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On Uniqueness of the Euler Limit of One-Component Lattice Gas Models

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Abstract. We investigate the interaction of one-dimensional asymmetric exclusion processes of opposite speeds, the exchange mechanism is combined with a spin-flip dynamics, and this asymmetric law is regularized by a nearest neighbor stirring of large intensity. At an intuitive level we can say that particles with ± 1 spins are subject to an external magnetic field, and the additional spin-flip dynamics results in a strong relaxation of total magnetization. Therefore this modification of the model of Fritz and Tóth (2004) admits particle number as the only conservation law, with hyperbolic scaling. By means of a two-step version of LSI based estimation techniques we prove that compensated compactness and the Lax entropy inequality imply the existence and uniqueness of the hydrodynamic limit even in a regime of shocks.

1. Introduction and main result

Most results of the microscopic theory of hydrodynamics concern diffusive models, see the monograph by Kipnis and Landim (1999). The relative entropy method of Yau (1991) is a general tool of scaling limits when the solution to the macroscopic equation is smooth enough, and hence unique. This method works even in the case of hyperbolic problems, see also Olla et al. (1993) and Tóth and Valkó (2003, 2005). Beyond shocks the specific structure of the microscopic system becomes more important. The *attractiveness* of the one-component models of Rezakhanlou (1991) implies also the uniqueness of the hydrodynamic limit, the proofs are based on powerful coupling techniques. The nice, more or less explicit calculations of Seppäläinen (1999) adopt direct PDE methods as the Hopf - Lax formula for solving the Burgers equation. Coupling methods can be extended to some wider

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classes of one-component sytems with a more general *order preserving* structure Seppäläinen (2000); Rezakhanlou (2002), see also Bahadoran (2004) for a synthesis and survey of such models with some further references.

In the last few years several efforts Fritz (2001, 2004); Fritz and Tóth (2004) have been done to derive the hydrodynamic limit of microscopic systems with a hyperbolic structure via *compensated compactness.* This fairly general method is not restricted to one-component systems allowing coupling. The stochastic version of compensated compactness yields existence of the limit along subsequences: all limit distributions of the scaled process are concentrated on a set of *weak solutions* to the macroscopic equations, the question of uniqueness of the limit is more problematic. To select the *physically admissible* element from the set of weak solutions, an *entropy condition* like that of Oleinik (1957) or Kružkov (1970) is needed; the latter can really be verified in the case of attractive and order preserving systems Rezakhanlou (1991); Bahadoran (2004). The asymmetric Ginzburg-Landau model of Fritz (2004) is attractive in the case of a convex potential only, the general problem has not been solved there. The two-component system of Fritz and Tóth (2004) is certainly not attractive or order preserving, we do not see any effective way of coupling in that case. Let us remark at this point that uniqueness of solutions to systems of conservation laws is difficult even at the level of partial differential equations, see the monograph of Bressan (2000) for a survey. These Oleinik type conditions of uniqueness are not easy to verify in the case of microscopic models, we have no results in this direction.

The situation is much better when we are facing with a single conservation law, even the weak version (2.16) of the Lax entropy inequality is sufficient for uniqueness of weak solutions to a single conservation law, see Kružkov's classical result as stated and proven in Bressan (2000): Theorem 6.2 and its Corollary 6.1. The result of A. Vasseur (2001) is also applicable, he proves existence of a strong trace of weak solutions on the $t = 0$ line under the *weak entropy condition* (2.16). Moreover, if the flux of a single conservation law in one space dimension is strictly convex, then the weak entropy condition for one strictly convex entropy is sufficient for weak uniqueness, see Panov (1994); De Lellis et al. (2004). This nice result implies also that the rate function Varadhan (2004) of large deviations for the totally asymmetric exclusion process has a unique minimum: the *entropy solution* to the *Burgers equation.* On the other hand, the initial condition of the DiPerna uniqueness theorem (DiPerna, 1985) for *measure valued solutions* has considerably been relaxed by Gallouët and Herbin (1993), cf. Rezakhanlou (1991) and the discussion after (2.12).

In this paper we prove existence and uniqueness of the hydrodynamic limit for a non-attractive hyperbolic model having only one conservation law. There are several models of this kind, for example asymmetric exclusion where a particle jumps to the nearest vacant site. These dynamics do not allow coupling unless the jump rate is a non-increasing function of the jump length, see Bahadoran (2004) for a fairly general exposition and discussion. The treatment of this problem might not require ideas that go far beyond Fritz (2001). Some other classes of lattice models with one or more conservation laws have been introduced and investigated by means of the relative entropy method in Tóth and Valkó (2003, 2005); the system of interacting exclusions that we have studied in Fritz and Tóth (2004) was proposed as a basic example in Tóth and Valkó (2003). Here we investigate this model with an additional relaxation mechanism. In the spirit of the theory of shock waves, instead of the usual assumption of periodic boundary conditions, we consider the system on the infinite line, results of Fritz and Tóth (2004) can also be extended in this way.

1.1. *The model.* The configuration space of the system is defined as $\Omega := \{-1, 0, +1\}^{\mathbb{Z}}$, it is equipped with the usual product topology and the associated Borel structure. For example, $\mathcal{F}_n$ denotes the σ-field generated by the variables $\{\omega_k : |k| \leq n\}$. In view of the physical interpretation, $\omega_k = 0$ means a vacant site $k \in \mathbb{Z}$, otherwise we have a ± 1 spin there. Due to the external field, particles of spin $+1$ jump to the right, -1 particles jump to the left, both at a unite rate with full exclusion. When particles of opposite spins meet, they are exchanged at rate 2, thus the generator, $\mathcal{L}_o$ of this component of the evolution is acting on cylinder functions φ as

$$\mathcal{L}_o\varphi(\omega) := \sum_{b\in\mathbb{Z}_*} c_b(\omega)\left(\varphi(\omega^b) - \varphi(\omega)\right) , \tag{1.1}$$

where $\mathbb{Z}_*$ is the set of bonds $b = (k, k+1)$ of $\mathbb{Z}$,

$$c_b(\omega) := \frac{1}{2}\left(\omega_k^2 + \omega_{k+1}^2 - \omega_k^2\omega_{k+1}^2 - \omega_k\omega_{k+1} + \omega_k - \omega_{k+1}\right),$$

and ω^b is obtained from $\omega \in \Omega$ by exchanging ω_k and ω_{k+1}; the rest of the configuration is not altered. Notice that the actual value of c_b is irrelevant if $\omega_k = \omega_{k+1}$; c_b of (1.1) can be replaced with

$$\tilde{c}_b(\omega) := \frac{1}{2}(\omega_k^2 + \omega_{k+1}^2 + \omega_k - \omega_{k+1}).$$

$\mathcal{L}_o$ is the main, asymmetric component of the evolution law investigated in Fritz and Tóth (2004), it preserves both particle number $N \sim \sum \omega_k^2$ and total magnetization $M \sim \sum \omega_k$. Due to the suitable choice of jump rates, all translation invariant product measures on Ω are stationary states for $\mathcal{L}_o$.

The Glauberian, spin-flip component of the evolution sends $\omega_k = 1 \to -\omega_k = -1$ at rate $1-\kappa$, and the rate of a transition $\omega_k = -1 \to -\omega_k = +1$ is $1+\kappa$, where $\kappa \in \mathbb{R}$ is a fixed parameter, $|\kappa| < 1$. The associated generator reads as

$$\mathcal{G}_\kappa\varphi(\omega) := \sum_{k\in\mathbb{Z}} c_k(\omega)\left(\varphi(\omega^k) - \varphi(\omega)\right) , \tag{1.2}$$

$$c_k(\omega) := \frac{1-\kappa}{2}\left(\omega_k^2 + \omega_k\right) + \frac{1+\kappa}{2}\left(\omega_k^2 - \omega_k\right) = \omega_k^2 - \kappa\omega_k ,$$

and ω^k is obtained from ω by changing the sign of ω_k; other spins remain as before the flip. Conservation of total magnetization is violated by $\mathcal{G}_\kappa$, but the number of particles is preserved, thus we get a one-parameter family λ_ρ, $0 < \rho < 1$ of stationary product measures specified by $\lambda_\rho[\omega_k^2 = 1] = \rho$ and $(1-\kappa)\lambda_\rho[\omega_k = 1] = (1+\kappa)\lambda_\rho[\omega_k = -1]$, i.e. the equilibrium expectation of ω_k is just $\lambda_\rho(\omega_k) = \kappa\rho$. The full generator of the process is now defined as $\mathcal{L} = \mathcal{L}_o + \alpha\mathcal{G}_\kappa + \sigma\,\mathcal{S}$, where α and σ are nonnegative parameters to be specified later,

$$\mathcal{S}\varphi(\omega) := \sum_{b\in\mathbb{Z}_*}\left(\varphi(\omega^b) - \varphi(\omega)\right) . \tag{1.3}$$

From a technical point of view, it is very important that σ, the intensity of stirring goes to $+\infty$ during the scaling procedure. An *elliptic perturbation* like $\sigma\mathcal{S}$ is widely employed in the theory and practice of hyperbolic systems of conservation laws

DiPerna (1983, 1985); Bressan (2000); Dafermos (2000); Serre (2000) to regularize approximate solutions; the method of *vanishing viscosity* is perhaps the most transparent analogy. Of course, the stirring modifies the microscopic evolution in a radical way, but it does not affect the macroscopic behavior of the system, see Fritz (2001, 2004); Fritz and Tóth (2004); Tóth and Valkó (2005). Unfortunately, we only have sufficient conditions on the strength σ of this *artificial viscosity.* The speed α of spin flips is almost arbitrary, we only need that it does not go to zero too fast. Another version of this model preserving total spin is mentioned at the end of the paper.

1.2. *Discussion.* Although coupling techniques are not applicable to the process generated by $\mathcal{L}$, it is very convenient to have product measures as its stationary states. More precisely, calculations of the next section imply that every translation invariant stationary measure is a superposition of product measures of type λ_ρ defined above. Since "conserved quantity" is a loose term, this is the precise meaning of the statement that *we have only one conservation law, namely that of particle number.* Indeed, according to the theory of Gibbs random fields, see Kipnis and Landim (1999) also for the basic notions of the theory of hydrodynamic limits, the measures λ_ρ are associated with the number of particles. Nevertheless, the description of stationary states does not play an explicit role in the forthcoming calculations, but it is really helpful in understanding what is going on.

Let $\eta_k := \omega_k^2$ denote our distinguished variable, and consider the microscopic current $\mathbf{j}_k$ of particle number along a bond $b = (k, k+1)$, i.e. $\mathcal{L}\eta_k = \mathbf{j}_{k-1} - \mathbf{j}_k$. Since $\mathcal{G}_\kappa\,\omega_k^2 = 0$, $\mathbf{j}_k = \mathbf{j}_k^o + \sigma\mathbf{j}_k^s$,

$$\mathbf{j}_k^o(\omega) := c_b(\omega)(\eta_k - \eta_{k+1}) = \tfrac{1}{2}\left(\omega_k + \omega_{k+1} - \omega_k\eta_{k+1} - \eta_k\omega_{k+1} + \eta_k - \eta_{k+1}\right), \tag{1.4}$$

and $\mathbf{j}_k^s(\omega) := \eta_k - \eta_{k+1}$. In view of the *principle of local equilibrium,* see e.g. Kipnis and Landim (1999), as $\lambda_\rho(\omega_k) = \kappa\rho$ and λ_ρ is a product measure, a Burgers equation

$$\partial_t\rho + \kappa\,\partial_x(\rho - \rho^2) = 0 \tag{1.5}$$

is expected for the hydrodynamic limit of particle density $\rho(t,x)$. To understand the claim from a point of view of dynamics, let $\mathbf{j}_k^\omega$ denote the flux of magnetization ω across a bond $(k, k+1)$,

$$\mathbf{j}_k^\omega = \frac{1}{2}\left(\eta_k + \eta_{k+1} - 2\omega_k\omega_{k+1} + \omega_k\eta_{k+1} - \eta_k\omega_{k+1} + (2\sigma+1)(\omega_k - \omega_{k+1})\right).$$

Since $\mathcal{G}_\kappa\omega_k = 2\kappa\eta_k - 2\omega_k$, $w_k := \kappa\eta_k - \omega_k$ satisfies

$$\mathcal{L}w_k + \kappa\,(\mathbf{j}_k - \mathbf{j}_{k-1}) - (\mathbf{j}_k^\omega - \mathbf{j}_{k-1}^\omega) = -2\alpha w_k\,, \tag{1.6}$$

a balance equation with a relaxation term on its right hand side. In the procedure of hydrodynamic limit time is speeded up, thus w should vanish in a mean sense. Of course, the materialization of this argument at the microscopic level requires additional tools. Just as in Fritz and Tóth (2004), first we apply the logarithmic Sobolev inequality due to stirring $\mathcal{S}$. The argument is then completed by substituting block averages of ω with those of $\kappa\eta$ via a second LSI related to the Glauberian generator $\mathcal{G}_\kappa$.

1.3. *Main result.* It is well known that the Burgers equation develops shocks, and uniqueness of *weak solutions* breaks down at the same time, thus we must be careful with definitions. A locally integrable $\rho : \mathbb{R}_+^2 \mapsto [0,1]$ is a weak solution to (1.5) with initial value $\rho_0 \in L^\infty(\mathbb{R})$ if

$$\int_0^\infty \int_{-\infty}^\infty \big(\rho\psi_t' + (\kappa\rho - \kappa\rho^2)\psi_x'\big)\, dx\, dt + \int_{-\infty}^\infty \psi(0,x)\rho_0(x)\, dx = 0 \tag{1.7}$$

for all $\psi \in C_c^1(\mathbb{R}^2)$. The notion of Lax entropy plays a fundamental role in the study of weak solutions. A couple $h, J \in C^1(\mathbb{R})$ is called a Lax entropy pair if $J'(\rho) = \kappa\,(1-2\rho)h'(\rho)$, that is $\partial_t h(\rho) + \partial_x J(\rho) = 0$ along classical solutions. A measurable and bounded $\rho(t,x)$ is a weak entropy solution to $\partial_t\rho + \kappa\partial_x(\rho - \rho^2) = 0$ with initial data ρ_0 if

$$\int_0^\infty \int_{-\infty}^\infty \big(h(\rho)\psi_t' + J(\rho)\psi_x'\big)\, dx\, dt + \int_{-\infty}^\infty h(\rho_0(x))\psi(0,x)\, dx \ \geq 0 \tag{1.8}$$

for all compactly supported $0 \leq \psi \in C_c^1(\mathbb{R}^2)$ and entropy pairs (h,J) with h convex. Since $h(\rho) \equiv \rho$ and $h(\rho) \equiv -\rho$ are both convex, all weak entropy solutions are weak solutions in the usual sense. Convexity of an entropy pair (h,J) means that h is a convex function.

At any level $\varepsilon > 0$ of scaling, the simplest version of the empirical process of particle density is defined as $\rho_\varepsilon(t,x) := \eta_k(t/\varepsilon) = \omega_k^2(t/\varepsilon)$ if $|x - \varepsilon k| < \varepsilon/2$, and

$$R_\varepsilon(\psi) := \int_0^\infty \int_{-\infty}^\infty \psi(t,x)\, \rho_\varepsilon(t,x)\, dx\, dt \tag{1.9}$$

is the scaled density field, where $\psi \in C_c(\mathbb{R}^2)$ is compactly supported. The initial conditions are specified in terms of a family $\mu_{\varepsilon,0}$ of probability measures, we are assuming that

$$\lim_{\varepsilon\to 0} \int_{-\infty}^\infty \varphi(x)\, \rho_\varepsilon(0,x)\, dx = \int_{-\infty}^\infty \varphi(x)\, \rho_0(x)\, dx \tag{1.10}$$

in probability for all $\varphi \in C_c(\mathbb{R})$, where $0 \leq \rho_0 \leq 1$ is a given measurable function. Here and below a subscript "c" refers to compactly supported functions, $\mathbb{R}_+ := [0,+\infty)$, $\mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}$ and $C_{co}^1(\mathbb{R}_+^2)$ is the space of continuously differentiable $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$ with compact support in the interior of $\mathbb{R}_+^2$. In its simplest form, our main result can be stated as

Theorem 1.1. *Suppose* (1.10) *and specify* $\sigma = \sigma(\varepsilon)$ *and* $\alpha = \alpha(\varepsilon)$ *such that* $\varepsilon\sigma(\varepsilon) \to 0$ *but* $\varepsilon\sigma^2(\varepsilon) \to +\infty$, *while* $\sigma(\varepsilon)\alpha(\varepsilon) \to +\infty$ *as* $\varepsilon \to 0$. *Then*

$$\lim_{\varepsilon\to 0} R_\varepsilon(\psi) = \int_0^\infty \int_{-\infty}^\infty \psi(t,x)\rho(t,x)\, dx\, dt$$

in probability for all $\psi \in C_c(\mathbb{R}^2)$; *this* $\rho(t,x)$ *is the uniquely specified weak entropy solution* (1.8) *to* $\partial_t\rho + \kappa\partial_x(\rho - \rho^2) = 0$ *with initial value* ρ_0.

Compensated compactness yields strong convergence of approximate solutions; an improved version of the above result is discussed at the end of the paper in this spirit. A trivial equation, $\partial_t\rho = 0$ is obtained when $\kappa = 0$, and *diffusive scaling* is the natural one in this particular case. Since the microscopic flux $\mathbf{j}_k$ is not a difference, we have got a non-gradient problem requiring methods of Varadhan (1993) that do not fit into the frames of the paper; we are going to return to this issue elsewhere.

2. An outline of the proof

The main ideas of the proof go back to Fritz (2001, 2004); Fritz and Tóth (2004), we have to develop compensated compactness and verify the entropy inequality (1.8) at the microscopic level as $\varepsilon \to 0$.

2.1. *Block averages.* As it is more or less obligatory in the microscopic theory of hydrodynamics, first we rewrite the empirical process in terms of block averages. For $l \in \mathbb{N}$ and any sequence ξ indexed by $\mathbb{Z}$ set

$$\bar{\xi}_{l,k} := \frac{1}{l}\sum_{j=0}^{l-1}\xi_{k+j} \quad \text{and} \quad \hat{\xi}_{l,k} := \frac{1}{l^2}\sum_{j=-l}^{l}(l-|j|)\,\xi_{k+j}\,. \tag{2.1}$$

The smooth averaging $\hat{\xi}_l$ is used to define a modified empirical process $\hat{\rho}_\varepsilon$ as $\hat{\rho}_\varepsilon(t,x) := \hat{\eta}_{l,k}(t/\varepsilon)$ if $|x-\varepsilon k| < \varepsilon/2$, while the usual arithmetic mean, $\bar{\xi}_l$ is preferred in computing canonical expectations. *Compensated compactness,* see Murat (1978); Tartar (1979) and DiPerna (1983, 1985); Dafermos (2000); Serre (2000) is a basic key word of the proof, this method yields strong convergence of suitably chosen approximate solutions. A stochastic version of the theory has been initiated in Fritz (2001, 2004); Fritz and Tóth (2004). In contrast to diffusive scaling problems, first we prove strong convergence for *mesoscopic block averages*, see last section: Concluding Remarks. The size $l = l(\varepsilon)$ of these blocks should be chosen in such a way that

$$\lim_{\varepsilon\to 0}\frac{\sigma(\varepsilon)}{\varepsilon l^3(\varepsilon)} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\frac{l(\varepsilon)}{\sigma(\varepsilon)} = 0\,, \tag{2.2}$$

thus $\varepsilon l^2(\varepsilon) \to +\infty$ as $\varepsilon \to 0$. Since $\varepsilon\sigma(\varepsilon) \to 0$ and $\varepsilon\sigma^2(\varepsilon) \to +\infty$, $\sigma^2 = o(l^3)$. We also see that $(\sigma/\varepsilon)^{1/3} = o(l) = o(\sigma)$, thus the integer part of $\varepsilon^{-1/4}\sqrt{\sigma(\varepsilon)}$ is an acceptable choice for l. From now on the block size $l = l(\varepsilon)$ is specified according to (2.2), and $\hat{P}_\varepsilon$ denotes the distribution of $\hat{\rho}_\varepsilon$ on $L^2_{\mathrm{loc}}(\mathbb{R}^2_+)$, the space of locally square integrable functions. This family is tight with respect to the weak topology of L^2_{loc}, we have to show that its weak limit exists, and it is concentrated on the unique entropy solution satisfying (1.8) for all convex entropy pairs. Since $\hat{\rho}_\varepsilon(t,\cdot)$ is a stochastically continuous process when $\varepsilon > 0$ is fixed, $\hat{P}_\varepsilon$ determines the initial distribution of $\hat{\rho}_\varepsilon$, but this relation might be lost in the hydrodynamic limit.

2.2. *Entropy production.* The microscopic version of entropy production $X_\varepsilon = \partial_t h + \partial_x J$ is defined as a distribution: for $\psi \in C^1_c(\mathbb{R}^2)$ and entropy pairs (h,J) we introduce

$$X_\varepsilon(\psi,h) := -\int_0^\infty\int_{-\infty}^\infty \big(h(\hat{\rho}_\varepsilon)\psi_t'(t,x) + J(\hat{\rho}_\varepsilon)\psi_x'(t,x)\big)\,dx\,dt\,. \tag{2.3}$$

This formula follows by a formal integration by part if $\psi \in C^1_c(\mathbb{R}^2_+)$ and $\psi(0,x) = 0$ $\forall\, x \in \mathbb{R}$. Calculating the stochastic differential of

$$H_\varepsilon(t,\psi,h) := \int_{-\infty}^\infty \psi(t,x)h(\hat{\rho}_\varepsilon(t,x))\,dx \tag{2.4}$$

we get a martingale $M_\varepsilon(t,\psi,h)$, see (4.19), such that

$$dH_\varepsilon = \int_{-\infty}^\infty \psi_t'(t,x)h(\hat{\rho}_\varepsilon)\,dx\,dt + \varepsilon^{-1}\mathfrak{L}H_\varepsilon\,dt + dM_\varepsilon\,,$$

whence

$$X_\varepsilon(\psi,h) = H_\varepsilon(0,\psi,h) + L_\varepsilon(\psi,h) + I_\varepsilon(\psi,h) + M_\varepsilon(\infty,\psi,h) + N_\varepsilon(\psi,h)\,, \qquad (2.5)$$

where $L_\varepsilon(\psi,h) = L_\varepsilon^o(\psi,h) + \sigma(\varepsilon)L_\varepsilon^s(\psi,h)$,

$$L_\varepsilon^o(\psi,h) := \frac{1}{\varepsilon}\int_0^\infty \int_{-\infty}^\infty \psi(t,x)\,\mathcal{L}_o h(\hat\rho_\varepsilon(t,x))\,dx\,dt\,, \qquad (2.6)$$

$$L_\varepsilon^s(\psi,h) := \frac{1}{\varepsilon}\int_0^\infty \int_{-\infty}^\infty \psi(t,x)\,\mathcal{S}h(\hat\rho_\varepsilon(t,x))\,dx\,dt\,, \qquad (2.7)$$

$$I_\varepsilon(\psi,h) := \frac{1}{\varepsilon}\int_0^\infty \int_{-\infty}^\infty \psi(t,x)\;\left(J(\hat\rho_\varepsilon(t,x)) - J(\hat\rho_\varepsilon(t,x-\varepsilon))\right)\,dx\,dt\,, \qquad (2.8)$$

and $N_\varepsilon(\psi,h)$ is a numerical error due to the lattice approximation of the space derivative, see (4.4).

The evaluation of X_ε is quite easy when $h(\rho)=\rho$ for all $0\le\rho\le 1$ because $\mathcal{L}\eta_k$ is a difference of the currents along adjacent bonds, thus rearranging sums by performing discrete integration by parts, the test function nicely absorbs the factor ε^{-1} of $\mathcal{L}$. However, a nonlinear entropy is not conserved by the microscopic evolution; in that case we have to do something more to remove the singularity.

The most crucial step of the proof is to show that L_ε^o and I_ε cancel each other when $\varepsilon\to 0$, that is the Lax entropy is partially conserved in the hydrodynamic limit; this will be shown by means of logarithmic Sobolev inequalities in Section 4. The Glauberian component, $\alpha(\varepsilon)\mathcal{G}_\kappa$ has no contribution to entropy production, the martingale component and the numerical error both vanish as $\varepsilon\to 0$. Finally, the viscous perturbation $\sigma(\varepsilon)\mathcal{S}$ is responsible for a preliminary version of the entropy condition (1.8): $\limsup_{\varepsilon\to 0} X_\varepsilon(\psi)\le 0$ in probability whenever $0\le\psi\in C_{co}^2(\mathbb{R}_+^2)$ and h is convex. Remember that so far we have considered the empirical process $\hat\rho_\varepsilon$ in $(0,+\infty)\times\mathbb{R}$ only, the question of initial values is a different issue.

2.3. *Measure-valued solutions.* The notion of Young measure DiPerna (1983, 1985) is a most convenient tool for the description of all limit distributions of our empirical process $\hat\rho_\varepsilon$. Let Θ denote the set of measurable families θ of probability measures; $\theta=\{\theta_{t,x}(d\rho)\}$ such that $\theta_{t,x}$ is a probability measure on $[0,1]$ for each $(t,x)\in\mathbb{R}_+^2$, and $\theta_{t,x}(h)$ is a measurable function of (t,x) if $h:[0,1]\mapsto\mathbb{R}$ is measurable. The abbreviation $\theta_{t,x}(h)$ for expectation of a function h with respect to $\theta_{t,x}$ will frequently be used also later on. We say that $\theta\in\Theta$ is a measure solution to the macroscopic equation $\partial_t\rho+\partial_x f(\rho)=0$ with initial value ρ_0 if

$$\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \theta_{t,x}(d\rho)\big(\rho\psi_t' + f(\rho)\psi_x'\big)\,dx\,dt + \int_{-\infty}^\infty \psi(0,x)\rho_0(x)\,dx = 0 \qquad (2.9)$$

for all $\psi\in C_c^1(\mathbb{R}^2)$. A measurable function $u:\mathbb{R}_+^2\mapsto[0,1]$ is represented by a family $\theta\in\Theta$ of *Dirac measures* such that $\theta_{t,x}$ is concentrated on the actual value $u(t,x)$ of u ; this θ is called the Young representation of u . Therefore any weak solution is a measure solution.

On the other hand, any $\theta\in\Theta$ can be identified as a locally finite measure m_θ by $dm_\theta := dt\,dx\,\theta_{t,x}(du)$ on $\mathbb{X}:=\mathbb{R}_+\times\mathbb{R}\times[0,1]$; let $M_\theta(\mathbb{X})$ denote the set of such measures m_θ equipped with the associated weak topology. In view of the Young representation, our empirical process $\hat\rho_\varepsilon$ can be considered as a random element $\hat m_{\theta,\varepsilon}$ of $M_\theta(\mathbb{X})$; let $\hat P_{\theta,\varepsilon}$ denote its distribution. This family is tight because $\hat\rho_\varepsilon$ is bounded, and as we shall see, any of its limit distributions $\hat{P}_\theta$ is concentrated on a set of measure solutions. The uniqueness of measure solutions is more problematic.

The probabilistic evaluation of entropy production yields the Lax inequality in a very weak form, as follows. With probability one with respect to any limit distribution of $\hat{P}_{\theta,\varepsilon}$ on $M_\theta(\mathbb{X})$ as $\varepsilon \to 0$, we have

$$\int_0^\infty \int_{-\infty}^\infty \left(\theta_{t,x}(h)\psi_t'(t,x) + \theta_{t,x}(J)\psi_x'(t,x)\right) dx\, dt \geq 0 \tag{2.10}$$

whenever (h,J) is a convex entropy pair and $0 \leq \psi \in C_c^1(\mathbb{R}_+^2)$.

Unfortunately, the initial condition (1.10) does not allow us to include the initial entropy into (2.10) when h is not linear, while the uniqueness result of Gallouët and Herbin (1993) requires

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty \left(\theta_{t,x}(h)\psi_t'(t,x) + \theta_{t,x}(J)\psi_x'(t,x)\right) dx\, dt \\ + \int_{-\infty}^\infty \psi(0,x)h(\rho_0(x))\, dx \geq 0\end{aligned} \tag{2.11}$$

for all convex entropies and $0 \leq \psi \in C_c^1(\mathbb{R}^2)$. For our purposes we ought to have (2.11) with probability one with respect to any limit distribution $\hat{P}_\theta$ of the Young representation of the empirical process $\hat{\rho}_\varepsilon$, which can only be derived from the strong initial condition

$$\lim_{\varepsilon \to 0} \int_{-r}^r |\hat{\rho}_\varepsilon(0,x) - \rho_0(x)|\, dx = 0 \quad \text{for all } r > 1 \tag{2.12}$$

in probability. Although (2.12) is not so bad in the case of stochastic models because we could assume that the initial distributions are well adjusted product measures, cf. Rezakhanlou (1991), we prefer the weaker condition (1.10) and compensated compactness. Indeed, (2.12) does not really simplify the proof, most tools of the next sections are needed in both cases. Moreover, in this setting we get a sharp result: strong convergence of the empirical process $\hat{\rho}_\varepsilon$ to the unique entropy solution.

2.4. *Compensated compactness.* This technique is used to show that measure solutions are, in fact weak solutions. In view of the stochastic version Fritz (2001, 2004); Fritz and Tóth (2004) of the Tartar–Murat theory of compensated compactness, we have to find a decomposition $X_\varepsilon = Y_\varepsilon + Z_\varepsilon$, and some random functionals $A_\varepsilon(\phi)$, $B_\varepsilon(\phi)$ such that $A_\varepsilon(\phi)$, $B_\varepsilon(\phi)$ do not depend on ψ, moreover

$$|Y_\varepsilon(\phi\psi, h)| \leq A_\varepsilon(\phi)\|\psi\|_{+1} \quad \text{and} \quad \lim_{\varepsilon \to 0} \mathsf{E} A_\varepsilon(\phi) = 0\,, \tag{2.13}$$

$$|Z_\varepsilon(\phi\psi, h)| \leq B_\varepsilon(\phi)\|\psi\| \quad \text{and} \quad \limsup_{\varepsilon \to 0} \mathsf{E} B_\varepsilon(\phi) < +\infty \tag{2.14}$$

for each $\psi \in C_c^1(\mathbb{R}^2)$ and $\phi \in C_{co}^2(\mathbb{R}_+^2)$, where $\|\psi\|$ is the uniform and $\|\psi\|_2$ is the $L^2(\mathbb{R}^2)$ norm of ψ, finally $\|\psi\|_{+1}$ is the H^{+1} norm, $\|\psi\|_{+1}^2 := \|\psi\|_2^2 + \|\psi_t'\|_2^2 + \|\psi_x'\|_2^2$. Let H^{-1} denote the dual of H^{+1} with respect to $L^2(\mathbb{R}^2)$; $\|\cdot\|_{-1}$ is its norm. (2.13) means that $Y_\varepsilon \to 0$ strongly in H_{loc}^{-1}, while (2.14) implies that Z_ε is locally bounded in the space of signed measures. More precisely, conditions (2.13) and (2.14) imply tightness of the underlying probability distributions, therefore the Skorohod embedding theorem allows us to realize the process on a huge probability space in such a way that convergence of all processes that are involved in the argument, holds true with probability one. Therefore the celebrated Div-Curl lemma (Murat, 1978; Tartar, 1979; Serre, 2000; Dafermos, 2000) of L. Tartar and F. Murat applies; in fact we obtain the following statement, see Proposition 2.1. in Fritz (2004) or Proposition 2. in Fritz and Tóth (2004) for more details. For any couple $(h_1, J_1), (h_2, J_2)$ of entropy pairs we have

$$\theta_{t,x}(h_1 J_2) - \theta_{t,x}(h_2 J_1) = \theta_{t,x}(h_1)\theta_{t,x}(J_2) - \theta_{t,x}(h_2)\theta_{t,x}(J_1) \tag{2.15}$$

for almost every $(t, x) \in \mathbb{R}_+^2$; this factorization property is valid with probability one with respect to any limit distribution, $\hat{P}_\theta$ of the Young representation of $\hat{\rho}_\varepsilon$.

Our main task now is the verification of conditions (2.13) and (2.14) above. Most terms on the right hand side of (2.5) will be split into further ones, and we shall show in Section 4 that each of them satisfies (2.13) or (2.14). The derivation of the weak entropy condition (2.10) for measure solutions goes in much the same way; but under the weak initial condition (1.10) we can not control the space integral of ψh at time zero.

The Div-Curl lemma implies the Dirac property of the Young measure in several situations Tartar (1979); DiPerna (1983, 1985); Dafermos (2000); Serre (2000). Since we are considering a single conservation law, to prove convergence of the empirical process $\hat{\rho}_\varepsilon$ to a set of weak solutions, it is sufficient to apply (2.15) to two entropy pairs only, say to the trivial $h_1(\rho) := \rho$, $J_1(\rho) := \kappa\rho(1-\rho)$, and to $h_2(\rho) := \rho^2$ with the associated $J_2(\rho) := \kappa\rho^2 - (4\kappa/3)\rho^3$. Therefore, as a direct consequence we also have the weak entropy condition for weak solutions:

$$\int_0^\infty \int_{-\infty}^\infty \big(h(\rho)\psi_t'(t,x) + J(\rho)\psi_x'(t,x)\big)\,dx\,dt \geq 0 \tag{2.16}$$

for all convex entropy pairs (h, J) and nonnegative $\psi \in C_c^1(\mathbb{R}_+^2)$ with probability one with respect to limit distributions $\hat{P}_\theta$ of the Young representation of $\hat{\rho}_\varepsilon$. The final step of the probabilistic argumentation is to conclude strong convergence of the empirical process in the $L^2_{\mathrm{loc}}(\mathbb{R}_+^2)$ sense. Hence (2.5) implies (1.7), thus we can refer to some advanced results on a single conservation law in one or more space dimensions Kružkov (1970); Bressan (2000); Chen and Rascle (2000); Vasseur (2001); Panov (1994); De Lellis et al. (2004) concerning uniqueness of weak entropy solutions.

2.5. *Entropy and LSI.* Probabilistic estimates of the proofs are mainly based on relative entropy of the evolved measure with respect to equilibrium, and on the associated Dirichlet form; these ideas go back to Guo et al. (1988), see also Varadhan (1993) and Yau (1997). Since we are going to treat hydrodynamic limit in infinite volume, we have to control entropy flux by means of its rate of production, see Fritz (1990, 2004) for a bit simpler problems. This a priori bound allows us to apply the robust logarithmic Sobolev inequality due to stirring $\sigma\mathcal{S}$, and also the second, Glauberian one. LSI based estimates result then in a replacement of the microscopic flux $\hat{\mathbf{j}}_{l,k}$ of $\hat{\eta}_{l,k}$ with the empirical estimator of its equilibrium expectation, see Lemma 3.2 and Lemma 3.5. The conditions (2.13) and (2.14) of the Div-Curl Lemma are verified in a similar way.

Most technical details of this estimation procedure have been elaborated in Fritz (2001, 2004); Fritz and Tóth (2004); our basic reference is Fritz and Tóth (2004). First we substitute the microscopic time derivative $\mathcal{L}_o h(\hat{\rho}_\varepsilon)$ with the spatial gradient of a *mesoscopic flux* depending on block averages $\bar{\eta}_{l,k}$ and $\bar{\omega}_{l,k}$ of the conserved quantities of the original, two-component model. The replacement of $\bar{\omega}_{l,k}$ by the empirical estimator $\kappa\,\bar{\eta}_{k,l}$ of its equilibrium expectation is based on a second LSI, it is related to the spin-flip component $\mathcal{G}_\kappa$. Although the spin-flip dynamics is not ergodic because empty sites are not affected, its Dirichlet form controls the distribution of $\bar{\omega}_{l,k}$ quite well, see Lemma 3.5. Since this step uses the explicit relation between block averages $\bar{\xi}$ and $\hat{\xi}$, which is not present in Fritz and Tóth (2004), for Reader's convenience we reproduce the main steps in terms of the present system of notations, elementary proofs of some known facts are also added.

3. Entropy, Dirichlet form and LSI

In this section we derive some fundamental estimates based on entropy and the associated Dirichlet forms. The parameters α, σ and l are almost arbitrary here, their dependence on the scaling parameter $\varepsilon > 0$ is not important. We only need $\alpha > 0$, $\sigma \geq 1$ and $l \in \mathbb{N}$.

3.1. *Entropy and its temporal derivative.* If μ and λ are probability measures on the same space, then entropy of μ relative to λ is defined by $S[\mu|\lambda] := \mu(\log f)$ if $\mu \ll \lambda$ and $f := d\mu/d\lambda$; as before, $\mu(\varphi) \equiv \mathsf{E}_\mu\varphi$ abbreviates expectation with respect to μ. A frequently used entropy inequality, $\mu(\varphi) \leq S[\mu|\lambda] + \log\lambda(e^\varphi)$ follows immediately by convexity. Since equality holds true if $\varphi = \log f$, we have another definition of relative entropy:

$$S[\mu|\lambda] := \sup_\varphi \{\mu(\varphi) - \log\lambda(e^\varphi) : \lambda(e^\varphi) < +\infty\}\ . \tag{3.1}$$

From $f\log(1/f) = 2f\log(f^{-1/2}) \leq 2f(f^{-1/2} - 1) = 2\sqrt{f} - 2f$ it follows that $\mathsf{E}_\lambda(\sqrt{f} - 1)^2 \leq S[\mu|\lambda]$.

Given a Markov generator $\mathcal{A}$, the Donsker-Varadhan rate function of large deviations is defined as

$$D[\mu|\mathcal{A}] := -\inf\left\{\int \frac{\mathcal{A}\psi}{\psi}\,d\mu\ : 0 < \psi \in \mathrm{Dom}(\mathcal{A})\right\}\ ; \tag{3.2}$$

$D[\mu|\mathcal{A}] = D(\sqrt{f})$ if $\mathcal{A}$ is self-adjoint in $L^2(\lambda)$ and $f = d\mu/d\lambda$, where $D(\varphi) := -\lambda(\varphi\mathcal{A}\varphi)$ is the Dirichlet form of $\mathcal{A}$. Remember that in view of their variational characterizations (3.1) and (3.2), both S and D are convex functionals of μ, and the definitions and relations above extend to conditional distributions and densities, too.

As a reference measure we can choose any of the equilibrium product measures $\lambda = \lambda_\rho$ with $0 < \rho < 1$ fixed, say $\rho = 1/2$. The evolved measure of the process is denoted by $\mu_{\varepsilon,t}$, and $\mu_{\varepsilon,t,n}$ is the restriction of $\mu_{\varepsilon,t}$ to $\mathcal{F}_n := \sigma\{\omega_k : k \in \Lambda^n\}$, $\Lambda^n := [-n,n] \cap \mathbb{Z}$. The sequence of local densities $f_n = f_{\varepsilon,t,n}$ is defined by $d\mu_{\varepsilon,t,n} = f_{\varepsilon,t,n}\,d\lambda$ such that $f_n : \Omega \mapsto \mathbb{R}_+$ is a martingale adapted to $\mathcal{F}_n$, i.e. $f_n = \mathsf{E}_\lambda(f_{n+1}|\mathcal{F}_n)$ λ-a.s. Entropy in the box Λ^n is defined as

$$S_n(t) := S[\mu_{\varepsilon,t,n}|\lambda] = \int \log f_{\varepsilon,t,n}\,d\mu_{\varepsilon,t}\ ,$$

and local versions of the Dirichlet form for $\mathcal{L}_o$, $\mathcal{S}$ and $\mathcal{G}_\kappa$ at $\varphi = \sqrt{f_{\varepsilon,t,n}}$ read as

$$D_n^o(t) := \frac{1}{2}\sum_{b\subset\Lambda^n}\int c_b(\omega)\,\left(\sqrt{f_{\varepsilon,t,n}(\omega^b)} - \sqrt{f_{\varepsilon,t,n}(\omega)}\right)^2\,\lambda(d\omega)\,, \tag{3.3}$$

$$D_n(t) := \frac{1}{2} \sum_{b \subset \Lambda^n} \int \left(\sqrt{f_{\varepsilon,t,n}(\omega^b)} - \sqrt{f_{\varepsilon,t,n}(\omega)} \right)^2 \lambda(d\omega) \,, \tag{3.4}$$

$$D_n^g(t) := \frac{1}{2} \sum_{k \in \Lambda^n} \int c_k(\omega) \left(\sqrt{f_{\varepsilon,t,n}(\omega^k)} - \sqrt{f_{\varepsilon,t,n}(\omega)} \right)^2 \lambda(d\omega) \,, \tag{3.5}$$

respectively. By convexity, both S_n and D_n are nondecreasing sequences. Moreover, $S_{n+1} = S_n + S_{n+1|n}$, where $S_{n+1|n} := E_\mu S[\mu_{n+1|n}|\lambda]$ is the conditional entropy given $\mathcal{F}_n$, i.e. $\mu_{n+1|n}$ denotes the conditional distribution of μ on $\mathcal{F}_{n+1}$ with respect to $\mathcal{F}_n$.

3.2. *Entropy flux.* Our basic a priori bound on local entropy and Dirichlet forms is the content of

Lemma 3.1. *If $\sigma \geq 1$ then we have a constant C_0 depending only on λ such that*

$$S_n(t) + 2\alpha \int_0^t D_n^g(\tau)\, d\tau + \sigma \int_0^t D_n(\tau)\, d\tau \leq C_0 \left(t + \sqrt{n^2 + \sigma t} \right)$$

for any initial distribution $\mu_{\varepsilon,0}$, $n \in \mathbb{N}$ and $t > 0$.

Proof. We follow the argument of Fritz (1990, 2004), our starting point is the Kolmogorov equation for the temporal derivative of entropy:

$$\begin{aligned}
\partial_t S_n &= \int (\partial_t + \mathcal{L}) \log f_{n,t}(\omega)\, \mu_t(d\omega) = \int f_{n+1}\, \mathcal{L} \log f_n\, \lambda(d\omega) \\
&= \sum_{k \in \mathbb{Z}} \int c_k f_n \log \frac{f_n^k}{f_n}\, \lambda(d\omega) + \sum_{b \in \mathbb{Z}_*} \int (c_b + \sigma) f_{n+1} \log \frac{f_n^b}{f_n}\, \lambda(d\omega) \,,
\end{aligned}$$

where abbreviations as $\varphi^b := \varphi(\omega^b)$ and $\varphi^k := \varphi(\omega^k)$ are used, and where it is not necessary, the dependence of densities on ε and t is omitted. Since $y \log(x/y) = 2y \log \sqrt{x/y} \leq 2\sqrt{y}\,(\sqrt{x} - \sqrt{y}) = x - y - (\sqrt{x} - \sqrt{y})^2$, we have

$$\begin{aligned}
f_{n+1} \log \frac{f_n^b}{f_n} &\leq \frac{f_{n+1}}{f_n} \left(f_n^b - f_n - \left(\sqrt{f_n^b} - \sqrt{f_n} \right)^2 \right) \\
&= f_n^b - f_n - \left(\sqrt{f_n^b} - \sqrt{f_n} \right)^2 + 2\Phi_{n,b}(t) \,,
\end{aligned}$$

where

$$\Phi_{n,b}(t) := \left(\frac{f_{n+1}}{f_n} - 1 \right) \left(\sqrt{f_n^b} - \sqrt{f_n} \right) \sqrt{f_n} \,.$$

The expectation of the sum of $c_b(f_n^b - f_n)$ vanishes by stationarity of λ with respect to $\mathcal{L}_0$, while $c_b \Phi_{n,b}$ has zero expectation with respect to λ unless $b \subset \partial\Lambda^n := \{-n-1, -n, n, n+1\}$. Finally, $1 \leq c_b \leq 2$ if $\omega^b \neq \omega$, consequently

$$\partial_t S_n(t) + 2\alpha D_n^g(t) + 2\sigma D_n(t) \leq B_n^o(t) + \sigma B_n(t) \,, \tag{3.6}$$

where

$$B_n^o(t) := -D_{\partial n}(t) + 4 \sum_{b \subset \partial\Lambda^n} \int |\Phi_{n,b}(t)|\, d\lambda \,,$$

$$D_{\partial n}(t) := \sum_{b \subset \partial\Lambda^n} \int \left(\sqrt{f_n^b} - \sqrt{f_n} \right)^2 d\lambda \,,$$

$$B_n(t) := \sum_{b\subset\partial\Lambda^n}\int f_{n+1}\log\frac{f_n^b}{f_n}\,d\lambda\,.$$

Observe now that we have a universal constant K_0 depending only on λ such that

$$\begin{aligned} B_n^o(t) &\le -D_{\partial n}(t) + 4K_0\sqrt{S_{n+1}(t)-S_n(t)}\,\sqrt{D_{\partial n}(t)} \\ &\le 4K_0^2\,(S_{n+1}(t)-S_n(t))\,. \end{aligned} \tag{3.7}$$

Indeed, $f_{n+1}=O(f_n)$ because $f_n=\mathsf{E}_\lambda[f_{n+1}|\mathcal{F}_n]$, moreover

$$\tfrac{f_{n+1}}{f_n}-1=\left(\sqrt{\tfrac{f_{n+1}}{f_n}}-1\right)\left(\sqrt{\tfrac{f_{n+1}}{f_n}}+1\right),$$

and from $\mathsf{E}_\lambda(\sqrt{f}-1)^2\le S[\mu|\lambda]$ for the conditional density f_{n+1}/f_n we get

$$\mathsf{E}_\lambda(\sqrt{f_{n+1}}-\sqrt{f_n})^2\le S_{n+1|n}=S_{n+1}-S_n\,.$$

Therefore supposing $1+\sqrt{f_{n+1}/f_n}\le K_0$, we obtain (3.7) by the Schwarz inequality. In the case of B_n we have $\lambda(f_n^b)=\lambda(f_n)=1$ for all $b\in\mathbb{Z}_*$, thus repeating the argument above, we have

$$\begin{aligned} B_n(t) &\le -D_{\partial n}(t)+K_0\sqrt{S_{n+1}(t)-S_n(t)}\,\sqrt{D_{\partial n}(t)} \\ &\le (K_0^2/4)\,(S_{n+1}(t)-S_n(t))\,. \end{aligned} \tag{3.8}$$

To complete the proof, we have to derive the following system of differential inequalities:

$$\begin{aligned} \partial_t S_n(t)+2\alpha D_n^g(t)+2\sigma D_n(t) &\le K_1\,(S_{n+1}(t)-S_n(t)) \\ &+\sigma K_1\sqrt{S_{n+1}(t)-S_n(t)}\,\sqrt{D_{n+1}(t)-D_n(t)}\,, \end{aligned} \tag{3.9}$$

where $\sigma\ge 1$ may be assumed, thus K_1 depends only on λ. This system admits an explicit solution implying the statement. Indeed, with $u_n:=S_n$ and $v_n:=D_n$ in Lemma 3 of Fritz (1990) we get

$$S_n(t)+2\alpha\int_0^t D_n^g(\tau)\,d\tau+\sigma\int_0^t D_n(\tau)\,d\tau\le\frac{M}{R}\sum_{m=0}^{\infty}\exp(-m/R)\,S_m(0)\,,$$

where $R:=Mt+(n^2+\sigma t)^{1/2}$, and M depends only on K_1. Since $S_n(0)=O(n)$, it is really sufficient to verify (3.9).

Unfortunately, B_n has a big factor σ, thus its trivial bound $S_{n+1}-S_n$ should be replaced with a better one. To get another inequality, observe that

$$B_n=\frac{1}{2}\sum_{b\subset\partial\Lambda^n}\int(f_{n+1}-f_{n+1}^b)\log(f_n^b/f_n)\,d\lambda$$

and

$$(f_{n+1}-f_{n+1}^b)\log\frac{f_n^b}{f_n}=\frac{2}{\alpha}\,(\sqrt{f_{n+1}}+\sqrt{f_{n+1}^b})(\sqrt{f_{n+1}}-\sqrt{f_{n+1}^b})(\sqrt{f_n^b}-\sqrt{f_n})\,,$$

where α is a number between $\sqrt{f_n}$ and $\sqrt{f_n^b}$. There is nothing to do if the left hand side is negative, thus either $f_n^b>f_n$ and $f_{n+1}>f_{n+1}^b$, or $f_n^b<f_n$ and $f_{n+1}<f_{n+1}^b$ may be assumed. Using $f_{n+1}=O(f_n)$ or $f_{n+1}^b=O(f_n^b)$ we get

$$(f_{n+1}-f_{n+1}^b)\log\frac{f_n^b}{f_n}\le K_2\left|\sqrt{f_{n+1}}-\sqrt{f_{n+1}^b}\right|\left|\sqrt{f_n^b}-\sqrt{f_n}\right|,$$

whence by the Schwarz inequality and convexity of D we get a second bound:

$$B_n(t) \leq K_2 \sqrt{D_{n+1}(t) - D_n(t)}\,\sqrt{D_{\partial n}(t)}\,. \tag{3.10}$$

Finally, if $B_n > 0$ then (3.8) implies $D_{\partial n} \leq (K_0^2/4)(S_{n+1} - S_n)$, thus (3.10) results in (3.9), which completes the proof. □

This lemma is the a priori bound we need to materialize hydrodynamic limit in infinite volume, it obviously applies also to the original model of Fritz and Tóth (2004). From now on we are assuming that $\sigma \geq 1$. This lemma shall be used when $t \approx \tau/\varepsilon$ and $n \approx r/\varepsilon$, $\tau, r \geq 1$, then $(r+\tau)/\varepsilon$ is the order of the bound.

3.3. *The first LSI.* The first replacement lemma for microscopic currents is based on the logarithmic Sobolev inequality Fritz and Tóth (2004) for stirring $\mathcal{S}$. Let $\lambda_{\rho,u}$ denote the product measure on Ω such that $\lambda_{\rho,u}(\eta_k) = \rho$ and $\lambda_{\rho,u}(\omega_k) = u$ for all $k \in \mathbb{Z}$, then $\lambda_{\rho,u}(\mathbf{j}_k^o) = \mathfrak{J}(\rho,u) := u - u\rho$.

Lemma 3.2. *There exists a universal constant C_1 such that*

$$\varepsilon^2 \sum_{|k|<r/\varepsilon} \int_0^{\tau/\varepsilon} \int \left(\bar{\mathbf{j}}_{l,k}^o - \mathfrak{J}(\bar{\eta}_{l,k}, \bar{\omega}_{l,k})\right)^2 d\mu_{\varepsilon,t}\,dt \leq C_1 \left(\frac{(r+\tau)\varepsilon l^2}{\sigma} + \frac{r\tau}{l}\right)$$

for all initial distributions $\mu_{\varepsilon,0}$, and $r, \tau > 1$.

Proof. In view of Lemma 3.1, this is essentially the first inequality of Proposition 1 in Fritz and Tóth (2004), for Reader's convenience we indicate the main steps of the argument. Let $\bar{\lambda}_{l,k}^{\rho,u}$ denote the canonical measure defined by $\bar{\lambda}_{l,k}^{\rho,u}(\varphi) = \mathsf{E}_\lambda[\varphi|\bar{\eta}_{l,k} = \rho, \bar{\omega}_{l,k} = u]$; it is just the uniform distribution on the subspace of codimension two specified by the conditions. In view of Proposition 4 of Fritz and Tóth (2004), we have a universal number $\aleph$ such that

$$S[\nu|\bar{\lambda}_{l,k}^{\rho,u}] \leq \aleph\, l^2 \sum_{b\subset[k,k+l-1]} \int \left(\sqrt{f^b} - \sqrt{f}\right)^2 d\bar{\lambda}_{l,k}^{\rho,u} \tag{3.11}$$

whenever $d\nu = f d\bar{\lambda}_{l,k}^{\rho,u}$. It is important that $\aleph$ does not depend on ρ and u. Consider now a function φ_k of $\omega_k^l := (\omega_k, \omega_{k+1}, ..., \omega_{k+l-1})$, then for $\beta > 0$ by the entropy inequality

$$\mu_{\varepsilon,t}(\varphi_k^2) \leq \frac{1}{\beta} \int S[\bar{\mu}_{t,l,k}^{\rho,u}|\bar{\lambda}_{l,k}^{\rho,u}]\,\tilde{\mu}_{\varepsilon,t}(d\rho,du) + \frac{1}{\beta}\int \log \bar{\lambda}_{l,k}^{\rho,u}(e^{\beta\varphi_k^2})\,\tilde{\mu}_{\varepsilon,t}(d\rho,du)\,,$$

where $\bar{\mu}_{t,l,k}^{\rho,u}$ is the conditional measure of $\mu_{\varepsilon,t}$ given $\bar{\eta}_{l,k} = \rho$ and $\bar{\omega}_{l,k} = u$, while $\tilde{\mu}_{\varepsilon,t}$ is the joint distribution of $\bar{\eta}_{l,k}$ and $\bar{\omega}_{l,k}$ under $\mu_{\varepsilon,t}$. The first term on the right hand side is estimated via (3.11), while Lemma 3.1 yields a bound for the Dirichlet form. Since D is convex, and the exchanges do not alter $\bar{\eta}_{l,k}$ or $\bar{\omega}_{l,k}$, we obtain a bound $O(r\varepsilon l^3/\beta\sigma)$ for this part of the sum.

The canonical exponential moment of the second term can be handled by means of the slightly sophisticated Lemma 9. of Fritz and Tóth (2004), see also Tóth and Valkó (2003) for the original proof. Here we present a nice, elementary argument. In general, the local version of the central limit theorem implies that if $0 < \ell < l$ and ℓ/l is bounded away from one ($\ell \leq (l+1)/2$, say), then canonical probabilities related to the interval $[k, k+\ell)$ are bounded by the corresponding grand canonical probabilities, which makes life easier. More precisely,

$$\bar{\lambda}_{l,k}^{\rho,u}[\omega_k^\ell = y_k^\ell] = O\left(\lambda_{\rho,u}[\omega_k^\ell = y_k^\ell]\right) \tag{3.12}$$

with a uniform bound. To prove this, for arbitrary nonnegative integers m_0, m_+ and m_- set

$$q(m_0,m_+,m_-) := \frac{(m_0+m_++m_-)!}{m_0!\,m_+!\,m_-!}\,\left(\frac{n_0}{l}\right)^{m_0}\,\left(\frac{n_+}{l}\right)^{m_+}\,\left(\frac{n_-}{l}\right)^{m_-}\,,$$

where n_0, n_+, n_- are the frequencies of 0 and ±1 in the sequence ω_k^l, i.e. $n_+ = l(\rho+u)/2$, $n_- = l(\rho-u)/2$ and $n_0 = l(1-\rho)$ with respect to the canonical measure $\bar\lambda_{l,k}^{\rho,u}$. Therefore if m_0, m_+ and m_- are the corresponding frequencies in the complementary sequence $\omega_{k+\ell}$, $\omega_{k+\ell+1}$, $\cdots$, ω_{k+l-1} of ω_k^ℓ, then we have

$$Q := \frac{\bar\lambda_{l,k}^{\rho,u}[\omega_k^\ell = y_k^\ell]}{\lambda_{\rho,u}[\omega_k^\ell = y_k^\ell]} = \frac{q(m_0,m_+,m_-)}{q(n_0,n_+,n_-)}\,,$$

whence

$$Q = \frac{(l-\ell)!\,n_0!\,n_+!\,n_-!}{l!\,m_0!\,m_+!\,m_-!}\,\left(\frac{n_0}{l}\right)^{m_0-n_0}\,\left(\frac{n_+}{l}\right)^{m_+-n_+}\,\left(\frac{n_-}{l}\right)^{m_--n_-}\,.$$

We may and do assume that $m_0m_+m_- > 0$, then by Stirling's formula and a direct calculation we obtain that Q is bounded by a constant multiple of

$$\tilde Q := \left(\tfrac{l-\ell}{l}\right)^{l-\ell+1/2}\,\left(\tfrac{n_0}{m_0}\right)^{m_0+1/2}\,\left(\tfrac{n_+}{m_+}\right)^{m_++1/2}\,\left(\tfrac{n_-}{m_-}\right)^{m_-+1/2}\,.$$

Since $\tilde Q = \bar Q R$, where

$$R := \left(\tfrac{m_0+1/2}{m_0}\right)^{m_0+1/2}\,\left(\tfrac{m_++1/2}{m_+}\right)^{m_++1/2}\,\left(\tfrac{m_-+1/2}{m_-}\right)^{m_-+1/2} \le (2e)^{3/2}\,,$$

we have to estimate

$$\bar Q := \left(\tfrac{l-\ell}{l}\right)^{l-\ell+1/2}\,\left(\tfrac{n_0}{m_0+1/2}\right)^{m_0+1/2}\,\left(\tfrac{n_+}{m_++1/2}\right)^{m_++1/2}\,\left(\tfrac{n_-}{m_-+1/2}\right)^{m_-+1/2}\,.$$

The inequality of geometric and arithmetic means implies

$$\bar Q \le \left(\tfrac{l-\ell}{l}\right)^{l-\ell+1/2}\left(\tfrac{l}{l-\ell+3/2}\right)^{l-\ell+3/2} \le \frac{l}{l-\ell}\,,$$

which completes the proof of (3.12).

Let $\varphi_k := \bar{\mathbf{j}}_{l,k}^o - u + u\rho$ with $\rho = \bar\eta_{l,k}$ and $u = \bar\omega_{l,k}$; the central limit theorem suggests that expectation of $\exp(\beta\varphi_k^2)$ is bounded if β/l is small enough. To do the calculation, we write $\exp(\beta\varphi_k^2) = \mathsf{E}_\theta\exp(\theta\varphi_k\sqrt{2\beta})$, where θ is a standard Gaussian variable, and E_θ denotes expectation with respect to its distribution. In this way the second bound has been reduced to a usual large deviation estimate. Indeed, let

$$F(z,\rho,u) := \log\int\exp(z\mathbf{j}_i^o - zu + zu\rho)\,\lambda_{\rho,u}(d\omega)\,,$$

then $F(0,\rho,u) = F_z'(0,\rho,u) = 0$ and $F_{zz}''(z,\rho,u) \le 4$ because it is the variance of a variable bounded by 2. Therefore

$$\int\exp(z\mathbf{j}_i^o - zu + zu\rho)\,\lambda_{\rho,u}(d\omega) \le \exp(2z^2)\,, \tag{3.13}$$

where $z := \sqrt{2\beta}(\theta/l)$ is the right choice.

To apply (3.12), first we split $[k,k+l)$ into two almost equal parts and decompose φ_k in the same way. These two terms can be separated by means of the Schwarz inequality, thus (3.12) applies to each of them. Since the individual currents $\mathbf{j}_i^o$ are not independent with respect to $\lambda_{\rho,u}$, we separate odd and even indices i in the individual subintervals by Schwarz, and use (3.13). Choosing β as a small multiple of l, the proof is now completed by a direct calculation. We see that $1/l$ is the order of the second, large deviation part of the bound. □

This is a sharp form of the so called *one block lemma* of Guo, Papanicolau and Varadhan (Guo et al., 1988), the explicit rate due to LSI is needed for the evaluation of entropy production X. Notice that, in view of (2.2), $\varepsilon l^2/\sigma$ is the leading term of the bound. The following version of the *two blocks lemma* of Guo et al. (1988); Kipnis and Landim (1999) is a particular case of the second inequality of Proposition 1 in Fritz and Tóth (2004); it follows easily from LSI by (3.12) as Lemma 3.2 did. For two sequences A and ξ indexed by $\mathbb{Z}$ let $A*\xi$ denote their convolution, i.e. $(A*\xi)_k := \sum_j A_j\xi_{k-j}$.

Lemma 3.3. *Let $m\in\mathbb{N}$ and $A_j\in\mathbb{R}$ for $j\in\mathbb{Z}$ such that $A_j=0$ unless $0\le j\le m$, $\sum_j A_j = 0$ and $\sum A_j^2\le a/m$. Then we have*

$$\varepsilon^2\sum_{|k|<r/\varepsilon}\int_0^{\tau/\varepsilon}\int (A*\eta)_k^2\,d\mu_{\varepsilon,t}\,dt \le C_2\left(\frac{a(r+\tau)\varepsilon m^2}{\sigma}+\frac{ar\tau}{m}\right)$$

for all $\mu_{\varepsilon,0}$ and $r,\tau>1$ with some universal constant C_2.

Interesting special cases of $A*\eta$ are differences like $\bar\eta_{l,k+m}-\bar\eta_{l,k}$, in particular with $m=l$, $\hat\eta_{l,k}-\bar\eta_{l,k}$ and $\hat\eta_{l,k+l}-\hat\eta_{l,k}$. The representation $A*\eta$ of differences of block averages with some A such that $\sum A_j=0$ is convenient for the calculation of the large deviation part of the bound. However, LSI is not optimal for the comparison of remote blocks. In that case we rewrite $A*\eta$ as $A*\eta = B*\nabla_1\eta$, where $\nabla_1\eta_j := \eta_{j+1}-\eta_j$, i.e. $(A*\eta)_k=\sum_j B_{k-j}(\eta_{j+1}-\eta_j)$, $A=\nabla_1 B$; and do some direct calculations in terms of the Dirichlet form, see Lemma IP and Lemma 2B in Fritz (2001).

Lemma 3.4. *Let $0<m<r/\varepsilon$ and $B_j\in\mathbb{R}$ for $j\in\mathbb{Z}$ such that $B_j=0$ unless $0\le j\le m$, then*

$$\varepsilon^2\sum_{|k|<r/\varepsilon}\int_0^{\tau/\varepsilon}\int (B*\nabla_1\eta)_k^2\,d\mu_{\varepsilon,t}\,dt \le C_2'\left(r\tau\|B\Delta_1B\|_1+\frac{r\varepsilon+\tau\varepsilon}{\sigma}\|B\|_1^2\right)$$

for all $\mu_{\varepsilon,0}$ and $r,\tau>1$, $\varepsilon\le 1$ with some universal constant C_2', where $\Delta_1B_j := B_{j+1}+B_{j-1}-2B_j$, $\|B\|_1 := \sum_j|B_j|$, $\|B\Delta_1B\|_1 := \sum_j|B_j\Delta_1B_j|$.

Proof. Let $b\in\mathbb{Z}_*$ denote the bond $b=(j,j+1)$ and set $\phi_k := (B*\nabla_1\eta)_k$ for brevity. Since our reference measure λ is exchangeable, for any measure μ on Ω with $d\mu = f\,d\lambda$ we have

$$\begin{aligned}\int(\eta_{j+1}-\eta_j)\phi_k\,d\mu &= \int\eta_j(\phi_k^b f^b-\phi_k f)\,d\lambda\\ &= \int\eta_j(\phi_k^b-\phi_k)\,d\mu+\int\eta_j\phi_k^b(f^b-f)\,d\lambda\end{aligned}$$

As before, we write $f^b-f=(\sqrt{f^b}+\sqrt{f})(\sqrt{f^b}-\sqrt{f})$, moreover

$$\int\eta_j\phi_k^b\sqrt{f^b}(\sqrt{f^b}-\sqrt{f})\,d\lambda = -\int\eta_{j+1}\phi_k\sqrt{f}(\sqrt{f^b}-\sqrt{f})\,d\lambda\,,$$

consequently

$$\int (\eta_{j+1}-\eta_j)\phi_k \, d\mu = \int \eta_j(\phi_k^b - \phi_k)\, d\mu$$
$$+ \int \Big(\eta_j(\phi_k^b-\phi_k) + (\eta_j-\eta_{j+1})\phi_k\Big)\sqrt{f}(\sqrt{f^b}-\sqrt{f})\, d\lambda\,.$$

On the other hand, as $\phi_k^2 = \phi_k \sum_j B_{k-j}(\eta_{j+1}-\eta_j)$, $1+D^{1/2} \le 2+D$ and $\phi_k^b - \phi_k = (\eta_j - \eta_{j+1})\Delta_1 B_{k-j}$, by a direct calculation using Schwarz we get

$$\sum_{|k|<r/\varepsilon} \mu(\phi_k^2) \le \sum_{|k|<r/\varepsilon}\sum_{j\in\mathbb{Z}} |B_{k-j}\Delta_1 B_{k-j}|(2+D[\mu|\mathcal{S}_j])$$
$$+ \sum_{|k|<r/\varepsilon}\sum_{j\in\mathbb{Z}} |B_{k-j}|\sqrt{\mu(\phi_k^2)D[\mu|\mathcal{S}_j]}\,,$$

where $\mathcal{S}_j\psi := \psi^b - \psi$ is the exchange operator across the bond $b=(j,j+1)$.

Now we are in a position to return to the original problem. Since the ℓ^2 norm of a convolution operator is bounded by the ℓ^1 norm of its kernel,

$$\Gamma := \sum_{|k|<r/\varepsilon} \int (B*\nabla_1\eta)_k^2 \, d\mu_{\varepsilon,t} \le \|B\Delta_1 B\|_1(2n + D_n(t)) + \|B\|_1\sqrt{\Gamma\, D_n(t)}\,,$$

whenever $n > m + 2r/\varepsilon$, whence

$$\sum_{|k|<r/\varepsilon} \int (B*\nabla_1\eta)_k^2 \, d\mu_{\varepsilon,t} \le 2\|B\Delta_1 B\|_1(2n+D_n(t)) + 2\|B\|_1^2 D_n(t) \tag{3.14}$$

because $y^2 \le a + by$ implies $y^2 \le 2a + 2b^2$, which completes the proof by Lemma 3.1. $\square$

Observe that $\sum_k A_k^2 \le \|B\Delta_1 B\|_1$, and $\|B\|_1 = O(m)$ if B is a bounded sequence, thus consequences of Lemma 3.3 and Lemma 3.4 are similar, although not identical. For instance, if $m = l$ then Lemma 3.4 yields the very same bound for $\phi_k = \bar\eta_{l,k+m} - \bar\eta_{l,k}$ that Lemma 3.3 does, but Lemma 3.4 is better when m is large, see Concluding Remarks at the end of the paper. Let us remark that without any change of the statement and its proof, we can replace the sequence η_k by ω_k ; this result is not needed here.

3.4. *The second LSI.* In order to replace $\bar\omega_{l,k}$ with its empirical estimator $\kappa\bar\eta_{l,k}$, a second LSI is needed, it is due to the Glauberian component $\mathcal{G}_\kappa$ of the evolution.

Lemma 3.5. *We have a universal constant C_3 such that if $r,\tau \ge 1$ then*

$$\varepsilon^2 \sum_{|k|<r/\varepsilon} \int_0^{\tau/\varepsilon} \int |\bar\omega_{l,k} - \kappa\,\bar\eta_{l,k}|^2 \, d\mu_{\varepsilon,t}\, dt \le C_3\left(\frac{r\varepsilon + \tau\varepsilon}{\alpha} + \frac{r\tau}{l}\right)$$

for all initial distributions $\mu_{\varepsilon,0}$, and $r,\tau > 1$.

Proof. As it was in the first part of this section, our reference measure is again $\lambda_\rho := \lambda_{\rho,u}$ with $u = \kappa\rho$, thus we have $p_0 := \lambda_\rho[\omega_k = 0] = 1-\rho$, $p_+ := \lambda_\rho[\omega_k = 1] = (\rho+\kappa\rho)/2$ and $p_- := \lambda_\rho[\omega_k = -1] = (\rho - \kappa\rho)/2$. We consider blocks of size l , the true distribution of ω_k^l will be denoted by μ_l . Observe now that the spin-flip dynamics of ω_k^l becomes ergodic if we fix the positions $\Gamma = \gamma \subset [k, k+l)$ of empty sites, let λ_l^γ and μ_l^γ denote the corresponding conditional distributions given $\Gamma = \gamma$, while ν_l is the distribution of the random set $\Gamma \subset [k, k+l)$. It is plain that λ_l^γ is a product measure such that

$$\lambda_+ := \lambda_l^\gamma[\omega_j = 1] = \frac{1+\kappa}{2} \quad \text{and} \quad \lambda_- := \lambda_l^\gamma[\omega_j = -1] = \frac{1-\kappa}{2}$$

if $j \notin \gamma$. On the other hand, if $f_+, f_- > 0$ and $\lambda_+ f_+ + \lambda_- f_- = 1$, then by convexity

$$\begin{aligned}
&\lambda_+ f_+ \log f_+ + \lambda_- f_- \log f_- \le 2\lambda_+ f_+(\sqrt{f_+} - 1) + 2\lambda_- f_-(\sqrt{f_-} - 1) \\
&\le 2\lambda_+(f_+ - 1)(\sqrt{f_+} - 1) + 2\lambda_-(f_- - 1)(\sqrt{f_-} - 1) \le \\
&\Big(2\lambda_+(\sqrt{f_+} + 1) + 2\lambda_-(\sqrt{f_-} + 1)\Big)\left(\sqrt{f_+} - \sqrt{f_-}\right)^2 \le 4\left(\sqrt{f_+} - \sqrt{f_-}\right)^2
\end{aligned}$$

because $f_+ > 1$ implies $f_- < 1$ and vice versa. Since the Dirichlet form of a single spin-flip is just $(1-\kappa^2)(\phi_+ - \phi_-)^2/2$, this means that we have LSI at single sites with a universal constant, whence

$$S(\mu_l^\gamma|\lambda_l^\gamma) \le \frac{8}{1-\kappa^2}\, D(\mu_l^\gamma|\mathcal{G}_\kappa) \tag{3.15}$$

because spin-flips at different sites are independent, and λ_l^γ is a product measure, see e.g. Lemma 2.2.11 in Saloff–Coste (1996) for the tensorization property of LSI. (3.15) is essentially the classical logarithmic Sobolev inequality of L. Gross (1976) for the binomial distribution.

From the entropy inequality for $\beta > 0$ by (3.15) we get

$$\mu(\varphi_k^2) \le \frac{8}{\beta - \beta\kappa^2}\int D(\mu_l^\gamma|\mathcal{G}_\kappa)\,\nu_l(d\gamma) + \frac{1}{\beta}\int \log \lambda_l^\gamma(e^{\beta\varphi_k^2})\,\nu_l(d\gamma)\,;$$

here $\varphi_k := \bar\omega_{l,k} - \kappa\bar\eta_{l,k}$. The Dirichlet form is estimated by means of Lemma 3.1, the contribution of these terms is $O(l\varepsilon/\alpha\beta)$. To calculate exponential moments, set again $\exp(\beta\varphi_k^2) = \mathsf{E}_\theta \exp(\theta\varphi_k\sqrt{2\beta})$, where θ is a standard Gaussian variable, and introduce $G(z) := \log \lambda_l^\gamma(e^{z(\omega_j - \kappa\eta_j)})$ with $j \in \gamma$. We have $G(0) = G'(0) = 0$ and $G''(z) \le 4$ as $|\omega_j - \kappa\eta_j| \le 2$, thus $G(z) \le 2z^2$, consequently

$$\lambda_l^\gamma(e^{\beta\varphi_k^2}) \le \mathsf{E}_\theta \exp(4n\beta l^{-2}\theta^2) = (1 - 8n\beta/l^2)^{-1/2}$$

if $8n\beta < l^2$, where $n \le l$ is the number of active sites. Specifying β as a small multiple of l, we obtain the statement. $\square$

Now we are in a position to replace the microscopic currents with the empirical estimator of their equilibrium expectations.

4. Estimation of entropy production

In this section the components L_ε, M_ε I_ε and N_ε of entropy production X_ε are evaluated, we are assuming that $h, J \in C^2(\mathbb{R})$. Several steps of the proofs have essentially been done in Fritz (2004); Fritz and Tóth (2004); Fritz and Tóth (2004) is our basic reference. Since there are some technical differences, for convenience we list the main points. Most calculations are done at the microscopic level, the following abbreviations reflect this picture. $F_k(t) := F(\hat\rho_\varepsilon(\varepsilon t, \varepsilon k)) = F(\hat\eta_{l,k}(t))$ is used for any function of the empirical process. The a priori bounds (2.13) and (2.14) we need for compensated compactness are localized by a smooth function

$\phi \in C^2_{co}(\mathbb{R}^2_+)$ of compact support. The integral mean of $\phi(\varepsilon t, x)\psi(\varepsilon t, x)$ over the space interval $|x - \varepsilon k| < \varepsilon/2$ will be denoted by $\psi_k(t)$,

$$\psi_k(t) := \frac{1}{\varepsilon} \int_{\varepsilon k - \varepsilon/2}^{\varepsilon k + \varepsilon/2} \varphi(\varepsilon t, x)\, dx\,,$$

where $\varphi(t,x) := \phi(t,x)\psi(t,x)$.

Finally, define $\nabla_\varepsilon \varphi(x) := \varepsilon^{-1}(\varphi(x+\varepsilon) - \varphi(x))$ for functions, in the case of sequences we write $\nabla_l \xi_k := l^{-1}(\xi_{k+l} - \xi_k)$, $\nabla_l^* \xi_k := l^{-1}(\xi_{k-l} - \xi_k)$, and $\Delta_l \xi_k := -\nabla_l^* \nabla_l \xi_k$. Note that ∇_l^* is the adjoint of ∇_l in $\ell^2(\mathbb{Z})$ and $\nabla_1 \hat{\xi}_{l,k} = \nabla_l \bar{\xi}_{l,k+1-l}$ while $\nabla_1^* \hat{\xi}_{l,k} = \nabla_l^* \bar{\xi}_{l,k}$. For $\nabla_1 \psi_k$ we have an identity,

$$\nabla_1 \psi_k(t) = \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} (\varepsilon - |x|) \varphi'_x(\varepsilon t, \varepsilon k + x + \varepsilon/2)\, dx\,, \tag{4.1}$$

where $\varphi = \phi\psi$, whence by the Schwarz inequality

$$(\nabla_1 \psi_k(t))^2 \le \frac{2\varepsilon}{3} \int_{\varepsilon k - \varepsilon/2}^{\varepsilon k + 3\varepsilon/2} \varphi'^2_x(\varepsilon t, x)\, dx\,. \tag{4.2}$$

A similar bound of $(\nabla_l \psi_k)^2$ follows easily because $\nabla_l \psi_k = \nabla_1 \bar{\psi}_{l,k}$, thus

$$(\nabla_l \psi_k(t))^2 \le \frac{1}{l} \sum_{j=k}^{k+l-1} (\psi_{j+1}(t) - \psi_j(t))^2\,; \tag{4.3}$$

such estimates are frequently used in the following calculations.

Our strategy is to derive (2.13) or (2.14) for components of entropy production first in terms of $\varphi = \phi\psi$ and its derivatives, but $\|\varphi\| = O(\|\psi\|)$ and $\|\varphi\|_{+1} = O(\|\psi\|_{+1})$ such that the constants depend only on ϕ, thus we need not worry too much about localization. In the rest of this section $O_\phi(\cdot)$ denotes a bound depending only on ϕ; dependence on h is not indicated because it is fixed here.

4.1. *The numerical error.* This is the easiest case, by a direct calculation

$$N_\varepsilon(\phi\psi, h) = \varepsilon \sum_{k\in\mathbb{Z}} \int_0^\infty (\nabla_1 \psi_k(t) - \varepsilon \nabla_\varepsilon \varphi(\varepsilon t, \varepsilon k - \varepsilon/2))\, J_k(t)\, dt\,, \tag{4.4}$$

where $\varphi = \phi\psi$.

Lemma 4.1. *The numerical error, N_ε satisfies* (2.13).

Proof. This looks elementary, but we must be a bit careful because ψ''_{xx} can not appear in the bound. Doing discrete integration by parts we obtain

$$N_\varepsilon(\phi\psi, h) = \varepsilon \sum_{k\in\mathbb{Z}} \int_0^\infty \delta_{\varepsilon,k}(t)\, \nabla_1^* J_k(t)\, dt\,,$$

where $\delta_{\varepsilon,k}(t) := \psi_k(t) - \varphi(\varepsilon t, \varepsilon k - \varepsilon/2)$ can be rewritten as

$$\delta_{\varepsilon,k}(t) = \frac{1}{2\varepsilon} \int_{-\varepsilon/2}^{\varepsilon/2} (\varepsilon - 2x) \varphi'_x(\varepsilon t, \varepsilon k + x)\, dx\,.$$

Now we factorize the sum of integrals into a product by means of the Cauchy and Schwarz inequalities, here and also several times later on, we are using the scheme

$$N_\varepsilon^2(\varphi,h) \le \left(\sum_{k\in\mathbb{Z}} \int_0^\infty \delta_{\varepsilon,k}^2(t)\,dt\right)\left(\varepsilon^2 \sum_{|k|\le r/\varepsilon} \int_0^{\tau/\varepsilon} \left(\nabla_1^* J_k(t)\right)^2\,dt\right),$$

where $r,\tau \ge 1$ are so large that $\phi(t,x)=0$ if $t>\tau$ or $|x|+\varepsilon>r$. On the other hand, by Schwarz again

$$\delta_{\varepsilon,k}^2(t) \le \frac{\varepsilon}{3}\int_{-\varepsilon/2}^{\varepsilon/2} \varphi_x'^2(\varepsilon t,\varepsilon k+x)\,dx\,,$$

while $\nabla_1^* J_k = J'(\xi_k)\nabla_l^*\bar\eta_{l,k}$ with some intermediate value ξ_k. Since J' is bounded and $\nabla_l^*\bar\eta_{l,k}=O(1/l)$, we get a bound in terms of φ_x', namely $N_\varepsilon(\varphi,h)=\|\varphi_x'\|_2\,O_\phi(1/l)$. Since $\|\psi\|_{+1}$ dominates both $\|\psi\|_2$ and $\|\psi_x'\|_2$, we have $\|\varphi_x'\|_2 \le \|\phi\|\|\psi_x'\|_2 + \|\phi_x'\|\|\psi\|_2 = O_\phi(\|\psi\|_{+1})$, consequently we have $N_\varepsilon = \|\psi\|_{+1}\,O_\phi(1/l)$. □

By means of Lemma 3.3 and (2.2) a slightly better bound, namely $\mathsf{E}|N_\varepsilon| = \|\psi\|_{+1}\,O_\phi((\varepsilon/\sigma)^{1/2})$ is obtained; notice that $\varepsilon/\sigma = o(1/l^2)$. In the case of unbounded spins, as in Fritz (2004), uniform bounds are not available, only the expectation of $|N_\varepsilon|$ can be estimated.

4.2. *The microscopic current.* The starting point of the estimation of L_ε is an identity,

$$\mathcal{L}h(\hat\eta_{l,k}) = h'(\hat\eta_{l,k})\,\mathcal{L}\hat\eta_{l,k} + \frac{1}{2}\sum_{b\in\mathbb{Z}_*} h''(\tilde\eta_{k,b})(c_b+\sigma)(\hat\eta_{l,k}^b-\hat\eta_{l,k})^2\,, \tag{4.5}$$

where $\tilde\eta_{k,b}$ is an intermediate value between $\hat\eta_{l,k}^b$ and $\hat\eta_{l,k}$. The second sum on the right hand side is a second order remainder, let Q_ε denote the resultant of these terms:

$$Q_\varepsilon(\phi\psi,h) := \frac{\varepsilon}{2}\sum_{k\in\mathbb{Z}}\sum_{b\in\mathbb{Z}_*}\int_0^\infty \psi_k(t)h''(\tilde\eta_{k,b})(c_b+\sigma)(\hat\eta_{l,k}^b-\hat\eta_{l,k})^2\,dt\,. \tag{4.6}$$

Since $|\hat\eta_{l,k}^b - \hat\eta_{l,k}| \le 2l^{-2}$, we have a uniform bound $Q_\varepsilon = \|\psi\|\,O_\phi(\sigma\varepsilon^{-1}l^{-3})$ depending only on ϕ and h, i.e. Q_ε satisfies (2.14). Similar remainders appear also in the definition (4.16) of Q_ε^i. The first condition of (2.2) prescribing that l, the size of our mesoscopic blocks is large enough is needed to show that these remainders do vanish as $\varepsilon\to 0$, the numerical error, $N_\varepsilon(\psi,h)$ and the martingale, $M_\varepsilon(t,\psi,h)$ behave in a similar manner. On the other hand, LSI is effective when $l(\varepsilon)$ is small, see the second condition of (2.2) and the next coming computations.

Observe now that $\mathcal{L}_o\hat\eta_{l,k} = \nabla_l^*\bar{\mathbf{j}}_{l,k}^o$, see (1.4) for the definition of $\mathbf{j}$. In view of Lemma 3.2 and Lemma 3.3 or 3.4 we decompose L_ε as $L_\varepsilon = W_\varepsilon + \sigma L_\varepsilon^{so} + V_\varepsilon + Q_\varepsilon$, where $W_\varepsilon = L_\varepsilon^{oo} - V_\varepsilon$ and

$$L_\varepsilon^{oo}(\phi\psi,h) := \varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)h_k'(t)\nabla_l^*\bar{\mathbf{j}}_{l,k}^o(\omega(t))\,dt\,, \tag{4.7}$$

$$L_\varepsilon^{so}(\phi\psi,h) := \varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)h_k'(t)\Delta_1\hat\eta_{l,k}(t)\,dt\,, \tag{4.8}$$

$$V_\varepsilon(\phi\psi,h) := \varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)h_k'(t)\nabla_l^*\mathfrak{J}(\bar\eta_{l,k}(t),\bar\omega_{l,k}(t))\,dt\,; \tag{4.9}$$

I_ε will be canceled by V_ε at the end of the argument.

Each of the sums above contains a space gradient, and $\sum x_k\nabla_l^* y_k = \sum y_k\nabla_l x_k$, moreover $\nabla_l(x_k y_k) = y_{k+l}\nabla_l x_k + x_k\nabla_l y_k$, therefore $W_\varepsilon = Y_\varepsilon^w + Z_\varepsilon^w$ and $L_\varepsilon^{so} = Y_\varepsilon^s + Z_\varepsilon^s$, where

$$Y_\varepsilon^w(\phi\psi,h) := \varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty (\nabla_l\psi_k)h_k'(t)(\bar{\mathbf{j}}_{l,k}^o - \mathfrak{J}(\bar\eta_{l,k},\bar\omega_{l,k}))\,dt\,, \tag{4.10}$$

$$Z_\varepsilon^w(\phi\psi,h) := \varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_{k+l}(t)(\nabla_l h_k')(\bar{\mathbf{j}}_{l,k}^o - \mathfrak{J}(\bar\eta_{l,k},\bar\omega_{l,k}))\,dt\,, \tag{4.11}$$

$$Y_\varepsilon^s(\phi\psi,h) := -\varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty (\nabla_1\psi_k(t))h_k'(t)\nabla_1\hat\eta_{l,k}(t)\,dt\,, \tag{4.12}$$

$$Z_\varepsilon^s(\phi\psi,h) := -\varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_{k+1}(t)(\nabla_1 h_k'(t))\nabla_1\hat\eta_{l,k}(t)\,dt\,. \tag{4.13}$$

The following bounds are more or less direct consequences of Lemma 3.2 and Lemma 3.3 or 3.4.

Lemma 4.2. *Y_ε^w and σY_ε^s satisfy (2.13), Q_ε, Z_ε^w and σZ_ε^s satisfy (2.14). The bounds of Q_ε and Z_ε^w vanish as $\varepsilon\to 0$, while $Z_\varepsilon^s\le 0$ if h is convex and $\phi\psi\ge 0$.*

Proof. Q_ε has a trivial deterministic bound, the estimation of Y_ε^w and Y_ε^s is fairly similar to that of N_ε. Separating $\nabla_l\psi_k$ and $\mathbf{j}^o - \mathfrak{J}$ by means of the Cauchy and Schwarz inequalities, (4.3) and Lemma 3.2 imply that $\mathsf{E}|Y_\varepsilon^w| = \|\psi\|_{+1}\,O_\phi(l\sqrt{\varepsilon/\sigma})$. In the same way, but now from (4.2) and Lemma 3.3 or 3.4, we get $\sigma(\varepsilon)\mathsf{E}|Y_\varepsilon^s| = \|\psi\|_{+1}\,O_\phi(\sqrt{\varepsilon\sigma})$.

In the case of Z_ε^w we separate $\nabla_l h'$ and $\mathbf{j}^o - \mathfrak{J}$ in the usual way. Since $\nabla_l h_k' = h''(\tilde\eta_k)\nabla_l\hat\eta_{l,k}$ with some intermediate value, and $l\nabla_l\hat\eta_{l,k} = (A*\eta)_k$, where the convolution kernel A satisfies $\sum A_j = 0$ and $\sum A_j^2 = O(1/l)$, Lemma 3.2 and Lemma 3.3 or 3.4 result in $\mathsf{E}|Z_\varepsilon^w| = \|\psi\|\,O_\phi(l/\sigma)$. Finally, the inequality $Z_\varepsilon^s\le 0$ is trivial if $h''\ge 0$ and $\phi\psi\ge 0$, in the general case $\sigma(\varepsilon)\mathsf{E}|Z_\varepsilon^s| = O_\phi(\|\psi\|)$ is a direct consequence of Lemma 3.3 with $(A*\eta)_k = \bar\eta_{l,k+l} - \bar\eta_{l,k}$. □

The gradient of the microscopic entropy flux can be written as

$$I_\varepsilon(\phi\psi,h) = -\varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)\nabla_1^* J_k(t)\,dt\,, \tag{4.14}$$

and $\nabla_1^* J_k = J_{k-1} - J_k = J_k'\,\nabla_l^*\bar\eta_{l,k} + (1/2)J''(\tilde\eta_k)(\nabla_l^*\bar\eta_{l,k})^2$ with some intermediate value $\tilde\eta_k$. Since $J'(\rho) = \kappa\,h'(\rho)(1-2\rho)$,

$$J_k'\,\nabla_l^*\bar\eta_{l,k} = \kappa\,h_k'\nabla_l^*\left(\bar\eta_{l,k} - (\bar\eta_{l,k})^2\right) + \kappa\,h_k'\left(\bar\eta_{l,k-l} + \bar\eta_{l,k} - 2\hat\eta_{l,k}\right)\nabla_l^*\bar\eta_{l,k}\,,$$

consequently $I_\varepsilon = I_\varepsilon^o - Q_\varepsilon^i$, where

$$I_\varepsilon^o(\phi\psi,h) := -\kappa\varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)h_k'(t)\nabla_l^*\left(\bar\eta_{l,k} - (\bar\eta_{l,k})^2\right)\,dt\,, \tag{4.15}$$

and Q^i_ε consists of the second order remainders above,

$$\begin{aligned} Q^i_\varepsilon(\phi\psi,h) &:= \frac{\varepsilon}{2}\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)h_k'(t)J''(\tilde\eta_k)(\nabla_l^*\bar\eta_{l,k})^2\,dt \\ &+\kappa\varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_k(t)h_k'(t)(\bar\eta_{l,k-l}+\bar\eta_{l,k}-2\hat\eta_{l,k})\,\nabla_l^*\bar\eta_{l,k}\,dt\,. \end{aligned} \tag{4.16}$$

The first sum of Q^i_ε can directly be estimated, $\|\psi\|(\varepsilon l^2)^{-1}$ is its order. Factorizing the second one, a bit larger bound, $\mathsf{E}|Q^i_\varepsilon| = \|\psi\|\,O_\phi(l/\sigma)$ follows by Lemma 3.3 or 3.4.

The crucial step of the argument is the replacement of the mesoscopic current, $\mathfrak{J}$ by its canonical expectation given $\bar\eta_{l,k}$. Just as before Lemma 4.2, we have $V_\varepsilon + I^o_\varepsilon = Y^i_\varepsilon + Z^i_\varepsilon$, where

$$Y^i_\varepsilon(\phi\psi,h) := \varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty (\nabla_l\psi_k)h_k'\left(\mathfrak{J}(\bar\eta_{l,k},\bar\omega_{l,k}) - \kappa\left(\bar\eta_{l,k}-\bar\eta_{l,k}^2\right)\right)\,dt\,, \tag{4.17}$$

$$Z^i_\varepsilon(\phi\psi,h) := \varepsilon\sum_{k\in\mathbb{Z}}\int_0^\infty \psi_{k+l}(\nabla_l h_k')\left(\mathfrak{J}(\bar\eta_{l,k},\bar\omega_{l,k}) - \kappa\left(\bar\eta_{l,k}-\bar\eta_{l,k}^2\right)\right)\,dt\,. \tag{4.18}$$

Since $\mathfrak{J}(\bar\eta_{l,k},\bar\omega_{l,k}) - \kappa\left(\bar\eta_{l,k}-\bar\eta_{l,k}^2\right) = (1-\bar\eta_{l,k})(\bar\omega_{l,k}-\kappa\,\bar\eta_{l,k})$, we are now in a position to apply Lemma 3.5, too.

Lemma 4.3. *Y^i_ε satisfies* (2.13), *Q^i_ε and Z^i_ε satisfy* (2.14) *with vanishing bounds.*

Proof. The treatment of Y^i_ε follows that of Y^w_ε, by means of (4.3) and Lemma 3.5 we get $\mathsf{E}|Y^i_\varepsilon| = \|\psi\|_{+1}\,O_\phi(\sqrt{\varepsilon/\alpha+1/l})$. Finally, as in the case of Z^w_ε, Lemma 3.3 and Lemma 3.5 imply $\mathsf{E}|Z^i_\varepsilon| = \|\psi\|\,O_\phi(\sqrt{1/\alpha\sigma+1/\varepsilon\sigma l})$, which completes the proof. □

4.3. *The martingale.* $M_\varepsilon(t,\psi,h)$ is delicate because it can not be written as integral of a function over space and time. The stochastic differential $dh(\hat\rho_\varepsilon) = \varepsilon^{-1}\mathfrak{L}h(\hat\rho_\varepsilon) + dm_\varepsilon$ defines a martingale $m_\varepsilon = m_\varepsilon(t,x)$ for each $x\in\mathbb{R}$ such that $m_\varepsilon(t,x) = m_\varepsilon(t,\varepsilon k)$ if $|x-\varepsilon k|<\varepsilon/2$ and

$$M_\varepsilon(t,\phi\psi,h) = \int_{-\infty}^\infty\int_0^t \psi(t,x)\,\phi(t,x)\,m_\varepsilon(dt,x)\,dx\,. \tag{4.19}$$

m_ε is identified by the intensity q_ε of its quadratic variation $\langle m_\varepsilon\rangle$,

$$\begin{aligned} q_\varepsilon(t,x) &:= \frac{1}{\varepsilon}\left(\mathfrak{L}h^2(\hat\rho_\varepsilon) - 2h(\hat\rho_\varepsilon)\mathfrak{L}h(\hat\rho_\varepsilon)\right) \\ &= \frac{1}{\varepsilon}\sum_{b\in\mathbb{Z}_*}(c_b+\sigma)\left(h(\hat\eta^b_{l,k}) - h(\hat\eta_{l,k})\right)^2 \end{aligned} \tag{4.20}$$

if $|x-\varepsilon k|<\varepsilon/2$, cf. (4.5). Since $d\langle m_\varepsilon\rangle = q_\varepsilon\,dt$, we have $\mathsf{E}\,m^2_\varepsilon(t,x) = \int_0^t \mathsf{E}\,q_\varepsilon(\tau,x)\,d\tau$.

Lemma 4.4. *$M_\varepsilon(\infty,\phi\psi,h)$ satisfies* (2.13).

Proof. Let $\dot m_\varepsilon(t,x)$ denote the time derivative of m_ε in the H^{-1} sense, we have to show that $\mathsf{E}\|\varphi\,\dot m_\varepsilon\|^2_{-1}\to 0$ as $\varepsilon\to 0$. Since $\|\partial_t(\varphi\,m_\varepsilon)\|_{-1}\le\|\varphi\,m_\varepsilon\|_2$ and $\|\varphi'_t\,m_\varepsilon\|_{-1}\le\|\varphi'_t\,m_\varepsilon\|_2$, the problem reduces to the calculation of q_ε. Just as in the case of Q_ε, see (4.20) and (4.5), independently of the configuration we have $q_\varepsilon(t,x) = O(\sigma/l^3\varepsilon)$, which completes the proof. □

Summarizing the results of this section, we see that entropy production $X_\varepsilon(\phi\psi,h)$, $\psi \in C_c^1(\mathbb{R}^2)$, $\phi \in C_{co}^2(\mathbb{R}_+^2)$ has been decomposed as $X_\varepsilon = Y_\varepsilon + Z_\varepsilon + \sigma Z_\varepsilon^s$, where $Y_\varepsilon := N_\varepsilon + Y_\varepsilon^w + \sigma Y_\varepsilon^s + Y_\varepsilon^i + M_\varepsilon$ vanishes in H_{loc}^{-1}, $Z_\varepsilon := Z_\varepsilon^w + Z_\varepsilon^i + Q_\varepsilon - Q_\varepsilon^i$ vanishes in the space of locally bounded measures, and σZ_ε^s is bounded in the same sense. Moreover, $Z_\varepsilon^s \le 0$ if $\psi, \phi \ge 0$ and h is convex. The rest of the proof is an application of the stochastic version of the theory of compensated compactness and PDE theory on uniqueness of weak solutions to a single conservation law.

5. Completion of the proof

Let us consider the distributions $\hat{P}_{\theta,\varepsilon}$, $\varepsilon > 0$ of the Young representation of the empirical process $\hat{\rho}_\varepsilon$, see Section 2 for definitions. This family is tight with respect to the weak topology of the space $M_\theta(\mathbb{X})$ of locally finite measures, denote $\hat{P}_\theta$ any of its limit distributions. In the proof of uniqueness the distribution of the empirical process shall be considered also as a probability measure on the space $L_{\mathrm{loc}}^2(\mathbb{R}_+^2)$.

5.1. *Convergence to weak solutions.* In the previous section conditions (2.13) and (2.14) on entropy production have been verified, consequently Proposition 2.1 of Fritz (2004) or Lemma 8 of Fritz and Tóth (2004) imply Tartar's factorization (2.15) $\hat{P}_\theta$-a.s. for all couples of smooth entropy pairs. Evaluating (2.15) for $h_1(\rho) := \rho$, $J_1(\rho) := \kappa\rho - \kappa\rho^2$ and $h_2(\rho) := \rho^2$, $J_2(\rho) = \kappa\rho^2 - 4\kappa\rho/3$ if $\rho \in [0,1]$, we get

$$\theta_{t,x}(\rho^4) - 4\theta_{t,x}(\rho)\theta_{t,x}(\rho^3) + 3\theta_{t,x}^2(\rho^2) = 0$$

almost everywhere with respect to (t,x) and $\hat{P}_\theta$. Observe that

$$\iint (u^4 + v^4 - 4u^3v - 4uv^3 + 6u^2v^2)\,\theta_{t,x}(du)\,\theta_{t,x}(dv) = \iint (u-v)^4\,\theta_{t,x}(du)\,\theta_{t,x}(dv) = 0 \quad \text{a.s}$$

is an equivalent form of the equation of factorization, thus the product of the measures $\theta_{t,x}(du)$ and $\theta_{t,x}(dv)$ is concentrated on the diagonal. This means that $\theta_{t,x}$ is a Dirac measure a.s., and any limit distribution $\hat{P}_\theta$ is concentrated on a set of weak solutions.

5.2. *Uniqueness of the limit.* Since $Z_\varepsilon^s(\psi,h) \le 0$ if $\psi \ge 0$ and h is convex, while all other terms of entropy production vanish in the limit, from (2.5) we obtain the weak Lax inequality (2.10) for measure solutions with probability one with respect to any limit distribution $\hat{P}_\theta$. As a consequence of compensated compactness, the weak inequality (2.10) for measure solutions turns into the weak inequality (2.16) for weak solutions; the initial value is not present in these inequalities. To get uniqueness, we have to show that these weak solutions all satisfy the given initial condition $\rho_0 \in L^\infty(\mathbb{R})$, that is (1.7) holds true a.s. for all limit distributions of the empirical process $\hat{\rho}_\varepsilon$ as specified below; our starting point is again (2.5), but now with $h(\rho) \equiv \rho$. The argument is an extension of the standard PDE proof in case of the vanishing viscosity limit, see e.g. Dafermos (2000); Serre (2000).

Besides $\hat{P}_{\varepsilon,\theta}$ and $\hat{P}_\varepsilon$, the distribution $\hat{P}_{\varepsilon,2}$ of $\hat{\rho}_\varepsilon^2 \in L_{\mathrm{loc}}^2(\mathbb{R}_+^2)$ plays also some role. Any of these families is tight with respect to the weak topology of the underlying spaces, thus Skorohod's embedding applies: for every sequence $\varepsilon(n) \to 0$ such that

$\hat{P}_{\varepsilon,\theta}$, $\hat{P}_\varepsilon$ and $\hat{P}_{\varepsilon,2}$ all converge in the weak sense along this subsequence, we have a probability space on which almost surely

$$\lim_{n\to\infty}\int_0^\infty\int_{-\infty}^\infty \psi(t,x)\hat{\rho}_{\varepsilon(n)}(t,x)\,dx\,dt = \int_0^\infty\int_{-\infty}^\infty \psi(t,x)\theta_{t,x}(\rho)\,dx\,dt\,,$$

and

$$\lim_{n\to\infty}\int_0^\infty\int_{-\infty}^\infty \psi(t,x)\hat{\rho}^2_{\varepsilon(n)}(t,x)\,dx\,dt = \int_0^\infty\int_{-\infty}^\infty \psi(t,x)\theta_{t,x}(\rho^2)\,dx\,dt$$

for all $\psi \in C_c(\mathbb{R}^2)$. Here $\theta_{t,x}$ is the limiting Young measure, i.e. $\theta_{t,x}(\rho^2)$ is just the a.s. weak limit of $\hat{\rho}^2_{\varepsilon(n)}$. However, $\theta_{t,x}(\rho^2) = \theta^2_{t,x}(\rho)$ a.s. in view of the previous subsection, consequently $\hat{\rho}_{\varepsilon(n)}$ is almost surely convergent also with respect to the strong topology of $L^2_{\mathrm{loc}}(\mathbb{R}^2_+)$ because weak convergence in a Hilbert space together with convergence of the norm imply strong convergence.

Now we are in a position to return to (2.5), set $0 \le \psi \in C^1_c(\mathbb{R}^2)$ and $h(\rho) \equiv \rho$. Since $Z^s_\varepsilon(\psi,h) = 0$ in this case, while Y_ε and Z_ε vanish in the limit, (1.10) implies (1.7). Perhaps Corollary 6.1 of Bressan (2000) or Main Theorem of Chen and Rascle (2000) are the most convenient references, these results tell us that the weak Lax inequality (2.16) and the weak equation (1.7) including the initial condition imply the uniqueness of weak solutions to a single conservation law as (1.5).

6. Concluding remarks

In view of the previous argument, the conclusion of Theorem 1.1 can considerably be improved as follows, conditions are not changed. Let ρ denote the entropy solution to (1.5) with initial condition ρ_0, then

$$\lim_{\varepsilon\to 0}\int_0^\tau\int_{-r}^r \mathsf{E}|\hat{\rho}_\varepsilon(t,x)-\rho(t,x)|^2\,dx\,dt = 0$$

for all $r,\tau > 0$. Notice that the statement is restricted to mesoscopic blocks of size $l(\varepsilon)$ as specified by (2.2). The bound of Lemma 3.3 is not sufficient to fill in the gap between large microscopic and mesoscopic block averages because for $(A*\eta)_k = \bar{\eta}_{p,k+m} - \bar{\eta}_{p,k}$ we only have

$$\varepsilon^2\sum_{|k|<r/\varepsilon}\int_0^{\tau/\varepsilon}\int(\bar{\eta}_{p,k+m}-\bar{\eta}_{p,k})^2\,d\mu_{\varepsilon,t}\,dt \le C_2\left(\frac{(r+\tau)\varepsilon m^3}{\sigma p}+\frac{r\tau}{p}\right)\,,$$

thus $m = O((\sigma/\varepsilon)^{1/3})$ is needed when $p \in \mathbb{N}$ is fixed and $\varepsilon \to 0$. However, (2.2) implies $(\sigma(\varepsilon)/\varepsilon)^{1/3} = o(l(\varepsilon))$; LSI is not optimal when m is large. Nevertheless

$$\bar{\eta}_{p,k+m}-\bar{\eta}_{p,k} = \frac{1}{p}\sum_{j=k}^{k+p-1}(\eta_{j+m}-\eta_j) = \frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{i=0}^{m-1}(\eta_{j+i+1}-\eta_{j+i})\,,$$

thus Lemma 3.4 implies

$$\varepsilon^2\sum_{|k|<r/\varepsilon}\int_0^{\tau/\varepsilon}\int(\bar{\eta}_{p,k+m}-\bar{\eta}_{p,k})^2\,d\mu_{\varepsilon,t}\,dt \le C_2'\left(\frac{(r+\tau)\varepsilon m^2}{\sigma}+\frac{r\tau}{p}\right)\,,$$

whence by the Cauchy inequality

$$\varepsilon^2\sum_{|k|<r/\varepsilon}\int_0^{\tau/\varepsilon}\int(\bar{\eta}_{m,k}-\bar{\eta}_{p,k})^2\,d\mu_{\varepsilon,t}\,dt \le C_2'\left(\frac{(r+\tau)\varepsilon m^2}{\sigma}+\frac{r\tau}{p}\right)\,,$$

at lest if m is a multiple of p. Since $\varepsilon l^2(\varepsilon)/\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$, we can choose $m \approx l(\varepsilon)$ to get

$$\lim_{p\to\infty} \lim_{\varepsilon\to 0} \int_0^\tau \int_{-r}^r \mathsf{E}|\bar\rho_{\varepsilon,p}(t,x) - \hat\rho_\varepsilon(t,x)|^2 \, dx \, dt = 0 \, ,$$

where $\bar\rho_{\varepsilon,p}(t,x) := \bar\eta_{p,k}(t/\varepsilon)$ if $|x - k\varepsilon| < \varepsilon/2$. Indeed, $\bar\eta_{l,k} \approx \hat\eta_{l,k}$ in a mean square sense, see Lemma 3.3 or 3.4, consequently

$$\lim_{p\to\infty} \lim_{\varepsilon\to 0} \int_0^\tau \int_{-r}^r \mathsf{E}|\bar\rho_{\varepsilon,p}(t,x) - \rho(t,x)|^2 \, dx \, dt = 0 \, .$$

There is another, not less interesting version of this model. Electrophoresis can be mimicked by replacing the spin-flip action $\omega_k \to -\omega_k$ with a creation-annihilation mechanism such that $(\omega_k, \omega_{k+1}) = (0,0) \to (-1,1)$ and $(\omega_k, \omega_{k+1}) = (-1,1) \to (0,0)$. Total charge $\sum \omega_k$ is the only conserved quantity in this case, the associated empirical process is defined as $\hat u_\varepsilon(t,x) := \hat\omega_{l,k}(t/\varepsilon)$ if $|x - \varepsilon k| < \varepsilon/2$. Stationary product measures, λ_u of this process are characterized by $\lambda_u(\omega_k) = u$ and $\lambda_u(\omega_k = 0) = (1/3)((4 - 3u^2)^{1/2} - 1)$, and the macroscopic equation for the limit of $\hat u_\varepsilon$ reads as

$$\partial_t u = \partial_x (u^2 + \tfrac{1}{3}\sqrt{4 - 3u^2}) \, . \tag{6.1}$$

A rigorous derivation of this equation could follow the argument of the paper, but the final crucial step, the replacement of $\bar\eta_{l,k}$ with its canonical expectation $4/3 - (1/3)(4 - 3\bar\omega_{l,k}^2)^{1/2}$ is problematic because that second LSI is not available. This system is also a relaxation scheme, perhaps methods of PDE theory Chen and Rascle (2000); Dafermos (2000) are helpful.

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