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On the Value – Volatility Relationship in a Real Options Model

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In the analytical real options approach, the most important proposition that the value of the investment opportunity increases as the volatility increases has been proved by assuming the convexity of the drift of the stochastic differential equation defined as the state variable. This paper demonstrates numerically that the convexity of the drift is not necessary for that proposition in the real options approach.

Keywords: investment analysis, option-pricing theory, finance, nonlinear stochastic differential equation

JEL Classification Numbers: D81, G31, O33

1. Introduction

The standard real options approach calculates the value of the investment opportunity when a firm faces uncertainty and the irreversibility of its investment expenditure. Since we can regard investment decision making under uncertainty as a financial option, the investment opportunity is similar to an American call option written on the underlying return on investment. The value of the investment opportunity can then be calculated by the option pricing theory of financial engineering (see e.g., Kijima (2002) and Chiarella (2002) for details about option pricing theory). The main result obtained in this framework is that the value as well as the optimal threshold for the investment decision of the firm increases as the underlying market uncertainty increases. An excellent overview of the real options approach is found in Dixit and Pindyck (1994) and Trigeorgis (1996).

It may be recalled that in almost all standard real options models, the stochastic process for the underlying uncertainty follows a linear stochastic differential equation (hereafter SDE), a especially geometric Brownian motion. A geometric Brownian motion is a diffusion process (a diffusion for short) with constant mean growth rate and constant volatility. In this case, we can obtain the value and the optimal threshold for the investment decision in closed form. However, it is a matter of course that the state variable follows a variety of SDE according to the economic situation in which the firm considers the investment opportunity. What is of great interest is whether or not the existing results in the real options approach remain unchanged by the introduction of a nonlinear SDE for the state variable[1)].

Real options models which the state variable follows a nonlinear SDE are not new. For example, Dixit and Pindyck (1994), Dixit et al. (1999), Alvarez and Stenbacka (2001) examined such a model. Alvarez and Stenbacka (2001) investigated a model by using a nonlinear SDE with general drift and volatility structure, and showed analytically the proposition that the value of the investment opportunity increases as the underlying volatility increases. Dixit and Pindyck (1994) examined the comparative static in the volatility on the optimal threshold when the state variable follows a nonlinear SDE. Dixit et al. (1999) examined the effect of the discount factor on the value when the underlying state variable follows a linear SDE, a nonlinear SDE, respectively.

In this paper, we investigate the impact of the volatility on the value of the investment opportunity when the underlying process follows a nonlinear SDE. We note that the sufficient condition on the analytical approach to prove the most important proposition that the value increases as the volatility increases, is the convexity of the drift on the state variable. When the drift on the underlying uncertainty is not convex in the state variable, we cannot prove analytically the most important proposition. We investigate whether the monotonicity of the volatility on the value is obtained or not by using a numerical method when the drift is not convex in the state variable. The reason that we have to use a numerical method is that we cannot obtain an analytical expression for the value function in our setting. The main contribution of this paper is to confirm that the result still holds without the assumption of convex drift on the underlying SDE.

The paper is organized as follows. Section 2 formulates the corporate investment problem and describes the assumption necessary for the analytical proof of the most important proposition in the real options approach. Section 3 examines the impact of the volatility on the value of the investment opportunity numerically when the state variable follows a SDE with concave drift. Section 4 concludes.

2. Some Preliminaries

Consider a firm having the possibility to make an irreversible investment that increases his profits. We assume that the firm is risk neutral. We denote a state variable that the revenue for the investment depends upon by $(X_t)_{t\in\mathbb{R}_+}$ that is defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$.

[1)] Recently, the basic model has been extended in various ways. One extension is to incorporate strategic interactions. See, e.g., Weeds (2002) or Kijima and Shibata (2002) for details in a real options model with strategic interactions. Another extension is to the situation of incomplete information. See e.g., Bernardo and Chowdhry (2002) or Shibata (2006) for details in a real options model with incomplete information.

It is assumed that the revenue process $(X_t)_{t\in\mathbb{R}_+}$ evolves according to

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}z_t, \quad X_0 =: x \in \mathbb{R}_{++}, \tag{1}$$

where both the infinitesimal drift $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ and the infinitesimal diffusion coefficient $\sigma : \mathbb{R}_+ \to \mathbb{R}_+\backslash\{0\}$ are assumed to be Lipschitz continuous in order to guarantee the existence and uniqueness of a strong solution for Equation (1). Here, $(z_t)_{t\in\mathbb{R}_+}$ denotes a one-dimensional $\mathbb{P}$–standard Brownian motion. We will also assume that both the lower boundary 0 and the upper boundary ∞ are natural for the revenue process $(X)_{t\in\mathbb{R}_+}$.

Suppose that the current time is $t \in \mathbb{R}_+$, and let τ be the stopping time at which the firm adopts the investment opportunity after time t. We denote the set of admissible strategies at time t (i.e., stopping times not less than t) by $\mathcal{T}_t$. The risk neutral discount factor is constant and equals $r \in \mathbb{R}_{++}$. The value function of the investment opportunity is given by

$$C(x) := \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}^x\left[\mathrm{e}^{-r\tau}\{X_\tau - I\}\,\middle|\mathcal{F}_t\right], \quad X_t =: x \in \mathbb{R}_{++}, \tag{2}$$

where $\mathbb{E}^x[\cdot|\mathcal{F}_t]$ denotes the conditional expectation operator evaluated at the initial state $x = X_t$ with respect to the risk neutral measure $\mathbb{P}$, and where $I \in \mathbb{R}_{++}$ denotes the sunk cost incurred at the moment τ.

The Bellman equation that the value function for the firm must satisfy is as follows:

$$C(x) = \max\left\{\mathbb{E}^x\left[\mathrm{e}^{-r\mathrm{d}t}C(X_{t+\mathrm{d}t})\middle|\mathcal{F}_t\right],\ x - I\right\}. \tag{3}$$

The first term in (3) represents the value of delaying the decision, the second term is the value when investing in the irreversible project. After some algebra, the value must satisfies the following ordinary differential equation (ODE):

$$\frac{1}{2}\sigma^2(x)C''(x) + \mu(x)C'(x) - rC(x) = 0, \quad x < x^*, \tag{4}$$

where x^* denotes the optimal threshold. The corresponding boundary conditions turn out to be

$$C(0) = 0, \quad C(x^*) = x^* - I, \quad C'(x^*) = 1. \tag{5}$$

The first condition is called the initial condition, the second condition is called the value-matching condition, and the final condition is called the smooth-pasting condition.

According to Theorem 2 or 3 in Alvarez and Stenbacka (2001), we state an important lemma in the real options model.

Lemma 1. *Assume that the drift term $\mu(x)$ on Equation* (1) *is convex in x. Then, the value function $C(x)$ is increasing and convex in x. Moreover, the value $C(x)$ increases as the volatility σ increases.*

Let us denote the diffusion process with volatility $\sigma_i(x)$ by $(X_t^i)_{t\in\mathbb{R}_+}$ $(i \in \{1,2\})$. Accordingly, the value function of the firm is denoted by $C_i(x)$. Lemma 1 implies that if $\sigma_1(x)$ dominates $\sigma_2(x)$, i.e., $\sigma_1(x) \geq \sigma_2(x)$ for all $x \in \mathbb{R}_+$, then the value function $C_1(x)$ also dominates $C_2(x)$ for all $x \in \mathbb{R}_+$.

For example, in the case of a geometric Brownian motion with drift $\mu(x) = \mu x$ and diffusion coefficient $\sigma(x) = \sigma x$, the value function is obtained as:

$$C(x) = \left(\frac{x}{x^*}\right)^{\beta} (x^* - I), \quad x < x^*, \tag{6}$$

where

$$x^* = \frac{\beta}{\beta - 1} I. \tag{7}$$

Here, β is a positive root to the quadratic equation $Q(k) = 0$, where is defined by:

$$Q(k) = \frac{1}{2}\sigma^2 k(k-1) + \mu k - r. \tag{8}$$

It is readily shown that[2)]

Lemma 2. *The value function $C(x)$ is increasing and convex in x. Moreover, the value $C(x)$ increases as the volatility σ increases.*

Also, it is readily verified that the optimal threshold x^* is increasing in volatility σ (see Dixit and Pindyck (1994) for details).

3. Main Results

3.1. Model

In this section, we assume that the revenue process, i.e., the state variable, follows a non-linear SDE with concave drift. The evolution of the revenue process is defined as:

$$\mathrm{d}X_t = \kappa X_t(m - X_t)\,\mathrm{d}t + \sigma X_t\,\mathrm{d}z_t, \quad X_0 =: x \in \mathbb{R}_{++}, \tag{9}$$

where κ, m, and σ are some constants (i.e., $\kappa, m, \sigma \in \mathbb{R}_+$). Equation (9) is often used as a model for the growth of a population size in a stochastic, crowded environment.

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2) Since $\beta > 1$, the value is convex in the state variable x. Next, let us prove the monotonicity of the value with respect to volatility. Differentiating Equation (6) with respect to σ gives

$$\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}\sigma} &= \left(\frac{\partial C}{\partial x^*}\frac{\partial x^*}{\partial \beta} + \frac{\partial C}{\partial \beta}\right)\frac{\partial \beta}{\partial \sigma} \\ &= \left(C(x)\cdot \log\left(\frac{x}{x^*}\right)\right)\frac{\partial \beta}{\partial \sigma}, \end{aligned}$$

where we have used the fact that $\frac{\partial C}{\partial x^*} = 0$. By differentiating Equation (8) with respect to σ, we obtain the result $\frac{\partial \beta}{\partial \sigma} < 0$. Hence we conclude $\frac{\mathrm{d}C}{\mathrm{d}\sigma} > 0$ on the event $\{x < x^*\}$.

The constant m is called the carrying capacity of the environment, the constant κ is a measure of the quantity of the environment, and the constant σ is a size of the noise in the system. Mathematically, Equation (9) implies that the logarithm of the revenue process $(X_t)_{t\in\mathbb{R}_+}$ follows a mean-reverting process. It should be noted that the drift $\mu(x)$ of (9) is not convex, but concave in x.

We can obtain the explicit solution of the non-linear SDE defined by Equation (9); (see, e.g., Gard (1988) or Øksendal (2003) for details). The solution of (9) is given by:

$$X_T \quad = \quad \frac{\mathrm{e}^{\left(\kappa m - \frac{1}{2}\sigma^2\right)T + \sigma z_T}}{x^{-1} + \int_0^T \kappa \mathrm{e}^{\left(\kappa m - \frac{1}{2}\sigma^2\right)t + \sigma z_t}\mathrm{d}t}. \tag{10}$$

The proof is simple[3]. Note that the explicit solution is not necessarily obtained in many nonlinear cases.

Similar to derivation of (4), the Bellman equation that the value function must satisfy in the continuation region is equal to:

$$\frac{1}{2}\sigma^2 x^2 C''(x) + \kappa(m - x)xC'(x) - rC(x) = 0, \quad x < x^*. \tag{11}$$

Also the value function must satisfy the boundary conditions (5) as before.

Since the value function and optimal threshold are unknown in this model, we have to obtain both the solution of Equation (11) and the optimal threshold simultaneously. This type of problem is known as a free-boundary value problem. As we cannot, however, obtain both the value function and optimal threshold in closed form under the given boundary condition when the state variable follows Equation (9), we examine the solution (i.e., the value and the optimal threshold) numerically in the next subsection.

3.2. Value of the Firm

In this subsection we use a numerical method to calculate the value of the firm and optimal threshold. The parameters have been chosen so that the value is not diverging. That is because our aim here is to focus on the shape of the value and the comparative statics. We set the basic parameters as $I = 5$, $m = 5$, $r = 0.05$, $\sigma = 0.1$, and $\kappa = 0.02$.

Figure 1 depicts the value $C(x)$ with respect to the initial state x[4]. Figure 1 shows that the value $C(x)$ is no longer convex for all x; it is concave for small value of x. Here, we conclude the following proposition:

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[3] By using the Ito formula for $y = X^{-1}$, the derived equation turns out to be linear in y:

$$\mathrm{d}y = \left\{\kappa + (\sigma^2 - \kappa m)y\right\}\mathrm{d}t - \sigma y\,\mathrm{d}z_t.$$

Here, the explicit closed form for y can be obtained. On the other hand, we can verify that Equation (10) is the unique (strong) solution of Equation (9). This result follows by the same arguments as Example 4.3 in Gard (1988).

[4] With these parameters, we find that the optimal threshold x^* is calculated as 7.5692.

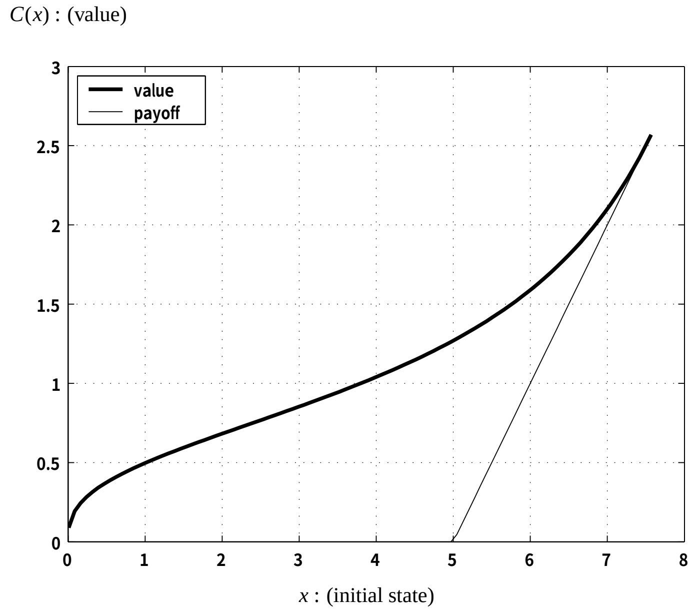

Figure 1 Value of the firm.

Proposition 1. *We assume that the state variable follows Equation* (9)*. Then, the value* $C(x)$ *is not necessarily convex in* x*.*

This result is exactly the same as that of Figures 5.12 and 5.13 in Dixit and Pindyck (1994). In the analytical real options model proposed by Alvarez and Stenbacka (2001), the uniform convexity of the state variable x on the value is inevitable in the proof of the most important proposition that the value increases as the volatility increases. Hence, according to analytical results on the real options model, we cannot prove the most important proposition in our setting. In what follows, we investigate whether the monotonicity of the volatility on the value is obtained or not by using numerical methods when the value is not convex for all x.

3.3. Comparative Statics

In this subsection, we examine the comparative statics of some parameters on the value of the investment opportunity. Figure 2 depicts the value for several choices of volatility σ. These numerical simulations suggest the following result.

Proposition 2. *The value* $C(x)$ *increases as the volatility* σ *increases even when the state variable follows the nonlinear SDE* (9)*, while it is not convex for all* x*.*

Surprisingly, this result demonstrates that the value $C(x)$ increases as the volatility σ increases even when the value $C(x)$ is concave for small value of x. This result is exactly the same as the proposition obtained by the model with the linear SDE in standard real options models; See McDonald and Siegel (1986), and Dixit and

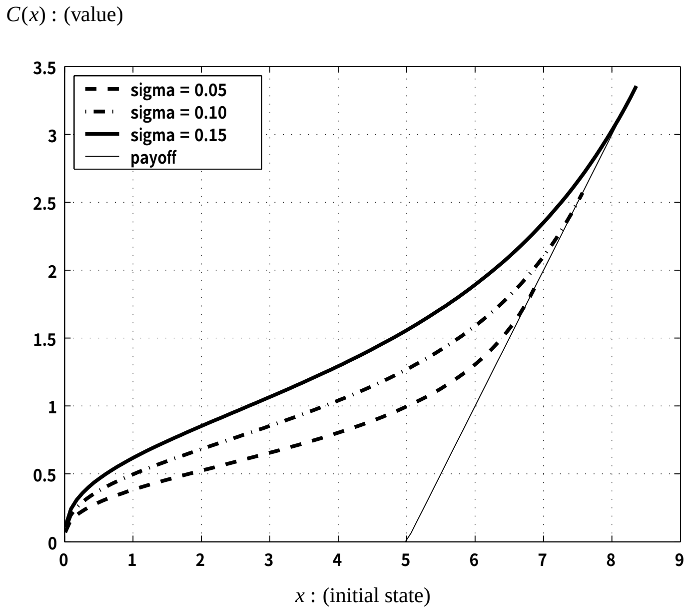

Figure 2 Comparative statics with respect to the volatility parameter σ.

Pindyck (1994). Such a monotonicity of the volatility on the value is the same as that of financial options developed originally by Black and Scholes (1973). Our numerical experiments indicate that the value increases as the volatility increases even when the real options model is extended to the model with the nonlinear SDE[5].

In Figure 3, we examine the value for different levels of κ. Numerical simulation results can be summarized as:

Proposition 3. *The value* $C(x)$ *does not increases monotonically as the parameter* κ *increases.*

In standard real options models, there are many cases in which the value is monotonically increasing or decreasing in some parameters. In contrast, Figure 3 shows that the value increases and decreases as κ increases when $x < m$, when $x > m$, respectively. The reason of this result can be explained as follows. When $x < m$, since the drift term on (9) is positive and the rate of change on the revenue is positive, the value increases as the parameter κ increases, and vice versa when $x > m$[6].

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[5] With these parameters, we find that the optimal thresholds are calculated as 6.8654, 7.5692, and 8.3556 for $\sigma = 0.05, 0.10$, and 0.15, respectively.

[6] In Figure 3, we see under our parameters that the optimal threshold x^* decreases as κ increases. The optimal thresholds are calculated as 6.8509, 6.4545 and 6.2848 for $\kappa = 0.00, 0.01$ and 0.02, respectively. This result is exactly the same as that of Figure 5.16 in Dixit and Pindyck (1994).

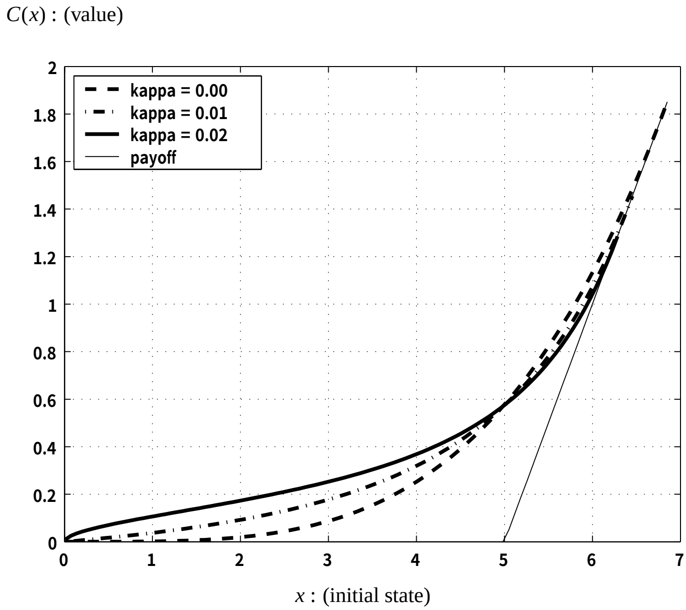

Figure 3 Comparative statics with respect to the speed parameter κ.

4. Conclusions

The standard real options framework has shown to be quite useful in the analysis of investment under uncertainty. The tools of option pricing theory can be applied to real investment analysis. However, the standard framework of formulating strategies is not necessarily relevant to real-world applications.

This article provides some analogous results for analyzing the monotonicity of volatility on the value of the investment opportunity. The result is found only numerically, and has potentially wide applications. For example, the convexity of the drift in the underlying SDE is not always necessary to guarantee that the value increases as the volatility increases in the real options framework.

We demonstrate that the monotonicity of volatility on the value is quite robust.

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