense to assume that outside the non-ideal region $D_R$, $E_\parallel = 0$. In fact $E \cdot B = 0$ is the (necessary) condition for ideal MHD[2], because this condition implies (i) magnetic topology conservation and (ii) the fact that without any restriction $X$ can be set constant everywhere outside the non-ideal domain $D_R$. There is another reason that $X$ must be zero or at least constant on the boundary: Assuming that the field is ideal, i.e. $E \cdot B = 0$, outside a non-ideal domain $D_R$ the equation $B \cdot \nabla X = 0$ cannot be fulfilled in the case

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[2]The necessary condition for the ideal transport of the magnetic field is only that $R$ could be written as a gradient.

of general 3-D magnetic fields. The reason for that is that no first integral $X$ exists in the case of ergodic fields (see the discussion and the references in the book of D'haeseleer, 1990).

The converging of $\mathbf{v} \to \mathbf{v}_p$ is therefore guaranteed by the boundary condition

$$X = \text{const} \quad \forall \, \mathbf{x} \in D \setminus D_R \text{ and especially on } \partial D_R \,, \tag{40}$$

only, which is discussed in similar form in Priest et al. (2003), but with a slightly different interpretation. Using the identities

$$E \cdot B = R \cdot B \,, \tag{41}$$

$$\nabla \cdot (XB) = B \cdot \nabla X \,, \tag{42}$$

we calculate

$$\int_D B \cdot \nabla X \, dV = \int_D R \cdot B \, dV \tag{43}$$

$$\Rightarrow \int_D \nabla \cdot (XB) \, dV = \int_D E \cdot B \, dV \tag{44}$$

$$\Rightarrow \int_{\partial D} X B \cdot dS = \int_{D_R} E \cdot B \, dV \tag{45}$$

$$\overset{(40)}{\Rightarrow} \quad X \underbrace{\int_{\partial D} B \cdot dS}_{\equiv 0} = \int_{D_R} E \cdot B \, dV \,. \tag{46}$$

One can see very easily that the left hand side due to the boundary conditions Eq. (40) can only be fulfilled, if and only if the right hand side integral in Eq. (46) vanishes identically. In contrast to the authors in Priest et al. (2003), who conclude that there will be no solution in general as $E \cdot B \neq 0$ in the non-ideal domain, we see at least a possibility to guarantee the existence of (continuous) flux velocities: This means that deviations from the 'ideal' condition $E \cdot B = 0$ integrated over the non-ideal domain should only be statistical fluctuations, canceling each other, so that the whole integral vanishes. Only the 'isotropic' turbulence for the case, that $E \cdot B \neq 0$ in $D_R$, would enable to fulfill the boundary condition for $X$. If the right hand side does not vanish, then the flux transporting velocity field inside the non-ideal region does not converge to the $E \times B$-drift ouside the non-ideal domain, as $X$ cannot vanish on the boundary of $D_R$. Thus the boundary condition cannot be fulfilled, $X$ is a non-vanishing function outside $D_R$. This implies a discontinuity of the flux transporting velocity, which can be interpreted as a 'shock' of the flux transporting velocity field. Thus the field lines are transported different at the boundary of the non-ideal domain and this gives an 'observer' the impression of field line tearing, i.e. of a discontinous transport of magnetic flux and magnetic field lines: magnetic reconnection takes place. Magnetic reconnection is a breakdown of magnetic topology

conservation, therefore the flux- or line-conserving velocity fields either must not exist or must have a discontinuity.

There is a connection of generalized helicity which is a generalized gauge independent concept of magnetic helicity and is based on the value of $\int (\boldsymbol{E} \cdot \boldsymbol{B}) \, dV$ (see the above boundary conditions) with the concept of flux conserving velocity fields. This concept was developped by Schindler and Hesse (1988) and has the advantage that it is gauge invariant. The changing of magnetic helicity in an ideal plasma with localized non-idealness $\boldsymbol{R}$ changes magnetic topology (definition for general magnetic reconnection) if and only if the integral over $\boldsymbol{E} \cdot \boldsymbol{B}$ does not vanish. This is equivalent to the problem of satisfying the boundary condition in Eq. (40) for the flux conserving velocity field with respect to the integral expression Eq. (46). Thus for non-reconnective flows

$$\frac{d}{dt} \int_{D} \left[ (\boldsymbol{A} + \boldsymbol{A}_0) \cdot (\boldsymbol{B} - \boldsymbol{B}_0) \right] dV$$

$$= (-2) \int_{D_R} (\boldsymbol{E} \cdot \boldsymbol{B}) \, dV = 0, \qquad (47)$$

where $B_0$ and $A_0$ are reference fields for the magnetic field $\boldsymbol{B}$ and the magnetic vector potential $\boldsymbol{A}$, chosen in such a way that boundary conditions and initial conditions like $d\boldsymbol{S} \cdot \partial \boldsymbol{B}/\partial t = \boldsymbol{0}$ on the boundary of the plasma far away from $D_R$, and also $\boldsymbol{E} \times d\boldsymbol{S} = \boldsymbol{0}$, where $d\boldsymbol{S}$ is the boundary of the plasma far away from $D_R$, are satisfied. $D_R$ is again the non-ideal domain. In our case no non-ideal domain exists, where $\boldsymbol{E} \cdot \boldsymbol{B} \neq 0$, as will be discussed in Sect. 6, i.e. $\boldsymbol{E} \cdot \boldsymbol{B} = 0$ everywhere. Thus the generalized magnetic helicity cannot be changed with the given non-idealness R in our manuscript.

### 4.2 The case: $\boldsymbol{B} = \boldsymbol{0}$ locally

Current sheets are not a problem in reality, as they have a finite width and e.g. with $\boldsymbol{R} = \eta \boldsymbol{j}$ there is in principle also a possibility to define a redefined velocity field which in general is not the plasma plasma bulk velocity, respectively plasma velocity. To find such a flux transporting velocity field is, of course, easiest if $\eta \boldsymbol{j}$ is a gradient in special systems. Only then $\mathbf{v} = \boldsymbol{v}_p$ is valid. Current sheets do not necessarily imply magnetic neutral sheets ($\boldsymbol{B} = \boldsymbol{0}$), even in symmetric systems, so that the large current can be generated by some magnetic shear component, i.e. components of the magnetic field in the invariant direction (direction of symmetry) or for 180 degrees shear without magnetic neutral line with a magnetic jump across a boundary line. Only in the case of 180 degrees shear with neutral sheets, e.g. antiparallel magnetic fields above and below a magnetic neutral line, and also in the case of an magnetic null line in 2-D (null point from the 2-D perspective) the method of Newcomb is in general not valid. In this case only parallel flows are flux conserving, see the discussion in Sect. 7 about null lines in 2-D ideal MHD flows.

For 3-D isolated null points of the magnetic field it is possible to find velocity fields, see the discussion in Titov and Hornig (2000) and Sect. 7.

How can we apply the above concept and discussion in the case of non-ideal plasmas to different space plasma regimes, for example the magnetopause region of the Earth, or for our interest, under heliospheric conditions, i.e. at the heliopause?

## 5 Application to the two fluid case/Hall-MHD

We will briefly show how flux conservation can be valid in a non-ideal plasma environment. In the vicinity of the Earth's magnetopause, the ion inertia length becomes comparable to typical lengthscales of the magnetopause. Therefore sometimes the following approximation of two-fluid theory is used, which is called Hall-MHD, see Freidberg (1987) and for discussion of applicability Dreher (1997). As it has only to serve as an example for the difference of flux- and field line conservation we drop the Ohmic term $\eta \boldsymbol{j}$. Then Hall Ohm's law is written as ($\boldsymbol{v}_i$ is the ion velocity, $P_e$ is the electron pressure):

$$\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B} = \frac{1}{n \, e} \left[ \boldsymbol{j} \times \boldsymbol{B} + \nabla P_e \right] \qquad (48)$$

$$\Rightarrow \qquad \boldsymbol{E} + \boldsymbol{v}_e \times \boldsymbol{B} = \frac{1}{n \, e} \nabla P_e \qquad (49)$$

$$\Rightarrow \qquad \boldsymbol{R} = \frac{1}{n \, e} \nabla P_e \qquad (50)$$

due to $\boldsymbol{j} = n \, e \, (\boldsymbol{v}_i - \boldsymbol{v}_e)$. The used relations take strictly the quasi-neutral and quasi-stationary conditions of the plasma into account.

If we assume a barotropic law: $P_e = P_e(n)$ it follows that

$$\nabla \times \boldsymbol{R} = \boldsymbol{0} \qquad (51)$$

which guarantees flux conservation. Thus the magnetic flux is frozen into the electron velocity $v_e$. Although in general the electron pressure $P_e$ is not constant on field lines, i.e. $\boldsymbol{E} \cdot \boldsymbol{B} \equiv \boldsymbol{B} \cdot \nabla P_e \neq 0$, implying that we here are talking about non-ideal MHD, the transport of the magnetic field is ideal, i.e. conserves magnetic flux.

A non-barotropic law $P_e \neq P_e(n)$ in general does not allow for flux conservation, as the Jacobian determinants of $P_e$ and $n$ do not vanish and therefore the following relation (which is equivalent to the Jacobian determinants $\partial (P_e, n)/\partial (x_i, x_j)$ with $i \neq j$ and $i, j = 1, 2, 3$)

$$\Rightarrow \qquad \nabla n \times \nabla P_e = \chi \boldsymbol{B} \qquad (52)$$

is only able to lead to line conservation as the Jacobian and therefore Eq. (52) does not allow vanishing $\chi$'s. The above equation results in

$$\boldsymbol{B} \cdot \nabla n = 0 \qquad (53)$$
$$\boldsymbol{B} \cdot \nabla P_e = 0 \qquad (54)$$

The equations Eqs. (53) and (54) imply, that the function $\chi$ is dependent on $P_e$ and $n$, the latter can be used as Euler potentials of the magnetic field. These conditions Eqs. (52–54) are severe restrictions of possible configurations, allowing line conservations.

## 6 Application to the heliospheric plasma flow

We now turn to the problem which initially/actually should be addressed, the problem of the multi-fluid interface of the outer heliosphere/VLISM region. In this case the non-ideal term $R$ is supposed to have the following shape (see Baranov and Fahr, 2003, and references therein):

$$R \equiv \frac{(1-\alpha)^2 c^2}{K_{ia}} \left[ \frac{\alpha \, \nabla P \times B}{1+\alpha} + \frac{B}{\mu_0} \times ((\nabla \times B) \times B) \right], \qquad (55)$$

where $K_{ia}$ is the resistance coefficient, discussed by Florinski and Zank (2003) and in references therein, that is proportional to the charge exchange collision rate, and $\alpha := n/n + n_a$ is the degree of ionization, with $n$, $n_a$ as number densities of the ions and neutral atoms.

Ohm's law, $E + v \times B = R$, can be written as

$$E + v \times B = 0 \qquad (56)$$

with

$$v = v - \frac{(1-\alpha)^2 c^2}{K_{ia}} \left[ \frac{\alpha \nabla P}{1+\alpha} - \frac{1}{\mu_0} ((\nabla \times B) \times B) \right]. \qquad (57)$$

This implies that it is possible for every parameter range of the transport coefficients to find a flux conserving velocity field given by Eq. (57). Therefore the non-ideal term given by Eq. (55) representing charge exchange processes, cannot lead a change of magnetic topology. Hence the charge exchange process described by Eq. (55) cannot be responsible for magnetic reconnection.

Initially the term in Eq. (55) was supposed to represent additional interactions (some kind of 'anomalous' collisions), to violate the frozen in paradigm for the heliosphere. The remaining question is therefore, whether there may be additional multi-fluid or charge exchange processes, which can lead to non-ideal terms, inducing magnetic reconnection. However, from Eq. (57) it is not clear in advance, which component of the plasma is responsible for the flux transport: the flux transporting velocity field **v** is in general not known, but must be determined from solving the whole set of multifluid equations. Thus **v** is neither a particular species velocity nor the plasma velocity. In the discussion between Baranov and Fahr (2003) and Florinski and Zank (2003), it seems that the frozen in velocity is approximately represented by the ion velocity for such a multifluid plasma with neutral particles.

Our paper should not be misunderstood as a continuation of the dispute between Florinski and Zank on one hand and Baranov and Fahr on the other hand. We only want to shed new light on the question what kind of shape Ohm's law must have to violate the frozen in condition seriously, leading to magnetic reconnection. Thus to refer to Baranov and Fahr, or Florinski and Zank is only to show with this example of an Ohm's law that there are open questions which need more intensive discussion and research.

Nevertheless, the $\nabla \times B$-term on the right hand side of Eq. (55) has maybe a value which is not negligible at the heliopause: Due to the fact that thin current sheets can exist at the location of the heliopause the $\nabla \times B$-term can change the value of the whole term by orders of magnitude. In simulations this behaviour can lead to numerical reconnection, as discussed by Ratkiewicz et al. (2004).

In spite of this interesting and important ansatz for finding points where reconnection probably will take place, we will see in the next section that the existence of singular points of the flow (= stagnation points) leads to the demand of finding a multi-fluid Ohm's law with terms suitable to violate the frozen in condition and to lead to *real* magnetic reconnection.

## 7 Stationary ideal MHD in the vicinity of a standard stagnation point

A singular point where the velocity of a flow vanishes (= stagnation point) is always sufficient and necessary for the determination of a separatrix, i.e. a contact surface or a so called 'pause'. Here two topologically distinct flows encounter. In the case of the interstellar medium flowing around the cavity of a stellar wind (= astrosphere) a so called 'astropause' forms (see e.g. Nickeler et al., 2006). In the case of the counterflow between the sun and the interstellar medium this separatrix is called the heliopause. The definition of an existing domain/structure as an astro- or a heliosphere requires that a quasi-stationary structure exists, which represents the mean physical values of such a counterflow configuration. It is therefore reasonable to assume $\partial/\partial t \approx 0$, neglecting 'oscillations' of the system. As an example we use the image of a region in the direct neighbourhood of the stagnation point in a 2-D ideal MHD flow. We briefly show and discuss the solution for such a scenario, discussed also in Nickeler and Fahr (2006). The logical conclusion which can be drawn from such a solution (with respect to heliospheric research), namely the breakdown of ideal MHD in the vicinity of a stagnation point, will be discussed in more detail in the remaining of this section.

In the vicinity of the heliopause nose, i.e. at the front stagnation point of the heliopause, the mass density should have a maximum due to the fact that at this location there exists a 'stagnation region'. The maximum in the density would lead to small gradients of density. Thus "incompressibility" is an approximation; this assumption does not directly contradict the observations of Voyager that the flow is compressible directly behind or in the vicinity of the termination shock.
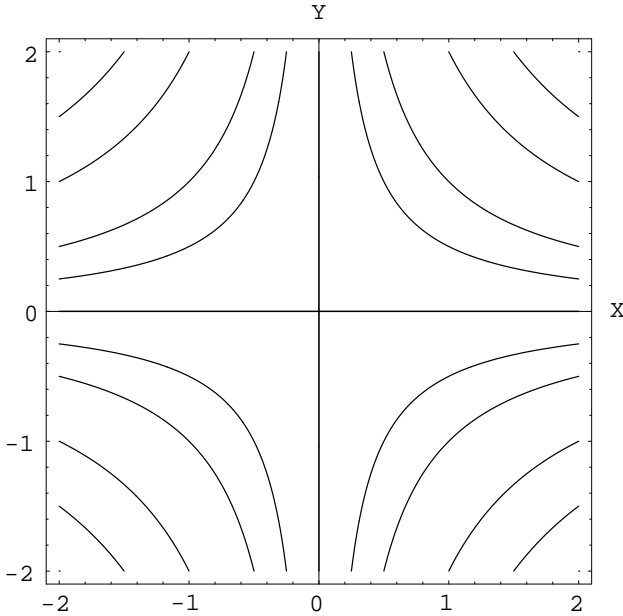
**Fig. 3.** Streamlines in the vicinity of a standard stagnation point. They also lead to a stagnation of the magnetic flux, but a divergent one.

Therefore we can demand that at least along the stream lines the density should be constant. Applying a two dimensional picture here should illustrate the problem of demanding the flux conserving or 'flux freezing constraint', i.e. that some plasma velocity transports the magnetic flux in such a way, that flux conservation is valid. We therefore introduce the streaming vector $\sqrt{\rho}\mathbf{v} := \mathbf{w} = \nabla\zeta \times \mathbf{e}_z$ to represent the incompressible flow, where $\mathbf{v}$ again is not necessarily the ion velocity or any other plasma velocity, $\rho(\zeta)$ is the mass density, and $\zeta(x, y)$ is a correspoding potential. Curves given by $\zeta(x, y) = \text{const}$ are streamlines and the mass density is an explicit function of the stream function $\zeta$. By introducing $\mathbf{B} = \nabla\alpha(x, y) \times \mathbf{e}_z$ the corresponding ideal Ohm's law can be written as

$$\frac{\partial(\zeta, \alpha)}{\partial(x, y)} = \frac{\partial\zeta}{\partial x}\frac{\partial\alpha}{\partial y} - \frac{\partial\zeta}{\partial y}\frac{\partial\alpha}{\partial x} = -\sqrt{\rho}\,E_0 \qquad (58)$$

where $E_0$ is the constant electric field in the invariant $z$-direction. A standard stagnation point in the incompressible case can be represented by using $\zeta = axy$, see the streamlines drawn in Fig. 3. Here $a$ is a normalization constant. The potential $\zeta$ represents the potential for the linearized velocity field around the stagnation point. This linearization of the velocity field in the vicinity of the stagnation point does not exclude globally asymmetric configurations.

The general solution of Eq. (58) leads to the occurrence of logarithmic singularities:

$$\alpha = -\frac{E_0}{2a}\sqrt{\rho}\left(k_1\ln(x/x_0)^2 + (1+k_1)\ln(y/y_0)^2\right). \quad (59)$$
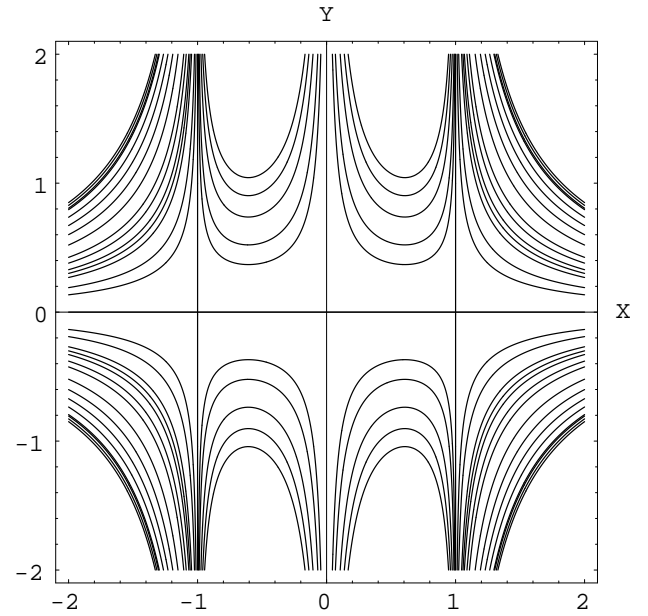
**Fig. 4.** Magnetic field lines resulting from a solution of Eq. (58) with the density $\rho \propto \zeta^4 \propto (xy)^4$. This solution represents the magnetic field for a prescribed velocity field in the vicinity of a stagnation point. The magnetic field structure and its topology is much more complex than that of the streaming, i.e. velocity field. The magnetic field shows additional separatrices, which are obviously crossed by the flow whose stream lines can be seen in Fig. 3. The plasma flow crossing magnetic separatrices is normally a critereon or at least a hint for magnetic reconnection in 2-D.

One can calculate the Laplacian of $\alpha$ and see, that the field is not a potential field, i.e. $\Delta\alpha = -\mu_0 j_z \neq 0$, where $j_z$ is the current density in $z$-direction. The singularities of the magnetic field (the magnetic field on the separatrices is infinity) can be removed by demanding a vanishing mass density on the separatrices: In this case the magnetic field would remain finite, but show a much more complex topology than the velocity field (see Fig. 4, for a mass density given by $\rho \propto \zeta^4 \propto (xy)^4$). The velocity field $\mathbf{v}$ diverges on the separatrices, but no mass is transported, i.e. $\sqrt{\rho}\mathbf{w} = \rho\mathbf{v} \to \mathbf{0}$. Additional separatrices occur, and the general solution given by Eq. (59) shows that singular current sheets exist in that domain. But even with density structures which converge much more quickly to zero density on the separatrices than that density used to calculate the magnetic field structure in Fig. 4, no solution of the Euler equation could be found by us (including isotropic plasma pressure) for the magnetic fields of the general solution given by Eq. (59) (see Nickeler and Fahr, 2006).

Such singularities for the magnetic field would also occur in compressible flows in 2-D Cartesian geometry. From stationary ideal Ohm's law it follows

$$v_x B_y - v_y B_x = -E_0, \qquad (60)$$

where $E_0 \neq 0$ is the constant component of the electric field in $z$-direction. Thus at the stagnation point the magnetic field, or at least one component has to be infinitely, which is not a reasonable physical result. This leads to the conclusion that ideal MHD cannot be used in the vicinity of a stagnation point. Only for a flow which is field aligned everywhere in the poloidal plane, implying that $E_0$ is zero, the problem could in principle be solved.

The diverging of the magnetic field seems to be a general problem for symmetric and stationary systems in ideal MHD, see e.g. Contopoulos (1996), where the vanishing electric field component into the invariant direction (direction of symmetry) leads to the demand of a pure field aligned flow or, for non-vanishing electric field, to an infinite magnetic field strength at the stagnation point. This can be seen by inspecting the poloidal part of Ohm's law,

$$v_z B_r - v_r B_z = \text{constant}/r \qquad (61)$$

which leads to flows being field aligned in pure 2-D axissymmetric systems, if the constant is zero. Here $\partial/\partial\phi = 0$ with $\phi$ as coordinate (angle around the axis), $r$ is the distance to the axis, and $z$ the $z$-coordinate. If the constant is not zero, for the stagnation region $v_z \to 0$ and $v_r \to 0$ at least one magnetic field component has to diverge, so it it is not possible to get regular solutions of the ideal MHD equations. Thus the above statements show clearly, that ideal MHD cannot be valid in the vicinity of stagnation points, as the magnetic field should, of course, be finite everywhere.

If components in the invariant direction exist, it is not necessary that the flow is field aligned. But the case that $\boldsymbol{v}_{\text{poloidal}} \parallel \boldsymbol{B}_{\text{poloidal}}$ is very special and excludes realistic symmetric ideal MHD flows which have an angle between the poloidal velocity and the poloidal magnetic field component. Solutions with $\boldsymbol{v}_{\text{poloidal}} \parallel \boldsymbol{B}_{\text{poloidal}}$ do represent situations which characterize very idealized situations, but at least it is doubtful, whether they occur in nature.

An extension to generic 3-D solution (without an ignorable coordinate respectively symmetry) of the ideal transport problem has been done by Titov and Hornig (2000). But their solution is restricted to a special magnetic topology in the vicinity of a 3-D stagnation point and gives no hint for a configuration without a null point of the magnetic field. This implies that this is a structure which can lead to magnetic reconnection, but does not likely lead to stationary or quasi-stationary MHD configurations.

## 8 Conclusions

In this article criteria for magnetic field line conservation and magnetic flux conservation are reviewed to show their importance for heliospheric physics. In the application to heliospheric plasma physics we showed, that the often used Ohm's law leads to a possibility to determine a flux conserving velocity field: However this velocity field depends on the solution of the whole set of multifluid equations, and is therefore neither a determined plasma species velocity nor a weighted mean of the plasma species velocities. Thus it would be interesting as a future perspective to get more insight into the relation between the plasma velocities (e.g. species velocities or bulk velocity) and the flux- or line-conserving velocity fields in the frame of multifluid theory.

It is also shown that in the vicinity of stagnation points within the frame of stationary MHD flows with symmetry, additional non-ideal terms have to be considered to avoid the occurence of magnetic singularities. If no non-ideal terms are present in Ohm's law, stationary or quasi stationary solutions show a divergent behaviour on the separatrices, or need a diverging velocity field, or, as one of our suggestions, an 'exotic' density distribution. This may give a hint that the ideal Ohm's la breaks down at the heliospheric nose or at stagnation points in general.

Therefore some open questions and unsolved problems remain:

– Is there any way to define a flux conserving velocity in a partially ionized plasma which is directly coupled to a plasma species velocity or to any other plasma velocity?

– Therefore: How is it possible to define magnetic reconnection in a partially ionized plasma?

Flux conservation is defined by the criterion of Newcomb (1958), but not sufficiently, ignoring the importance of the expression $\boldsymbol{E} \cdot \boldsymbol{B}$ and a minimal flux preserving velocity field as boundary condition, as was shown in Sect. 4. For the case of the heliosphere or for any other partially ionized plasma the up to now used model of e.g. Baranov and Fahr (2003) lacks a term which can violate magnetic flux respectively magnetic topology conservation, due to the fact that $\boldsymbol{E} \cdot \boldsymbol{B} = 0$.

– Is there any additional term which for physical reasons has to be added to Ohm's law, so that this term leads to a violation of the frozen in paradigm and therefore to magnetic reconnection processes, e.g. due to charge exchange processes?

The transport equations used by Baranov and Fahr (2003), Florinski and Zank (2003), based on the research of Schlüter (1958), Kulikovskii and Lyubimov (1965) and Cowling (1976) are only one possibility how the interaction of neutrals, and ions and electrons can look like. The "friction force" $F_{ij}$ in these models is represented as linear relation, which is the most simplest form one can think of (even complicated enough) by a constant coefficient $K_{ij}$: $F_{ij} = \pm K_{ij}(v_i - v_j)$. Maybe things can be much more complicated, thus there is enough "material" for discussion, but this is beyond the scope of this paper.

– Is ideal MHD valid in the vicinity of a 3-D stagnation point or does the ideal theory break down there?

To answer the last question, results from numerical simulations should be analyzed carefully with respect to the topology of field and streamlines in the vicinity of the stagnation point. The calculation of the characteristic surfaces in 3-D can help to find out whether regular and stable solutions of flux conserving flows in the vicinity of a stagnation point can be found. This investigation must be left for future work.

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