DEMOGRAPHIC RESEARCH

Formal Relationships 3

The effect of changes in fertility on the age distribution of stable populations

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Abstract

When fertility increases in a previously stable population, the new stable population will be younger than the old one, and the two age distributions will cross at the mean age of the populations.

1. Relationship

Consider a stable population with a growth rate r. Suppose that, in that population, age-specific fertility rates change to new levels and stay constant thereafter, while mortality remains constant. Eventually, a new stable population with a different growth rate will emerge. For example, if age-specific fertility rates increase, the growth rate of the new stable population will be higher than that of the old one. The effect of this fertility-induced change in r on the proportionate age distribution of the stable population can be formally expressed as follows:

$$(1) \qquad \frac{d \ln[c(a)]}{dr} = A_P - a$$

where $c(a)\,da$ is the proportion of the stable population in the age interval a to $a + da$, and A_P is the mean age of the stable population.

In Equation (1), the derivative of $\ln[c(a)]$ is positive at ages below A_P and negative at ages above A_P. This means that, in the case of a fertility-induced increase in r, the proportion of the population at ages below A_P will increase, while the proportion at ages above A_P will decrease. At age A_P, the derivative is zero, so the proportion of the population at that age remains unchanged. In other words, the new stable population will be younger than the initial one, and the two age distributions will cross at the mean age of the populations. Conversely, in the case of a fertility-induced decrease in r, the new stable population will be older. Equation (1) also shows that the further away an age is from A_P, the larger the relative change in the age distribution.

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2. Proof

The age distribution of a stable population, $c(a)$, is given by

(2) $$c(a) = be^{-ra}p(a)$$

where b is the crude birth rate, r is the population growth rate, and $p(a)$ is the proportion surviving from birth to age a.

The impact of an infinitesimal, fertility-induced change in r on $c(a)$ is analyzed by developing the derivative of $\ln[c(a)]$ with respect to r:

(3) $$\frac{d \ln[c(a)]}{dr} = \frac{d\left[\ln b - ra + \ln p(a)\right]}{dr}$$

Since $b = 1/\int_0^\infty e^{-ra}p(a)da$ in a stable population, equation (3) becomes:

(4) $$\frac{d \ln[c(a)]}{dr} = \frac{d\left[\ln 1 - \ln \int_0^\infty e^{-ra}p(a)da - ra + \ln p(a)\right]}{dr}.$$

This demonstration is about the impact on $c(a)$ of a fertility-induced change in r, so $p(a)$ is not affected and remains constant. Therefore, equation (4) simplifies as follows:

(5) $$\begin{aligned}
\frac{d \ln[c(a)]}{dr} &= \frac{-d \ln \int_0^\infty e^{-ra}p(a)da}{dr} - \frac{dra}{dr} \\
&= \frac{\int_0^\infty ae^{-ra}p(a)da}{\int_0^\infty e^{-ra}p(a)da} - a = \frac{\int_0^\infty ac(a)da}{\int_0^\infty c(a)da} - a \\
&= A_P - a
\end{aligned}$$

Q.E.D.

3. Related relationship

Equation (1) pertains to an infinitesimal change in r producing an infinitesimal change in A_P. When fertility changes are more substantial and produce non-infinitesimal changes in r, the old and new stable populations will have different mean ages. For these non-infinitesimal changes in r, the crossing age $\hat{A}$ is located between the mean ages of the old and new stable populations. For a pair of intrinsic growth rates that are in the range of r values usually observed in human populations, the crossing age can be approximated by the mean of the mean ages of the two stable populations:

(6) $$\hat{A} \approx \frac{A_P^1 + A_P^2}{2}$$

where A_P^1 and A_P^2 are the mean ages of the old and new stable populations.

4.   History

Equation (1) and its proof first appear in Keyfitz (1968:186, equation (7.4.3)). A version of equation (1) is also discussed in Coale (1972:28, equation (2.27)). However, the foundations of Equation (1) can be traced back to Lotka (1938, 1939), who derived an expression for the crossing age, or "center of tilt", between the age distributions of two stable populations with identical mortality but different values of r (Lotka 1939, equation (56)). He also showed that, for differences in r in the range of .01, the crossing age can be well approximated using equation (6) (Lotka 1938:164-165), and demonstrated that, in the case of an instantaneous change in r, the crossing age is A_P (1939, equation (58)). Later, Coale considered a wide range of life tables and estimated that, when the difference in r between the two stable populations is less than .02, the exact value of the crossing age exceeds the approximation given in equation (6) by at most .16 years (Coale 1972:60, fn.7). Equation (1) is also discussed in Preston, Heuveline, and Guillot (2001:156-157).

5.   Applications

Equation (1) elegantly shows that, in the long run, fertility increases produce younger populations, while fertility decreases produce older populations. By expressing the value of the crossing age, equation (1) also shows the ages at which changes in the age distribution are occurring. Lotka used his expression for the "center of tilt" to make the following practical point:

> "(...) as we pass from greater to lesser values of r, the natural rate of increase, the proportion of the population comprised within the age groups at the young end and at the old end of the curve of the population change materially, but the age group or groups centered about the average age of the population, i.e., essentially the group of productive workers, undergoes relatively small changes." (Lotka 1938:166).

In reality, fertility changes occur in populations that are never perfectly stable. Moreover, mortality often varies in tandem with fertility. In such cases, equation (1) will not be applicable, and the effect of changes in fertility on the population age distribution needs to be studied with other approaches, for example using counterfactual projections or decomposition methods (Keyfitz 1968:189-193). Nonetheless, the overall message of equation (1), i.e., that fertility increases tend to make populations younger, holds in a variety of situations.

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