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ESTIMATES FOR THE DERIVATIVE OF DIFFUSION SEMIGROUPS

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Abstract

Let $\{P_t\}_{t\geq 0}$ be the transition semigroup of a diffusion process. It is known that P_t sends continuous functions into differentiable functions so we can write DP_tf. But what happens with this derivative when $t \to 0$ and $P_0 f = f$ is only continuous ?. We give estimates for the supremum norm of the Fréchet derivative of the semigroups associated with the operators $\mathcal{A} + V$ and $\mathcal{A} + Z\cdot\nabla$ where $\mathcal{A}$ is the generator of a diffusion process, V is a potential and Z is a vector field.

1 Introduction

Consider the following stochastic differential equation on $\mathbb{R}^n$

$$\begin{aligned} dX_t &= \mathbb{X}(X_t)\,dB_t + \mathbb{A}(X_t)\,dt\,, \\ X_0 &= x \in \mathbb{R}^n\,, \end{aligned}$$

for $t \geq 0$, where the first integral is an Itô stochastic integral and the second is a Riemann integral. Here $\{B_t\}_{t\geq 0}$ is Brownian motion on $\mathbb{R}^m$ and the equality holds almost everywhere. The coefficients of this equation are the mapping $\mathbb{X}\colon \mathbb{R}^n \to \mathbb{L}(\mathbb{R}^m;\mathbb{R}^n)$ and the vector field $\mathbb{A}\colon \mathbb{R}^n \to \mathbb{R}^n$. Assume standard regularity conditions on these coefficients so that there exists a strong solution $\{X_t\}_{t\geq 0}$ to our equation. We write X_t^x for X_t when we want to make clear its dependence on the initial value x. It is known that under further assumptions on the coefficients of our equation, the mapping $x \mapsto X_t^x$ is differentiable (see for instance [1]).

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Assume coefficients $\mathbb{X}$ and $\mathbb{A}$ are smooth enough and consider the associated derivative equation

$$\begin{aligned} dV_t &= D\mathbb{X}(X_t)(V_t)\,dB_t + DA(X_t)(V_t)\,dt\,, \\ V_0 &= v \in \mathbb{R}^n\,, \end{aligned}$$

whose solution $\{V_t\}_{t\geq 0}$ is the derivative of the mapping $x \mapsto X_t^x$ at x in the direction v. We will assume that there exists a smooth map $\mathbb{Y}\colon \mathbb{R}^n \to \mathbb{L}(\mathbb{R}^n;\mathbb{R}^m)$ such that $\mathbb{Y}(x)$ is the right inverse of $\mathbb{X}(x)$. That is, $\mathbb{X}(x)\mathbb{Y}(x) = I_{\mathbb{R}^n}$ for all x in $\mathbb{R}^n$. We shall also assume that the process $\{\mathbb{Y}(X_t)(V_t)\}_{t\geq 0}$ belongs to $L^2([0,t])$ for each $t > 0$, that is, $\int_0^t |\mathbb{Y}(X_s)(V_s)|^2\,ds < \infty$ and thus we can write $\int_0^t \mathbb{Y}(X_s)(V_s)\,dB_s$.
With all above assumptions, we have from [6], the following result (see Appendix for a proof)

Theorem 1 *For every $t > 0$ there exist positive constants k and a such that*

$$\mathbb{E}\,|\int_0^t \mathbb{Y}(X_s)(V_s)\,dB_s| \leq k\sqrt{e^{at}-1}\,. \tag{1}$$

We are interested in small values for t. Thus, since $\sqrt{e^{at}-1} = O(\sqrt{t})$ as $t \to 0$, we have that there exist a positive constant N such that $\sqrt{e^{at}-1} \leq N\sqrt{t}$ for small t. Hence for sufficiently small t, we have the following estimate

$$\mathbb{E}\,|\int_0^t \mathbb{Y}(X_s)(V_s)\,dB_s| \leq c\sqrt{t}\,, \tag{2}$$

where c is a positive constant.
Let now $BC^r(\mathbb{R}^n)$ be the Banach space of bounded measurable functions on $\mathbb{R}^n$ which are r-times continuously differentiable with bounded derivatives. The norm of this space is given by the supremum norm of the function plus the supremum norm of each of its r derivatives. In particular $B(\mathbb{R}^n)$ is the Banach space of bounded measurable functions on $\mathbb{R}^n$ with supremum norm $\|f\|_\infty = \sup_{x\in\mathbb{R}^n} |f(x)|$. Suppose our diffusion process $\{X_t\}_{t\geq 0}$ has transition probabilities $P(t,x,\Gamma)$. Then this induces a semigroup of operators $\{P_t\}_{t\geq 0}$ as follows. For every $t \geq 0$ we define on $B(\mathbb{R}^n)$ the bounded linear operator

$$(P_t f)(x) = \int_{\mathbb{R}^n} f(y)\,P(t,x,dy) = \mathbb{E}(f(X_t^x))\,. \tag{3}$$

The semigroup $\{P_t\}_{t\geq 0}$ is a strongly continuous semigroup on $BC^0(\mathbb{R}^n)$. Denote by $\mathcal{A}$ its infinitesimal generator.
It is known that $\{P_t\}_{t\geq 0}$ is a strong Feller semigroup, that is, P_t sends continuous functions into differentiable functions. In fact, under above assumptions, a formula for the derivative of $P_t f$ is known (see [4] or [5]).

Theorem 2 *If $f \in BC^2(\mathbb{R}^n)$ then the derivative of $P_t f\colon \mathbb{R}^n \to \mathbb{R}$ is given by*

$$D(P_t f)(x)(v) = \frac{1}{t}\mathbb{E}\{f(X_t)\int_0^t \mathbb{Y}(X_s)(V_s)\,dB_s\}\,. \tag{4}$$

Higher derivatives in a more general setting are given in [4]. See also [2] for a general formula of this derivative in the context of a stochastic control system. Observe the mapping $f \mapsto \frac{1}{t}\mathbb{E}\{f(X_t)\int_0^t \mathbb{Y}(X_s)(V_s)\, dB_s\}$ defines a bounded linear functional on $BC^2(\mathbb{R}^n)$. Hence there exists a unique extension on $BC^0(\mathbb{R}^n)$. Since the expression of this linear functional does not depend on the derivatives of f, it has the same expression for any f in $BC^0(\mathbb{R}^n)$.
From last theorem we obtain

$$\begin{aligned} |DP_t f(x)(v)| &\leq \frac{1}{t}\|f\|_\infty \mathbb{E}|\int_0^t \mathbb{Y}(X_s)(V_s)\, dB_s| \\ &\leq \frac{1}{t}\|f\|_\infty k\sqrt{e^{at}-1}\,. \end{aligned}$$

And hence for small t

$$\|DP_t f\|_\infty \leq \frac{c\|f\|_\infty}{\sqrt{t}}\,. \tag{5}$$

Observe that, as expected, our estimate goes to infinity as t approaches 0 since $P_0 f = f$ is not necessarily differentiable. Also the rate at which it goes to infinity is not faster than $\frac{1}{\sqrt{t}}$ does.

2 Potential

Let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a bounded measurable function. We shall perturb the generator $\mathcal{A}$ by adding to it the function V. We define the linear operator

$$\mathcal{A}^V = \mathcal{A} + V\,,$$

with the same domain as for $\mathcal{A}$. A semigroup $\{P_t^V\}_{t\geq 0}$ having $\mathcal{A}^V$ as generator is given by the Feynman-Kac formula $P_t^V f = \mathbb{E}\{f(X_t)e^{\int_0^t V(X_u)\, du}\}$. We will find a similar estimate as (5) for $\|DP_t^V f\|_\infty$. We first derive a recursive formula that will help us calculate the derivative of $P_t^V f$. We have

$$\begin{aligned} P_t^V f &= P_t f + \left[\, P_{t-s}P_s^V f\,\right]_{s=0}^{s=t} \\ &= P_t f + \int_0^t \frac{\partial}{\partial s}(P_{t-s}P_s^V f)\, ds \\ &= P_t f + \int_0^t [-\mathcal{A}(P_{t-s}P_s^V f) + P_{t-s}((\mathcal{A}+V)P_s^V f)]\, ds\,. \end{aligned}$$

Hence

$$P_t^V f = P_t f + \int_0^t P_{t-s}(VP_s^V f)\, ds\,.$$

Now we use our formula for differentiation (4) to calculate the derivative of this semigroup. We have

$$\begin{aligned} DP_t^V f(x)(v) &= DP_t f(x)(v) + \int_0^t DP_{t-s}(VP_s^V f)(x)(v)\,ds \\ &= \frac{1}{t}\mathbb{E}\{f(X_t)\int_0^t \mathbb{Y}(X_s)(V_s)\,dB_s\} \\ &\quad + \int_0^t \frac{1}{t-s}\mathbb{E}\{V(X_{t-s})P_s^V f(X_{t-s})\int_0^{t-s}\mathbb{Y}(X_u)(V_u)\,dB_u\}ds\,. \end{aligned}$$

Then, by the Feynman-Kac formula and the Markov property we have

$$\begin{aligned} DP_t^V f(x)(v) &= \frac{1}{t}\mathbb{E}\{f(X_t)\int_0^t \mathbb{Y}(X_s)(V_s)\,dB_s\} \\ &+ \int_0^t \frac{1}{t-s}\mathbb{E}\{V(X_{t-s})\mathbb{E}\{f(X_t)e^{\int_{t-s}^t V(X_u)du}\}\int_0^{t-s}\mathbb{Y}(X_u)(V_u)\,dB_u\}\,ds\,, \end{aligned}$$

from which we obtain

$$\begin{aligned} \|DP_t^V f\|_\infty &\le \frac{1}{t}\|f\|_\infty \mathbb{E}|\int_0^t \mathbb{Y}(X_s)(V_s)\,dB_s| \\ &\quad + \|V\|_\infty\|f\|_\infty \int_0^t \frac{e^{s\|V\|_\infty}}{t-s}\mathbb{E}|\int_0^{t-s}\mathbb{Y}(X_u)(V_u)\,dB_u|\,ds\,. \end{aligned}$$

And hence for small t

$$\begin{aligned} \|DP_t^V f\|_\infty &\le \frac{c\|f\|_\infty}{\sqrt{t}} + \|V\|_\infty\|f\|_\infty e^{t\|V\|_\infty}\int_0^t \frac{c}{\sqrt{t-s}}\,ds \\ &= \frac{c\|f\|_\infty}{\sqrt{t}} + 2c\sqrt{t}\,\|V\|_\infty\|f\|_\infty e^{t\|V\|_\infty}\,. \end{aligned}$$

Observe again that our estimate goes to infinity as $t \to 0$.

3 Bounded Smooth Drift

Let $Z : \mathbb{R}^n \to \mathbb{R}^n$ be a bounded smooth vector field. We shall consider another perturbation to the generator $\mathcal{A}$. This time we define the linear operator

$$\mathcal{A}^Z = \mathcal{A} + Z \cdot \nabla\,.$$

The existence of a semigroup $\{P_t^Z\}_{t\ge 0}$ having $\mathcal{A}^Z$ as infinitesimal generator is guaranteed by the regularity of Z. Indeed, if we write $Z(x) = (Z^1(x), \dots, Z^n(x))$, then the operator $\mathcal{A}^Z$ can be written as

$$\mathcal{A}^Z = \frac{1}{2}\sum_{i,j=1}^n (\mathbb{X}(x)\mathbb{X}(x)^*)^{ij}\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^n (\mathbb{A}^i(x) + Z^i(x))\frac{\partial}{\partial x^i}\,,$$

and this operator is the infinitesimal generator associated with the equation

$$\begin{aligned} dX_t &= \mathbb{X}(X_t)dB_t + [\mathbb{A}(X_t) + Z(X_t)]dt\,, \\ X_0 &= x \in \mathbb{R}^n\,. \end{aligned}$$

Thanks to the smoothness of Z, this equation yields a diffusion process $(X_t^{x,Z})_{t\in T}$ and hence the semigroup $P_t^Z f(x) = \mathbb{E}(f(X_t^{x,Z}))$. Then previous estimate applies also to $\|DP_t^Z f\|_\infty$. But we can do better because we can find the explicit dependence of the estimate upon Z as follows. As before, we first find a recursive formula for this semigroup.

$$\begin{aligned} P_t^Z f &= P_t f + \left[P_{t-s} P_s^Z f \right]_{s=0}^{s=t} \\ &= P_t f + \int_0^t \frac{\partial}{\partial s}(P_{t-s} P_s^Z f)\, ds \\ &= P_t f + \int_0^t [-\mathcal{A}(P_{t-s} P_s^Z f) + P_{t-s}((\mathcal{A} + Z\cdot\nabla) P_s^Z f)]\, ds\,. \end{aligned}$$

Hence

$$P_t^Z f = P_t f + \int_0^t P_{t-s}(Z\cdot\nabla P_s^Z f)\, ds\,. \tag{6}$$

We can now calculate its derivative as follows

$$\begin{aligned} DP_t^Z f(x)(v) &= DP_t f(x)(v) + \int_0^t DP_{t-s}(Z\cdot\nabla P_s^Z f)(x)(v)\, ds \\ &= \frac{1}{t}\mathbb{E}\{f(X_t)\int_0^t \mathbb{Y}(X_s)(V_s)\, dB_s\} \\ &\quad + \int_0^t \frac{1}{t-s}\mathbb{E}\{Z(X_{t-s})\cdot\nabla P_s^Z f(X_{t-s})\int_0^{t-s}\mathbb{Y}(X_u)(V_u)\, dB_u\}\, ds\,. \end{aligned}$$

We now find an estimate for the supremum norm of this derivative. Taking modulus we obtain

$$\begin{aligned} |DP_t^Z f(x)(v)| &\le \frac{1}{t}\|f\|_\infty \mathbb{E}\,|\int_0^t \mathbb{Y}(X_s)(V_s)\, dB_s| \\ &\quad + \int_0^t \frac{1}{t-s}\mathbb{E}\,|DP_s^Z f(X_{t-s})(Z(X_{t-s}))|\;|\int_0^{t-s}\mathbb{Y}(X_u)(V_u)\, dB_u|\, ds\,. \end{aligned}$$

Hence for small t

$$\|DP_t^Z f\|_\infty \le \frac{c}{\sqrt{t}}\|f\|_\infty + c\|Z\|_\infty \int_0^t \frac{\|DP_s^Z f\|_\infty}{\sqrt{t-s}}\, ds\,.$$

We now solve this inequality. If we iterate once we obtain

$$\begin{aligned} \|DP_t^Z f\|_\infty &\le \frac{c}{\sqrt{t}}\|f\|_\infty + c^2\|Z\|_\infty\|f\|_\infty \int_0^t \frac{ds}{\sqrt{s(t-s)}} \\ &\quad + c^2\|Z\|_\infty^2 \int_0^t\int_0^s \frac{\|DP_u^Z f\|_\infty}{\sqrt{(t-s)(s-u)}}\, du\, ds\,. \end{aligned}$$

By Fubini's theorem, the double integral becomes

$$\int_0^t \|DP_u^Z f\|_\infty \int_u^t \frac{ds}{\sqrt{(t-s)(s-u)}}\, du\,,$$

and then we observe that

$$\int_u^t \frac{ds}{\sqrt{(t-s)(s-u)}} = 2\tan^{-1}\sqrt{\frac{s-u}{t-s}}\,\Bigg|_u^t = \pi\,.$$

The case $u = 0$ solves also the first integral. Hence our inequality reduces to

$$\|DP_t^Z f\|_\infty \le \frac{c}{\sqrt{t}}\|f\|_\infty + c^2\pi\|Z\|_\infty\|f\|_\infty + c^2\pi\|Z\|_\infty^2\int_0^t \|DP_u^Z f\|_\infty\,du\,.$$

We now apply Gronwall's inequality. After some simplifications (extending the integral up to infinity) we finally obtain the estimate

$$\|DP_t^Z f\|_\infty \le \frac{c}{\sqrt{t}}\|f\|_\infty + 2c^2\pi\|Z\|_\infty\|f\|_\infty e^{t(c^2\pi\|Z\|_\infty)} \tag{7}$$

As expected, our estimate goes to infinity as $t \to 0$ since $P_0^Z f = f$ is not necessarily differentiable.

4 Bounded Uniformly Continuous Drift

We now find a similar estimate when $Z\colon \mathbb{R}^n \to \mathbb{R}^n$ is only bounded and uniformly continuous. We look again at the operator $\mathcal{A}^Z = \mathcal{A} + Z\cdot\nabla$. The problem here is that in this case we do not have the semigroup $\{P_t^Z f\}_{t\ge 0}$ since the stochastic equation with the added nonsmooth drift Z might not have a strong solution. So we cannot even talk about its derivative. To solve this problem we proceed by approximation.

4.1 Existence of Semigroup

Since $Z \in BC^0(\mathbb{R}^n;\mathbb{R}^n)$ is uniformly continuous, and $BC^\infty(\mathbb{R}^n;\mathbb{R}^n)$ is dense in $BC^0(\mathbb{R}^n;\mathbb{R}^n)$, there exists a sequence $\{Z_i\}_{i=1}^\infty$ in $BC^\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that Z_i converges to Z uniformly. Thus, for every $i \in \mathbf{N}$, we have the semigroup $\{P_t^{Z_i}\}_{t\ge 0}$ since our stochastic equation with the added smooth drift Z_i has a strong solution.
For every $t \ge 0$ and $f \in BC^0(\mathbb{R}^n)$ fixed, the sequence of functions $\{P_t^{Z_i} f\}_{i=1}^\infty$ is a Cauchy sequence in the Banach space $BC^0(\mathbb{R}^n)$. We will prove this fact later. Let us denote its limit by $P_t^Z f$. All properties required for $P_t^Z f$ are inherited from those of the semigroup $P_t^{Z_i} f$. Indeed by simply writing $P_t^Z f = \lim_{i\to\infty} P_t^{Z_i} f$ and using an interchange of limits we can prove

1. $f \mapsto P_t^Z f$ is a bounded linear operator.
2. $t \mapsto P_t^Z f$ is a contraction semigroup of operators.
3. $\{P_t^Z\}_{t\ge 0}$ has generator $\mathcal{A}^Z = \mathcal{A} + Z\cdot\nabla$.

Now, suppose for a moment that the sequence of derivatives $\{DP_t^{Z_i} f\}_{i=1}^\infty$ converges uniformly. Then we would have $D(\lim_{i\to\infty} P_t^{Z_i} f) = \lim_{i\to\infty} DP_t^{Z_i} f$. This proves that $P_t^Z f$ is differentiable. Then, for any $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that if $i \ge N$, $\|DP_t^Z f\|_\infty \le \|DP_t^{Z_i} f\|_\infty + \epsilon$ and therefore our estimate also applies to $\|DP_t^Z f\|_\infty$.

4.2 Uniform Convergence of Derivatives

We now prove that the sequence $\{DP_t^{Z_i}f\}_{i=1}^{\infty}$ converges uniformly. We use again our recursive formula (6) to obtain

$$P_t^{Z_i}f - P_t^{Z_j}f = \int_0^t P_{t-s}(Z_i \cdot \nabla P_s^{Z_i}f - Z_j \cdot \nabla P_s^{Z_j}f)\, ds\,.$$

We now use our formula for differentiation (4). After differentiating and taking modulus we obtain

$$\begin{aligned}
&|D(P_t^{Z_i}f - P_t^{Z_j}f)(x)(v)| \\
\leq\; & \int_0^t \frac{1}{t-s}\mathbb{E}\{|DP_s^{Z_i}f(X_{t-s})(Z_i(X_{t-s})) - DP_s^{Z_j}f(X_{t-s})(Z_j(X_{t-s}))| \\
& |\int_0^{t-s} \mathbb{Y}(X_u)(V_u)\, dB_u|\}\, ds\,.
\end{aligned}$$

Thus

$$\begin{aligned}
\|D(P_t^{Z_i}f - P_t^{Z_j}f)\|_\infty \quad \leq \quad & \|Z_i - Z_j\|_\infty \int_0^t \frac{k}{t-s}\sqrt{e^{a(t-s)}-1}\|DP_s^{Z_i}f\|_\infty\, ds \\
& + \|Z_j\|_\infty \int_0^t \frac{k}{t-s}\sqrt{e^{a(t-s)}-1}\,\|DP_s^{Z_i}f - DP_s^{Z_j}f\|_\infty\, ds\,.
\end{aligned}$$

And hence for small t

$$\begin{aligned}
\|D(P_t^{Z_i}f - P_t^{Z_j}f)\|_\infty \quad \leq \quad & \|Z_i - Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}}\|DP_s^{Z_i}f\|_\infty\, ds \\
& + \|Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}}\|DP_s^{Z_i}f - DP_s^{Z_j}f\|_\infty\, ds\,.
\end{aligned}$$

Now write estimate (7) as

$$\|DP_t^{Z_i}f\|_\infty \leq \frac{A}{\sqrt{t}} + B\, e^{C\,t}\,,$$

for some positive constants A, B and C depending on $\|f\|_\infty$ and $\|Z_i\|_\infty$. Thus, substituting this in our last estimate gives

$$\begin{aligned}
\|D(P_t^{Z_i}f - P_t^{Z_j}f)\|_\infty \quad \leq \quad & \|Z_i - Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}}\frac{A}{\sqrt{s}}\, ds \\
& + \|Z_i - Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}} B\, e^{C\,s}\, ds \\
& + \|Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}}\|DP_s^{Z_i}f - DP_s^{Z_j}f\|_\infty\, ds\,.
\end{aligned}$$

The first two integrals are bounded if we allow t to move within a finite interval $(0, T]$. Thus there exist positive constants M_1 and M_2 such that

$$\begin{aligned}
\|D(P_t^{Z_i}f - P_t^{Z_j}f)\|_\infty \quad \leq \quad & M_1\|Z_i - Z_j\|_\infty \\
& + M_2 \int_0^t \frac{1}{\sqrt{t-s}}\|DP_s^{Z_i}f - DP_s^{Z_j}f\|_\infty\, ds\,.
\end{aligned}$$

Now iterate this to obtain

$$\begin{aligned}\|D(P_t^{Z_i}f - P_t^{Z_j}f)\|_\infty \;\le\;& M_1\|Z_i - Z_j\|_\infty \\ &+ M_2M_1\|Z_i - Z_j\|_\infty \int_0^t \frac{1}{\sqrt{t-s}}\,ds \\ &+ M_2^2\int_0^t\int_0^s \frac{\|D(P_u^{Z_i}f - P_u^{Z_j}f)\|_\infty}{\sqrt{(t-s)(s-u)}}\,du\,ds\,.\end{aligned}$$

As before the double integral reduces to

$$\pi M_2^2\int_0^t \|D(P_u^{Z_i}f - P_u^{Z_j}f)\|_\infty\,du\,.$$

Collecting constants into new constants M and N we arrive at

$$\|D(P_t^{Z_i}f - P_t^{Z_j}f)\|_\infty \le M\|Z_i - Z_j\|_\infty + \; N\int_0^t \|D(P_u^{Z_i}f - P_u^{Z_j}f)\|_\infty\,du\,.$$

Now we apply again Gronwall's inequality to obtain

$$\|D(P_t^{Z_i}f - P_t^{Z_j}f)\|_\infty \le M\|Z_i - Z_j\|_\infty + \; NM\|Z_i - Z_j\|_\infty \int_0^t e^{N(t-u)}\,du\,.$$

The right hand side goes to zero as $\|Z_i - Z_j\|_\infty \to 0$. This proves the sequence $\{DP_t^{Z_i}\}_{i=1}^\infty$ is uniformly convergent.

4.3 Uniform Convergence of Semigroups

We finally prove that the sequence of functions $\{P_t^{Z_i}f\}_{i=1}^\infty$ is a Cauchy sequence. From our recursive formula (6) we obtain

$$\begin{aligned}\|P_t^{Z_i}f - P_t^{Z_j}f\|_\infty \;\le\;& \int_0^t \|Z_i\cdot\nabla P_s^{Z_i}f - Z_j\cdot\nabla P_s^{Z_j}f\|_\infty\,ds \\ \le\;& \int_0^t \|Z_i - Z_j\|_\infty\|DP_s^{Z_i}f\|_\infty\,ds \\ &+ \int_0^t \|Z_j\|_\infty\|DP_s^{Z_i}f - DP_s^{Z_j}f\|_\infty\,ds \\ \le\;& \|Z_i - Z_j\|_\infty\int_0^t k\sqrt{e^{as}-1}\;ds \\ &+\|Z_j\|_\infty\int_0^t \|DP_s^{Z_i}f - DP_s^{Z_j}f\|_\infty\,ds\,.\end{aligned}$$

The first integral is bounded so the first part goes to zero as $\|Z_i - Z_j\|_\infty$ approaches 0. The second part also goes to zero since we just proved the sequence of derivatives is a Cauchy sequence. Hence $\{P_t^{Z_i}f\}_{i=1}^\infty$ is uniformly convergent.
Thus, with this approximating procedure we found a semigroup for the operator $\mathcal{A} + Z\cdot\nabla$ for Z bounded and uniformly continuous and we proved it is differentiable and that our estimate also applies to its derivative.

Observe that the boundedness and uniform continuity of Z are required in order to ensure the existence of a sequence of smooth vector fields Z_i uniformly convergent to Z. Alternative assumptions on Z that guarantee the existence of such approximating sequence may be used.

Appendix

We here give a proof of inequality (1) which is taken from [6]. By using the Cauchy-Schwarz inequality and then the isometric property, we have

$$\begin{aligned}\mathbb{E}\,|\int_0^t \mathbb{Y}(x_s)(v_s)\,dB_s| &\leq (\mathbb{E}|\int_0^t \mathbb{Y}(x_s)(v_s)\,dB_s|^2\,)^{1/2}\\ &= (\mathbb{E}\int_0^t \|\mathbb{Y}(x_s)(v_s)\|^2\,ds\,)^{1/2}\\ &\leq \|\mathbb{Y}\|_\infty^2(\int_0^t \mathbb{E}\|v_s\|^2\,ds\,)^{1/2}\,. \qquad (8)\end{aligned}$$

We will estimate the right-hand side of the last inequality. Itô's formula applied to the function $f(\cdot) = \|\cdot\|^2 \colon \mathbb{R}^n \to \mathbb{R}$ and the semimartingale $(v_t)_{t\geq 0}$ yields

$$\|v_s\|^2 = \|v_0\|^2 + 2\int_0^s v_u\,dv_u + \sum_{i=1}^n \int_0^s d\langle v^i\rangle_u\,.$$

Therefore

$$\begin{aligned}\mathbb{E}\,\|v_s\|^2 &= \|v_0\|^2 + \mathbb{E}\langle v\rangle_s\\ &= \|v_0\|^2 + \mathbb{E}\int_0^s [D\mathbb{X}(x_u)(v_u)]\,[D\mathbb{X}(x_u)(v_u)]^*\,du\\ &= \|v_0\|^2 + \mathbb{E}\int_0^s \|D\mathbb{X}(x_u)(v_u)\|^2\,du\\ &\leq \|v_0\|^2 + \|D\mathbb{X}\|_\infty^2 \int_0^s \mathbb{E}\,\|v_u\|^2\,du\,.\end{aligned}$$

We now use Gronwall's inequality to obtain

$$\mathbb{E}\,\|v_s\|^2 \leq \|v_0\|^2 e^{\|D\mathbb{X}\|_\infty\, s}\,.$$

Integrating from 0 to t yields

$$\int_0^t \mathbb{E}\,\|v_s\|^2\,ds \leq \frac{\|v_0\|^2}{\|D\mathbb{X}\|_\infty}(e^{\|D\mathbb{X}\|_\infty\, t} - 1)\,,$$

and thus, substituting in (8) we obtain

$$\mathbb{E}\,|\int_0^t \mathbb{Y}(x_s)(v_s)\,dB_s| \leq \|\mathbb{Y}\|_\infty^2 \frac{\|v_0\|}{\|D\mathbb{X}\|_\infty^{1/2}}\sqrt{e^{t\,\|D\mathbb{X}\|_\infty} - 1}\,.$$

This proves inequality (1).

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