Hurst exponent estimation of Fractional Lévy Motion

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Abstract. In this paper, we build an estimator of the Hurst exponent of a fractional Lévy motion. The stochastic process is observed with random noise errors in the following framework: continuous time and discrete observation times. In both cases, we prove consistency of our wavelet type estimator. Moreover we perform some simulations in order to study numerically the asymptotic behaviour of this estimate.

1. Introduction

In this paper we present results on an asymptotic analysis of a wavelet type estimator of the self-similarity (Hurst) parameter of a Real Harmonizable Fractional Lévy Motion (RHFLM). In particular, we show consistency of this estimate in a noisy regression framework. This enables to detect RHFLM in noisy data and to use it in practical settings.

It is well known that standard Brownian Motion fails in explaining certain statistical time series arising from finance and turbulence theory. Hence Mandelbrot and Van Ness, in 1968, introduced a stochastic motion of a quite different nature, the Fractional Brownian Motion (in short FBM). The FBM of index H is the only centered Gaussian field, vanishing at zero, with stationary increments and self-similar with index H. Its Hurst exponent governs its properties. The main difference between both processes is that, now the increments are not independent and the process can model short or long range dependent data.

Nevertheless, in image modeling, in finance or in biology, the processes are rarely Gaussian, which prevents the use of Gaussian fields. Hence, Benassi et al., in 2002, introduce Real Harmonizable Fractional Lévy Motions (in short RHFLM) to model such processes. Let us recall that a RHFLM X_H of index H $(0 < H < 1)$ is defined as the stochastic integral

$$X_H(x) = \int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{H+d/2}} L(d\xi), \quad x \in \mathbb{R}^d, \tag{1.1}$$

where $\| \cdot \|$ is the Euclidean norm and $L(d\xi)$ is a Lévy random measure in the sense of Benassi et al. (2002).

Such processes are non Gaussian, locally asymptotically self-similar with Hurst exponent H and are well fitted to mimic most of the irregular phenomena that can be observed in turbulence experiments, provided the parameter H is well chosen.

So, estimating the Hurst exponent of a RHFLM is the key issue in order to analyze data and to model real observations by such a process. More precisely, we want to be able to estimate the Hurst exponent of a RHFLM observed only at discrete times in a white noise framework. In their work, Benassi et al. (2002) propose an estimator of Hurst exponent based on some generalized quadratic variation of the process. This estimator has first been introduced by Istas and Lang (1997) in a Gaussian framework. Such estimators have also been studied in Benassi et al. (1998b) in the case of some self-similar Gaussian random fields or by Benassi et al. (1998a), Benassi et al. (2000), Ayache and Lévy Véhel (2004), Ayache et al. (2005) and Lacaux (2004a) in the case of multifractional random fields.

However, such estimation techniques are unable to handle noisy observations and to estimate this exponent when data are blurred by a Gaussian white noise, as we will see later in this paper. Our purpose is hence to construct a robust estimator and to detect RHFLM in noisy data.

For this, we consider wavelet analysis of this stochastic process. Work on properties of wavelet coefficients of fractional Brownian motion was pioneered in the papers by Flandrin and Abry (1999) or Flandrin (2000). Statistical properties of wavelet type estimators for Hurst exponent were studied in Delbeke and Van Assche (1998) or Bardet (2002) in the case of FBM. Such estimators have also been introduced in the case of linear fractional stable motion in Delbeke and Abry (2000), Stoev et al. (2002) and Pipiras et al.. In Abry et al. (2003) are highlighted the properties of wavelet coefficients for self-similar processes or long-range dependent processes. For multifractal processes, such estimators are studied in Jaffard (2000), Gamboa and Loubes (2005a) or Gamboa and Loubes (2005b) for example. They are based on a regression of the log-variance of the wavelet coefficients versus scale. Other authors study also this issue, for instance in Kurchenko (2002) or Hall et al. (2000).

The aim of this paper is to introduce an estimator of the Hurst exponent based on wavelet type coefficients and prove its asymptotic behaviour in the case of RHFLMs. Contrary to other work, the coefficients of a RHFLM are not Gaussian neither independent. Hence we are facing a difficult issue since work in this direction either relies on independence of the coefficients or its Gaussian properties to prove consistency of the estimator, see for instance the work by Morales and Kolaczyk.

Indeed, in such papers, the authors often generalize Gaussian type limit theorems to the case of weak dependent random variables using results in Doukhan (2003) for instance. In Pipiras et al., the asymptotic normality of the estimator relies on properties of stable moving average sequence.

Unfortunately, in the case of coefficients of a RHFLM, we are not in such cases and do not have powerfull probabilistic tools at hand. Hence we rely directly on the properties of RHFLM to get asymptotic results. Nevertheless, we prove almost sure consistency of the moment type estimator in the presence of Gaussian noise. This enables us to compute the estimator and analyze its performance for simulated data.

The paper falls into the following parts. Section 2 is devoted to a wavelet type representation of RHFLM. In Section 3, we construct estimators of the Hurst exponent of a RHFLM. Then, in Section 4, we provide an estimator of the Hurst exponent in a regression framework. Section 5 deals with a numerical study of our estimators for simulated data.

2. RHFLM and wavelet bases

RHFLM is one of the canonical example of fractional non Gaussian process, with a large presence in both the theoretical literature and applications. Wavelets have become a standard tool for the modeling and analysis of such signals.

Let us first recall the definition and the properties of a RHFLM.
RHFLM:

Benassi et al. (2002) have defined RHFLMs by substituting in the harmonizable representation of the FBM to the Wiener measure $W(d\xi)$ a Lévy random measure $L(d\xi)$. Heuristically, a Lévy random measure is linked with the increments of a Lévy process. Also, the non-Brownian part of a Lévy random measure is defined thanks to a Poisson random measure. We first recall the definition of a Lévy random measure $L(d\xi)$ in the sense of Benassi et al. (2002).

The non-Brownian part $M(d\xi)$ of the Lévy random measure $L(d\xi)$ is represented by a Poisson random measure $N(d\xi, dz)$ on $\mathbb{R} \times \mathbb{C}$ whose mean measure $n(d\xi, dz) = d\xi\, \nu(dz)$ is such that $\nu(\{0\}) = 0$ and

$$\forall\, p \geqslant 2, \quad \int_{\mathbb{C}} |z|^p\, \nu(dz) < +\infty.$$

Here, $\nu(dz)$ is a non vanishing rotationally invariant measure, i.e.

$$P(\nu(dz)) = d\theta\, \nu_\rho(d\rho), \tag{2.1}$$

where $d\theta$ is the uniform measure on $[0, 2\pi)$ and $P\big(\rho e^{i\theta}\big) = (\theta, \rho) \in [0, 2\pi) \times \mathbb{R}_*^+$.

The random measure $M(d\xi)$ is defined by

$$\int_{\mathbb{R}} f(\xi)\, M(d\xi) = \int_{\mathbb{R} \times \mathbb{C}} [f(\xi)z + f(-\xi)\bar{z}]\, (N - n)(d\xi, dz),$$

where $f \in L^2(\mathbb{R}^d)$. If $\overline{f(\xi)} = f(-\xi)$, then $\int_{\mathbb{R}} f(\xi)\, M(d\xi)$ is a real-valued symmetric infinitely divisible random variable and for every $u \in \mathbb{R}$

$$\mathbb{E}\Big[e^{iu\int f(\xi)\, M(d\xi)}\Big] = \exp\left[\int_{\mathbb{R}^d \times \mathbb{C}} \left[e^{2iu\Re(f(\xi)z)} - 1 - 2iu\Re\left(f(\xi)\, z\right)\right] d\xi\, \nu(dz)\right]. \tag{2.2}$$

Here, $M(d\xi)$ is only the non-Brownian part of the Lévy random measure $L(d\xi)$. Finally,

$$L(d\xi) = aM(d\xi) + bW(d\xi),$$

where $(a, b) \in \mathbb{R}^2$ and $W(d\xi)$ is a Wiener measure independent of $M(d\xi)$. Taking $L(d\xi) = W(d\xi)$, the field X_H defined by (1.1) is a FBM. The Wiener measure $W(d\xi)$ is a complex random measure which ensures that X_H is a real-valued field, see Samorodnitsky and Taqqu (1994); Cohen (1999) for reference on complex Gaussian random measure. If $f \in L^2(\mathbb{R})$ and $\overline{f(\xi)} = f(-\xi)$, $\int f(\xi) L(d\xi)$ is a symmetric real-valued random variable,

$$\mathbb{E}\Big[\mathrm{e}^{iu \int f(\xi) L(d\xi)}\Big] = \exp\left(-\frac{b^2 u^2 \|f\|_2^2}{2}\right) \mathbb{E}\Big[\mathrm{e}^{iau \int f(\xi) M(d\xi)}\Big], \quad u \in \mathbb{R} \qquad (2.3)$$

and

$$\mathbb{E}\left[\left|\int_{\mathbb{R}} f(\xi) L(d\xi)\right|^2\right] = A\|f\|_{L^2(\mathbb{R})}^2, \qquad (2.4)$$

with $A = a^2 \int_{\mathbb{C}} |z|^2 \nu(dz) + b^2$. The isometry property (2.4) allows us to evaluate the second order moment of the wavelet type coefficients of a RHFLM.

In the following, we assume that $L(d\xi)$ is a non vanishing measure, i.e. $(a, b) \neq (0, 0)$.

A RHFLM with index H, defined by

$$X_H(x) = \int_{\mathbb{R}} \frac{\mathrm{e}^{-ix\xi} - 1}{|\xi|^{H+1/2}} L(d\xi), \quad x \in \mathbb{R},$$

has stationary increments and the same structure of covariance as a FBM. Note that the Brownian part of X_H is a FBM with index H independent from its non-Brownian part.

In addition, as in the case of FBM, RHFLMs have locally Hölder sample paths and their pointwise Hölder at point x is equal almost surely to H. Whereas a FBM is self-similar with index H, in general, a RHFLM is only locally self-similar and looks like locally a FBM with index H. An important difference between RHFLM and FBM is that as soon the RHFLM is non-Gaussian, it does not have moment of every order. However, as noticed in the following, the wavelet type coefficients of a RHFLM have moments of every order. This key property allows us to study, in Section 3, moment type estimators based on the wavelet type coefficients of a RHFLM.

Weakness of quadratic variation estimator:

Consider the following standard regression model with $N + 1$ observations. We observe at discrete times l/N, $l = 0, \ldots, N$ a noisy RHFLM

$$Y_H\left(\frac{l}{N}\right) = X_H\left(\frac{l}{N}\right) + \sigma_N \varepsilon_l, \; l = 0, \ldots, N, \qquad (2.5)$$

where ε_l are an i.i.d sample of Gaussian random variables $\mathcal{N}(0, 1)$ independent from X_H and σ_N is the noise level.

In their work, Benassi et al. (2002) consider for $K > 0$ and a sequence a_k, $k =$

$0, \ldots, K$ the quadratic variation

$$V_N = \frac{1}{N-K+1} \sum_{p=0}^{N-K} \left[\sum_{k=0}^{K} a_k X_H \left(\frac{k+p}{N} \right) \right]^2$$

$$= \frac{1}{N-K+1} \sum_{p=0}^{N-K} \left[\Delta X_{p,N} \right]^2 .$$

If observed without observation errors, this technique enables to build a weakly consistent estimate of Hurst exponent. When observed in practice with Gaussian errors, let us apply such method and define the noisy quadratic variation as

$$W_N = \frac{1}{N-K+1} \sum_{p=0}^{N-K} \left[\Delta Y_{p,N} \right]^2 \qquad (2.6)$$

$$= \frac{1}{N-K+1} \sum_{p=0}^{N-K} \left[\Delta X_{p,N} + \sigma_N \sum_{k=0}^{K} a_k \varepsilon_{k+p} \right]^2 .$$

Write $\xi_p = \sum_{k=0}^{K} a_k \varepsilon_{k+p} \sim \mathcal{N}(0, \sum_{k=0}^{K} a_k^2)$. Now, if we consider the expectancy of (2.6), we get that

$$\mathbf{E}(W_N) = \mathbf{E}(V_N) + \sigma_N^2 \mathbf{E}(\xi_1^2)$$

$$= N^{-2H} c(H) + \sigma_N^2 \sum_{k=0}^{K} a_k^2,$$

where $c(H) > 0$ only depends on H.

As a result, in order to get consistency of the noisy quadratic variation, we need to ensure that

$$\sigma_N^2 N = O(N^{1-2H}),$$

which implies that $\sigma_N = O(1/N^H), 0 < H < 1$. In a standard regression framework, when the RHFLM is observed at equispaced times, the noise level is of order $\sigma_N = O(1/\sqrt{N})$. Hence the method described in Benassi et al. (2002) does not provide a consistent estimator of the Hurst exponent when $H > 1/2$ in the regression framework. Moreover, when estimating Hurst exponent, we can not consider a method depending on the values of the parameter of interest. As a result we pay attention in the next section to wavelet type estimators.

Wavelet bases:

Let $\psi(\cdot)$ be a function with compact support, continuous derivative and r vanishing moments, i.e.

$$\forall m = 0, \ldots, r-1, \int t^m \psi(t) \, dt = 0.$$

We assume that $\psi \neq 0$, i.e. that its support $\mathrm{supp}\psi \neq \emptyset$. Define the rescaled function at scale j and location k as $\psi_{jk}(t) = 2^{j/2} \psi(2^j t - k)$. Then given a process $(Y(t))_{t \in \mathbb{R}}$ observed at continuous time, the corresponding wavelet type coefficients of the process are defined by

$$w_{jk} = \int Y(t) \psi_{jk}(t) dt = 2^{-j/2} \int Y \left(\frac{u+k}{2^j} \right) \psi(u) du. \qquad (2.7)$$

When we observe Y at discrete times, the coefficients w_{jk} are not known. In this case, we will approximate them thanks to a discretization of the integral (2.7). For ease of writing, we assume that the compact support of ψ is included in $[0, 1]$ and that we observe Y at discrete times l/N, $l = 0 \ldots N$. However, as in Bardet (2002), up to change the discretization, all the following results can be stated for ψ a wavelet function with compact support. We refer to Mallat (1998) and Daubechies (1992) for general references about wavelet bases and their properties. In this framework, we will replace the coefficients w_{jk} by the corresponding coefficients of the discretized process defined by

$$w_{jk}^n = \frac{1}{2^{j/2}n} \sum_{p=1}^{n} Y\left(\frac{p+nk}{2^j n}\right) \psi(\frac{p}{n}). \tag{2.8}$$

Remark that w_{jk}^n is a discretization of constant step of the integral (2.7). The parameter n will be chosen such that w_{jk}^n can be computed knowing Y at times l/N, $l = 0, \ldots, N$.

Let us first assume that we observe a RHFLM in continous time. Some key properties of its wavelet type coefficients are given in the following lemma. As in the case of LFSM (Delbeke and Van Assche, 1998), these coefficients can be rewritten as stochastic integral with respect to the Lévy random measure $L(d\xi)$. The coefficients of a LFSM are stable random variables and then do not have a finite second order moment. In our frawemork, w_{jk} is an infinitely divisible random variable with moments of every order. Since a RHFLM X_H has stationary increments, its coefficients w_{jk} are stationary in k for each fixed j according to Abry et al. (2003). We provide here a direct proof of this result. Note that our framework is not studied in Abry et al. (2003) since X_H is not (in general) a self-similar process. Then, contrary to Abry et al. (2003), w_{jk} is not a self-similar process in j. However, we can establish the asymptotics of w_{jk} as $j \to +\infty$.

Lemma 2.1 (wavelet type coefficient of RHFLM). *Let $(X_H(t))_{t\in\mathbb{R}}$ be a RHFLM with Hurst exponent H. Define*

$$w_{jk} = 2^{j/2} \int X_H(t)\psi(2^j t - k)dt, \tag{2.9}$$

where the function ψ has $r \geq 2$ vanishing moments. We have the following properties:

(1) $w_{jk} = \int_{\mathbb{R}} \frac{\overline{\widehat{\psi_{jk}}(\xi)}}{|\xi|^{H+1/2}} L(d\xi)$ *a.s. where* $\widehat{f}(\xi) = \int_{\mathbb{R}} \exp(it\xi)f(t)dt$.

(2) *(Stationarity)* $w_{jk} \sim w_{j0}$*, for $k = 0, \ldots, 2^j - 1$, but the coefficients are not independent.*

(3) *(Moments) Set $N_0 \sim \mathcal{N}(0, \mathbf{E}(w_{00}^2))$. For all integer $p \geqslant 0$,*

$$\mathbf{E}(w_{jk}^{2p+1}) = 0$$

and as $j \to +\infty$, for each fixed k

$$2^{pj+2pH}\mathbf{E}(w_{jk}^{2p}) \to \mathbf{E}(N_0^{2p}).$$

(4) *(Asymptotic Normality) As $j \to +\infty$,*

$$\sqrt{2^{j(1+2H)}}w_{jk} \xrightarrow{(\mathcal{D})} \mathcal{N}\left(0, \mathbf{E}(w_{00}^2)\right).$$

Proof.   • In view of the definition of the process X_H,

$$w_{jk} = \int_{\mathbb{R}} \psi_{jk}(t) \left(\int_{\mathbb{R}} \frac{\mathrm{e}^{-it\xi} - 1}{|\xi|^{H+1/2}} L\,(d\xi) \right) dt.$$

Since $\psi(\cdot)$ has $r \geq 2$ vanishing moments,

$$\int_{\mathbb{R}} \psi_{jk}(t)dt = 0, \tag{2.10}$$

and then the first statement is immediate provided we can exchange the integral in ξ with the integral in t. However, this exchange is not trivial since the stochastic integral in ξ is defined in a L^2 framework. We first point out that since ψ is a function with compact support, $\widehat{\psi_{jk}}$ is a $\mathcal{C}^{\infty}$-function which vanishes at infinity. Define

$$g_{jk}(\xi) = \frac{\overline{\widehat{\psi_{jk}}(\xi)}}{|\xi|^{H+1/2}}, \quad \xi \in \mathbb{R}\backslash\{0\}.$$

Since ψ satisfies (2.10), g is a square integrable function and

$$\alpha_{jk} = \int_{\mathbb{R}} \frac{\overline{\widehat{\psi_{jk}}(\xi)}}{|\xi|^{H+1/2}} L(d\xi)$$

is well defined. Then, using the isometry property (2.4) and the independence of the random measures $M\,(d\xi)$ and $W\,(d\xi)$, one easily proves owing to the Fubini Theorem that

$$\mathbb{E}\left(|w_{jk} - \alpha_{jk}|^2 \right) = 0,$$

which leads to the first statement.
• By definition of ψ_{jk},

$$w_{jk} = 2^{-j/2} \int_{\mathbb{R}} \frac{\overline{\widehat{\psi}(2^{-j}\xi)}}{|\xi|^{H+1/2}} \exp\left(2^{-j}ik\xi \right) L(d\xi).$$

Recall that the control measure $\nu(dz)$ is invariant by rotation. Then, (2.2) and (2.3) gives

$$w_{jk} \stackrel{(\mathcal{D})}{=} 2^{-j/2} \int_{\mathbb{R}} \frac{\overline{\widehat{\psi}(2^{-j}\xi)}}{|\xi|^{H+1/2}} L\,(d\xi) = w_{j0}.$$

• Since ψ has $r \geq 2$ vanishing moments,

$$\int_{\mathbb{R}} t\psi(t)dt = 0,$$

and then, as $\xi \to 0$, $\widehat{\psi_{jk}}(\xi) = O\left(\xi^2\right)$ and $g_{jk} \in L^p\,(\mathbb{R})$ for every $p \geq 2$. The wavelet type coefficient w_{jk} has moments of every order in view of Proposition 2.2 in Benassi et al. (2002). We first point out that by definition of $L\,(d\xi)$, w_{jk} is a real-valued symmetric random variable which implies

$$\forall p \in \mathbb{N},\, \mathbb{E}\left(w_{jk}^{2p+1} \right) = 0.$$

In order to obtain the asymptotic behaviour of $\mathbb{E}\left(w_{jk}^{2p}\right)$ as $j \to +\infty$, we apply Proposition 2.2 in Benassi et al. (2002) which links this moment to the deterministic L^{2q}-norms

$$\|g_{jk}\|_{2q}^{2q} = \int_{\mathbb{R}} |g_{jk}(\xi)|^{2q} d\xi = 2^{j(1-2q-2qH)} \|g_{00}\|_{2q}^{2q}.$$

Let us first assume that $L(d\xi) = M(d\xi)$. Then Proposition 2.2 in Benassi et al. (2002) leads to

$$\mathbf{E}(w_{jk}^{2p}) = 2^{-2pj-2pH} \sum_{m=1}^{p} 2^{jm}(2\pi)^m \sum_{L_m} \prod_{q=1}^{m} \frac{(2l_q)!\|g_{00}\|_{2l_q}^{2l_q} \int_0^{+\infty} \rho^{2l_q} \nu_\rho(d\rho)}{(l_q!)^2},$$

where $\sum_{L_m}$ stands for the sum over the set of partitions L_m of $\{1, \ldots, 2p\}$ in m subsets K_q such that the cardinality of K_q is $2l_q$ with $l_q \geqslant 1$. Therefore, as $j \to +\infty$,

$$2^{pj+2pH}\mathbf{E}(w_{jk}^{2p}) \to (2\pi)^p \sum_{L_p} \prod_{q=1}^{p} \frac{(2l_q)!\|g_{00}\|_{2l_q}^{2l_q} \int_0^{+\infty} \rho^{2l_q} \nu_\rho(d\rho)}{(l_q!)^2}. \qquad (2.11)$$

Note that by definition of $\sum_{L_p}$, in (2.11), for each q, $l_q = 1$, which leads to

$$2^{pj+2pH}\mathbf{E}(w_{jk}^{2p}) \to (2\pi)^p 2^p \|g_{00}\|_2^{2p} \left(\int_0^{+\infty} \rho^2 \nu_\rho(d\rho)\right)^{2p} \mathrm{card}(L_p).$$

Hence, as $j \to +\infty$,

$$2^{pj+2pH}\mathbf{E}(w_{jk}^{2p}) \to \mathrm{card}(L_p)\left(\mathbb{E}\left(w_{00}^2\right)\right)^p.$$

Since $\mathrm{card}(L_p) = (2p)!/(2^p p!)$, as $j \to +\infty$,

$$2^{pj+2pH}\mathbf{E}(w_{jk}^{2p}) \to \mathbb{E}\left(N_0^{2p}\right).$$

Therefore we have proved the third statement in the case where $L(d\xi) = M(d\xi)$. Note that this statement is evident when $L(d\xi) = W(d\xi)$. Hence, the independence between $W(d\xi)$ and $M(d\xi)$ gives the conclusion for any Lévy random measure $L(d\xi) = aM(d\xi) + bW(d\xi)$.

- By stationarity (see second statement), we can assume $k = 0$. Equations (2.2) and (2.3) gives the characteristic function of w_{j0}. A simple change of variable ($\lambda = 2^{-j}\xi$) and a dominated convergence argument leads to the conclusion.

$$\square$$

Hence a RHFLM is indexed by the single parameter H. It controls both the correlation structure of the process and the smoothness of its sample paths. In the two following sections, we define a moment based estimator and present main results concerning its asymptotic behaviour in both cases where the random process is observed directly or in a regression framework.

3. Estimation procedure and properties without observation noise

The moment based estimator derives from the observation that the statistical second order moment of the wavelet type coefficients w_{jk} of a RHFLM obey a certain scaling property. Namely, we get

$$\mathbf{E}(w_{jk}^2) = 2^{-j(1+2H)}C_H^2, \tag{3.1}$$

where C_H is given by

$$
\begin{aligned}
C_H^2 &= \left(a^2 + b^2 \int_{\mathbb{C}} |z|^2 \, \nu\,(dz)\right) \int_{\mathbb{R}} \frac{|\widehat{\psi}(\lambda)|^2}{|\lambda|^{1+2H}} d\lambda \\
&= A \int_{\mathbb{R}} \frac{|\widehat{\psi}(\lambda)|^2}{|\lambda|^{1+2H}} d\lambda \neq 0,
\end{aligned}
$$

since ψ is a continous function with $\mathrm{supp}\psi \neq \emptyset$.

If we observe a continous path of the RHFLM, hence it is possible to compute directly the true wavelet type coefficients of the random process. However, as in Bardet (2002), we also tackle the case where the RHFLM is observed at discrete times, which induces in the estimation, a discretization error. In both cases, we construct a consistent estimator.

3.1. *Continuous time observations without noise.*

Suppose we observe the wavelet type coefficients of a RHFLM with unknown Hurst exponent. The coefficients are defined as a growing array of random variables

$$w_{jk}, \ k = 0, \ldots, 2^j - 1, \ j = 1, \ldots, J,$$

where J is the maximum number of levels in the wavelet type expansion.

Theorem 3.1 (Consistency of moments wavelet type estimator)**.** *Consider the Hurst exponent estimator of a RHFLM defined as*

$$\hat{H}_J = \frac{-1}{2J} \log_2 \left(\sum_{k=0}^{2^J - 1} w_{Jk}^2 \right). \tag{3.2}$$

Provided ψ has $r \geq 2$ vanishing moments, the following asymptotics holds

$$\hat{H}_J \overset{J \to +\infty}{\longrightarrow} H, \ a.s.$$

Proof. Define for all $J > 0$

$$V_J = \sum_{k=0}^{2^J - 1} w_{Jk}^2, \quad \tilde{V}_J = 2^{2JH}V_J.$$

Hence,

$$\mathbf{E}(V_J) = 2^{-2JH}C_H^2, \quad \mathbf{E}\left(\tilde{V}_J\right) = C_H^2 = \mathbf{E}\left(w_{00}^2\right).$$

Applying Formula 20 in Benassi et al. (2002),

$$\mathrm{Var}(V_J) = 2B2^{-J-4JH} \int \frac{|\widehat{\psi}(\lambda)|^4}{|\lambda|^{2+4H}} d\lambda$$

$$+ 2A^2 2^{-J-4H} \sum_{l=-2^J+1}^{2^J-1} \left(\int \frac{e^{il\lambda}|\widehat{\psi}(\lambda)|^2}{|\lambda|^{1+2H}} d\lambda \right)^2$$

$$+ 2^{-2J-4JH} B \sum_{l=-2^J+1}^{2^J-1} \int \frac{e^{2il\lambda}|\widehat{\psi}(\lambda)|^4}{|\lambda|^{2+4H}} d\lambda$$

$$= (I) + (II) + (III),$$

with $A = 4a^2 \int_{\mathbb{C}} |z|^2 \, \nu(dz) + b^2$ and $B = 4a^2 \int_{\mathbb{C}} |z|^2 \, \nu(dz)$. For the second term, we first point out that $\widehat{\psi}$ is continuously differentiable. Then using that

$$\lim_{|\lambda| \to +\infty} \widehat{\psi}(\lambda) = 0$$

and $\widehat{\psi}(0) = 0$, partial integration yields that

$$\int \frac{e^{il\lambda}|\widehat{\psi}(\lambda)|^2}{|\lambda|^{1+2H}} d\lambda = \frac{-1}{il} \int e^{il\lambda} \frac{d}{d\lambda} \left(\frac{|\widehat{\psi}(\lambda)|^2}{|\lambda|^{1+2H}} \right) d\lambda.$$

Hence we can conclude that

$$\left| \int \frac{e^{il\lambda}|\widehat{\psi}(\lambda)|^2}{|\lambda|^{1+2H}} d\lambda \right|^2 \leqslant \frac{c}{1+l^2},$$

for c a given finite positive constant and every $\lambda \in \mathbb{R}$.

Let us recall that $\int t\psi(t)dt = 0$, so that $\widehat{\psi}(\xi) = O\left(|\xi|^2\right)$ as $\xi \to 0$. The third term is handled the same way (owing two integrations by parts), which enables us to conclude that

$$\left| \int \frac{e^{2il\lambda}|\widehat{\psi}(\lambda)|^4}{|\lambda|^{2+4H}} d\lambda \right| \leqslant \frac{c}{1+l^2}, \ \forall l \in \mathbb{R}.$$

Finally, we obtain that the order of the variance is given by the terms (I) and (II) so

$$\mathrm{Var} V_J \sim D_H 2^{-J-4JH},$$

with

$$0 < D_H = 2B \int_{\mathbb{R}} \frac{|\widehat{\psi}(\lambda)|^4}{|\lambda|^{2+4H}} d\lambda + 2A^2 \sum_{l=-\infty}^{\infty} \left(\int_{\mathbb{R}} \frac{e^{il\lambda}|\widehat{\psi}(\lambda)|^2}{|\lambda|^{1+2H}} d\lambda \right)^2$$

$$= 2B \int_{\mathbb{R}} \frac{|\widehat{\psi}(\lambda)|^4}{|\lambda|^{2+4H}} d\lambda + 4A^2 \sum_{l=1}^{\infty} \left(\int_{\mathbb{R}} \frac{\cos(l\lambda)|\widehat{\psi}(\lambda)|^2}{|\lambda|^{1+2H}} d\lambda \right)^2 + 2C_H^4 < +\infty.$$

This implies after renormalization that

$$\mathrm{Var} \tilde{V}_J \overset{+\infty}{\sim} D_H 2^{-J} \quad => \sum_{J=1}^{+\infty} \mathrm{Var} \tilde{V}_J < +\infty.$$

The Borel Cantelli lemma yields that

$$2^{2JH} V_J \overset{J \to +\infty}{\longrightarrow} \mathbf{E}(w_{00}^2) \neq 0, \ a.s.$$

Finally we obtain the result

$$\frac{-1}{2J}\log_2(V_J) \stackrel{J\to+\infty}{\longrightarrow} H \ a.s.$$

$\square$

3.2. *Discretized version.*

Suppose we observe $X_H(l/N)$, $l = 0\ldots N$. The size of data is then $N+1$ and the wavelet type coefficients w_{jk} can not be evaluated. However, the discretized coefficients

$$w_{jk}^n = \frac{1}{2^{j/2}n}\sum_{p=1}^{n} X_H\left(\frac{p+nk}{2^j n}\right)\psi(\frac{p}{n}), \quad k = 0\ldots 2^j - 1, \quad j = 1\ldots J$$

can be computed taking $n = n(j)$ such that $2^{-j}N/n(j) \in \mathbb{N}$. Also, we replace w_{Jk} by w_{Jk}^n in (3.2) and then obtain a new estimator which can be computed. Next theorem gives its consistency. Note that the parameter of discretization n depends on j and that the minimal size of data we need is $2^J n(J) + 1$. In the following, we then take $N = 2^J n(J)$.

Theorem 3.2. *Consider the Hurst exponent estimator of a RHFLM defined as*

$$\hat{H}_J = \frac{-1}{2J}\log_2\left(\sum_{k=0}^{2^J-1}(w_{Jk}^n)^2\right), \tag{3.3}$$

with $n = n(J) = 2^J$ and with size of data $N + 1 = 2^{2J} + 1$. Provided ψ has $r \geq 2$ vanishing moment, the following asymptotics holds

$$\hat{H}_J \stackrel{J\to+\infty}{\longrightarrow} H, \ a.s.$$

Before proving Theorem 3.2, we state a result about the difference $w_{jk} - w_{jk}^n$ in norm L^2. In particular, a rate of convergence of w_{jk}^n in norm L^2 is given for each fixed j and k. Moreover, from the result established, we will deduced a comparison between the estimators defined by (3.2) and (3.3) and how to link n and J in (3.3), and then the size of data $N + 1$ and J, in order to obtain the asymptotic behaviour of (3.3).

Lemma 3.3. *There exists a constant $C > 0$ such that for every (j, k, n), $j \in \mathbb{N}\backslash\{0\}$, $k = 0\ldots 2^j - 1$, $n \in \mathbb{N}\backslash\{0\}$,*

$$2^{2jH}\mathbf{E}\left[\left(w_{jk} - w_{jk}^n\right)^2\right] \leq C\left(\frac{1}{2^j n^{2H}} + \frac{2^{2jH}}{2^j n^2}\right).$$

Proof of Lemma 3.3. Let $f_H(x,\xi) = \dfrac{e^{-ix\xi} - 1}{|\xi|^{H+1/2}}$ and

$$h_n(k,\xi) = \int_0^1 f_H(u+k,\xi)\psi(u)\,du - \frac{1}{n}\sum_{p=1}^{n} f_H\left(\frac{p}{n} + k, \xi\right)\psi\left(\frac{p}{n}\right).$$

Since

$$w_{jk} - w_{jk}^n = 2^{-j/2}\int_{\mathbb{R}} h_n\left(k, 2^{-j}\xi\right) L(d\xi),$$

in view of (2.4),

$$\mathbb{E}\left[\left(w_{jk} - w_{jk}^n\right)^2\right] = 2^{-j}A \int_{\mathbb{R}} \left|h_n\left(k, 2^{-j}\xi\right)\right|^2 d\xi$$
$$= 2^{-j-2jH}A \int_{\mathbb{R}} |h_n(k,\lambda)|^2 d\xi.$$

Then let us write

$$h_n(k,\lambda) = \sum_{p=1}^n h_{n,p}(k,\lambda),$$

with

$$h_{n,p}(k,\lambda) = \int_{(p-1)/n}^{p/n} \left(f_H(u+k,\lambda)\psi(u) - f_H\left(\frac{p}{n}+k,\lambda\right)\psi\left(\frac{p}{n}\right)\right) du.$$

By the Minkowski inequality,

$$2^{2jH}\mathbf{E}\left[\left(w_{jk} - w_{jk}^n\right)^2\right] \leq 2^{-j}A\left(\sum_{p=1}^n \|h_{n,p}(k,\cdot)\|_{L^2(\mathbb{R})}\right)^2. \qquad (3.4)$$

Moreover, the Cauchy-Schwarz inequality implies that

$$|h_{n,p}(k,\lambda)|^2 \leq \frac{1}{n} \int_{(p-1)/n}^{p/n} \left|f_H(u+k,\lambda)\psi(u) - f_H\left(\frac{p}{n}+k,\lambda\right)\psi\left(\frac{p}{n}\right)\right|^2 du.$$

Therefore, applying the Minkowski inequality, we obtain:

$$\|h_{n,p}(k,\cdot)\|_{L^2(\mathbb{R})} \leq A_{n,p}(k) + B_{n,p}(k),$$

where

$$A_{n,p}(k) = \frac{1}{\sqrt{n}}\left(\int_{(p-1)/n}^{p/n} \left\|f(u+k,\cdot) - f\left(\frac{p}{n}+k,\cdot\right)\right\|_{L^2(\mathbb{R})}^2 \psi^2\left(\frac{p}{n}\right) du\right)^{1/2}$$

and

$$B_{n,p}(k) = \frac{1}{\sqrt{n}}\left(\int_{(p-1)/n}^{p/n} \|f(u+k,\cdot)\|_{L^2(\mathbb{R})}^2 \left|\psi(u) - \psi\left(\frac{p}{n}\right)\right|^2 du\right)^{1/2}.$$

Let us first point out that

$$\left\|f(u+k,\cdot) - f\left(\frac{p}{n}+k,\cdot\right)\right\|_{L^2(\mathbb{R})}^2 = \left\|f\left(u - \frac{p}{n},\cdot\right)\right\|_{L^2(\mathbb{R})}^2$$
$$= \left|u - \frac{p}{n}\right|^{2H} \|f(1,\cdot)\|_{L^2(\mathbb{R})}^2.$$

As a result, there exists a finite constant M_1 such that

$$A_{n,p}(k) = \frac{M_1}{n^{H+1}}\left|\psi\left(\frac{p}{n}\right)\right|,$$

for every n, p, j and k.

In addition,

$$B_{n,p}(k) = \frac{1}{\sqrt{n}}\|f(1,\cdot)\|_{L^2(\mathbb{R})}\left(\int_{(p-1)/n}^{p/n} |u+k|^{2H}\left|\psi(u) - \psi\left(\frac{p}{n}\right)\right|^2 du\right)^{1/2}.$$

Then owing to a Taylor expansion, one proves that there exists a constant M_2 such that for every n, p, j and k,

$$B_{n,p}(k) \le \frac{M_2 \, (k+1)^H}{n^2} \sup_{(p-1)/n \le x \le p/n} |\psi'(x)| \le \frac{2^{jH} M_2}{n^2} \sup_{(p-1)/n \le x \le p/n} |\psi'(x)|.$$

Then, in view of (3.4),

$$2^{jH} \mathbb{E}\Big[\big(w_{jk} - w_{jk}^n\big)^2\Big]$$

$$\le 2^{-j} A \left(\frac{M_1}{n^{1+H}} \sum_{p=1}^{n} \left|\psi\Big(\frac{p}{n}\Big)\right| + \frac{2^{jH} M_2}{n^2} \sum_{p=1}^{n} \sup_{(p-1)/n \le x \le p/n} |\psi'(x)| \right)^2.$$

We then point out that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{n} \left|\psi\Big(\frac{p}{n}\Big)\right| = \int_0^1 |\psi|(u)\, du < +\infty$$

and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{n} \sup_{(p-1)/n \le x \le p/n} |\psi'(x)| = \int_0^1 |\psi'|(u)\, du < +\infty,$$

which concludes the proof. □

Let us now prove Theorem 3.2.

Proof of Theorem 3.2. Let

$$\tilde{V}_{J,n} = 2^{2JH} \sum_{k=0}^{2^J - 1} (w_{Jk}^n)^2.$$

We compare $\tilde{V}_{J,n}$ to $\tilde{V}_J$ defined in proof of Theorem 3.1. Define

$$\tilde{D}_{J,n} = \sqrt{\tilde{V}_{J,n}} - \sqrt{\tilde{V}_J}.$$

Since

$$\left|\tilde{D}_{J,n}\right|^2 \le 2^{2JH} \sum_{k=0}^{2^J - 1} (w_{Jk} - w_{Jk}^n)^2,$$

by lemma 3.3,

$$\mathbb{E}\left[\left|\tilde{D}_{J,n}\right|^2\right] \le C \left(\frac{1}{n^{2H}} + \frac{2^{2JH}}{n^2} \right),$$

where C is a finite constant. Taking $n = n(J) = 2^J$, the Borel Cantelli lemma yields that as $J \to +\infty$,

$$\tilde{D}_{J,2^J} \to 0, a.s.$$

Recall that , as $J \to +\infty$,

$$\tilde{V}_J \to \mathbb{E}\big(w_{00}^2\big) \quad a.s.$$

Then,

$$\tilde{V}_{J,2^J} \to \mathbb{E}\big(w_{00}^2\big) \quad a.s,$$

which leads to the convergence of $\hat{H}_J$ defined by (3.3). □

4. Regression framework

Suppose we observe noisy data from a RHFLM X_H

$$Y_H\left(\frac{l}{N}\right) = X_H\left(\frac{l}{N}\right) + \sigma_N\varepsilon_l,\ l = 0,\ldots,N, \tag{4.1}$$

where ε_l are an i.i.d sample of Gaussian random variables $\mathcal{N}(0,1)$ and σ_N is the noise level. We point out that $\sigma_N = O(N^{-\frac{1}{2}})$. This framework is the usual regression setting well studied in statistics, which models natural observations of a phenomenom, observed at discrete times.

The discretized coefficients w_{jk}^n of X_H can not be observed. However, the discretized coefficients of Y_H

$$d_{jk}^n = \frac{1}{2^{j/2}n}\sum_{p=1}^{n}Y_H\left(\frac{p+nk}{2^jn}\right)\psi(\frac{p}{n}),\quad k = 0\ldots2^j-1,\quad j = 1\ldots J,$$

can be computed, taking $N = 2^{2J}$ and $n = n(j)$ such that $2^{-j}N/n(j) \in \mathbb{N}$ as in Section 3.

Theorem 4.1. *Consider the estimator of the Hurst coefficient of a RHFLM defined as*

$$\hat{H}_J = \frac{-1}{2J}\log_2\left(\sum_{k=0}^{2^J-1}(d_{Jk}^n)^2\right), \tag{4.2}$$

with $n = n(J) = 2^J$ and $N = 2^{2J}$. Provided ψ has at least $r \geq 2$ vanishing moments, the following asymptotics holds

$$\hat{H}_J \overset{J\to+\infty}{\Longrightarrow} H,\ a.s.$$

Proof. We proceed as in the proof of Theorem 3.2. Let us recall that

$$\tilde{V}_{J,n} = 2^{2JH}\sum_{k=0}^{2^J-1}(w_{Jk}^n)^2$$

and define

$$\tilde{W}_{J,n} = 2^{2JH}\sum_{k=0}^{2^J-1}(d_{Jk}^n)^2.$$

Then,

$$\mathbb{E}\left(\left|\sqrt{\tilde{W}_{J,n}} - \sqrt{\tilde{V}_{J,n}}\right|^2\right) \leq 2^{2JH}\sum_{k=0}^{2^J-1}\mathbb{E}\left[(d_{Jk}^n - w_{Jk}^n)^2\right].$$

By definition of Y_H,

$$d_{Jk}^n - w_{Jk}^n = \frac{1}{2^{J/2}n}\sigma_N\sum_{p=1}^{n}\psi\left(\frac{p}{n}\right)\varepsilon_{p+nk} \sim \mathcal{N}\left(0,\frac{\sigma_N^2}{2^Jn^2}\sum_{p=1}^{n}\psi^2\left(\frac{p}{n}\right)\right).$$

Hence,

$$\mathbb{E}\left(\left|\sqrt{\tilde{W}_{J,n}} - \sqrt{\tilde{V}_{J,n}}\right|^2\right) \leq \frac{2^{2JH}\sigma_N^2}{n^2}\sum_{p=1}^{n}\psi^2\left(\frac{p}{n}\right).$$

Since $\lim_{n\to+\infty} \frac{1}{n}\sum_{p=1}^{n} \psi^2\left(\frac{p}{n}\right) = \int_{\mathbb{R}} \psi^2(u)du = 1$, there exists a constant $C \in (0,+\infty)$ such that for every J and n,

$$\mathbb{E}\left(\left|\sqrt{\tilde{W}_{J,n}} - \sqrt{\tilde{V}_{J,n}}\right|^2\right) \leq \frac{2^{2JH}C\sigma_N^2}{n}.$$

Also, taking $n = n(J) = 2^J$, the Borel Cantelli Lemma yields that

$$\sqrt{\tilde{W}_{J,2^J}} - \sqrt{\tilde{V}_{J,2^J}} \overset{J\to+\infty}{\longrightarrow} 0 \quad \text{a.s.},$$

as soon as $\sigma_N = O(N^{-1/2})$ with $N = 2^{2J}$. Then, since (see proof of Theorem 3.2),

$$\tilde{V}_{J,2^J} \overset{J\to+\infty}{\longrightarrow} \mathbb{E}\left(w_{00}^2\right) \quad \text{a.s.},$$

we finally obtain that

$$\hat{H}_J = -\frac{1}{2^J}\log_2\left(2^{-2JH}\tilde{W}_{J,2^J}\right) \overset{J\to+\infty}{\longrightarrow} H \quad \text{a.s.},$$

as soon as $\sigma_N = O(N^{-1/2})$ with $N = 2^J n(J) = 2^{2J}$. $\qquad\square$

As a result, we obtain consistency of the estimators in the practical case where the RHFLM is observed at discrete times. This result can be used to build an estimator of the Hurst exponent when studying the outcome of an experiment though to behave like a RHFLM. Once this parameter is estimated it becomes possible to try to model the data by a realization of a RHFLM. Hence, it should be of interest to construct a test based on the estimator of the Hurst exponent. So the asymptotic distribution of the estimator is needed. However, such a result is very difficult to obtain. On the one hand, the nature of the observations prevents the use of Central Limit type theorems. On the other hand, evaluating the distance between the distribution of the coefficients of a RHFLM and a Gaussian distribution is also far beyond the scope of this paper due since the calculations can only be handled using (2.2), preventing the use of weak dependency theorems. Nevertheless a consistency result is a first step in a modeling attempt with RHFLM.

5. Numerical study

In this section, consider the issue of estimating the Hurst exponent of a RHFLM observed with a Gaussian white noise. In the simulations we present in Figure 5 and 5.2 simulated data in straight lines, obtained using the procedure described in Lacaux (2004b), together with the noisy paths in dotted lines.

First we aim at studying the rate of convergence of the wavelet type estimator. The estimator of the Hurst exponent is constructed using rescaled Daubechies wavelet $Db(4)$ with $r = 4$ vanishing moments. We construct the estimator of the Hurst exponent for the two previous RHFLMs with the two different values of the Hurst exponent $H = .4$ and $H = .7$. The data are dyadic $N = 2^{2J}$ and we increase the data from $J = 2$ to $J = 7$. For the noisy data, we perform 30 replications of the estimation procedure and plot the boxplots for the absolute error for different values of J in Figure 5 and Figure 5.4. In both cases, we can see that the convergence is achieved with $N > 2^8$ observations. Such study highlights consistency of the estimate. But, due to the relative slowness of the rate of convergence together with

the lack of an asymptotic distribution law, it is difficult to estimate the efficiency
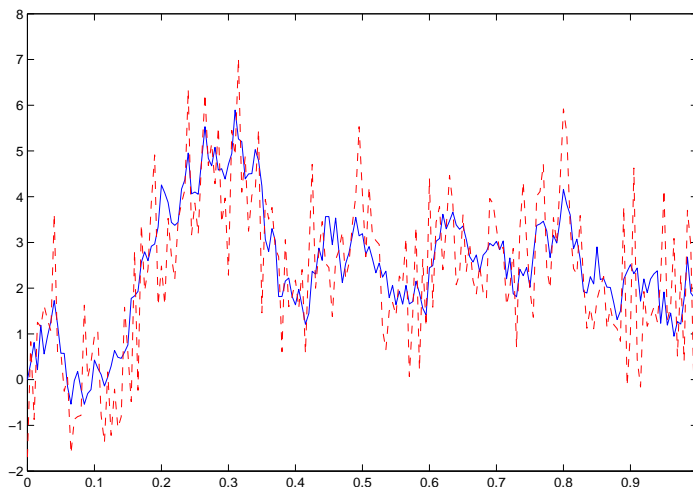of such an estimation procedure for real data.

FIGURE 5.1. Raw and Noisy Data with $H = .4$

However we can compare the efficiency of our estimator with the estimator given
in Benassi et al. (2002). We compute this estimator and the estimator constructed
in Section 3, when $H = .4$ for two cases, whether or not the RHFLM is observed
with a Gaussian white noise. We take $N = 2^{10}$ observations and consider 30
replications. We plot the distribution of both estimators, first the distribution of
the wavelet type estimator and then the distribution of the quadratic variation
based estimator. We observe in Figure 5.5 that without noise, the two estimators
give the same kind of results. But, as expected, in a noisy setup, in Figure 6,
the estimator based on quadratic variation does not concentrate around the true
value of the Hurst parameter. This result is also highlighted by the boxplot of the
absolute error of the two estimators, in Figure 5.7. The absolute error of estimation
is important in the noisy setup for the quadratic type estimator.

Finally, we study the relative influence in the choice of the mother wavelet by
computing the distribution of the absolute estimation error of the estimation of
$H = .4$ with noisy data for 30 simulations and two different sets of coefficients,
obtained for a rescaled Daubechies wavelet with $r = 4$ and for $r = 2$. We can see
in Figures 5.8 and Figures 5.9 that the distribution are similar. So as expected and
as usual in estimation with wavelet type estimators, the choice of the wavelet does
not play a particular role in the estimation as long as the initial assumptions are
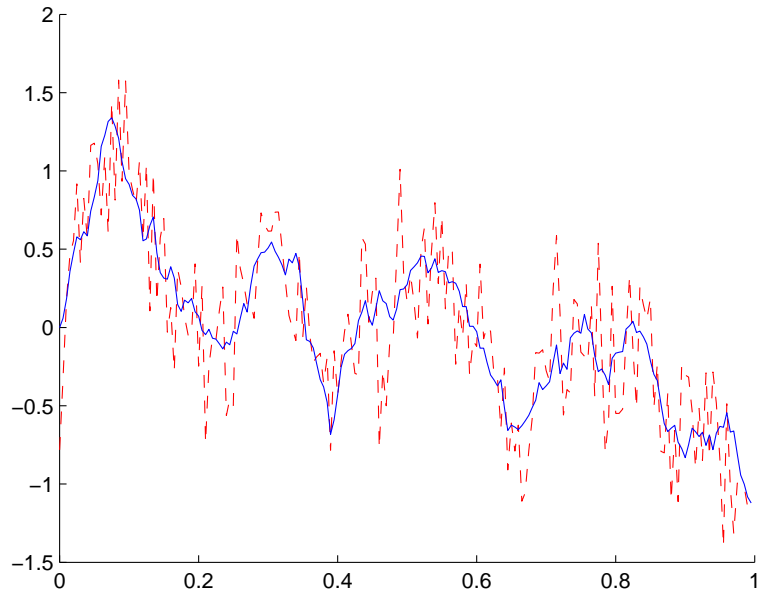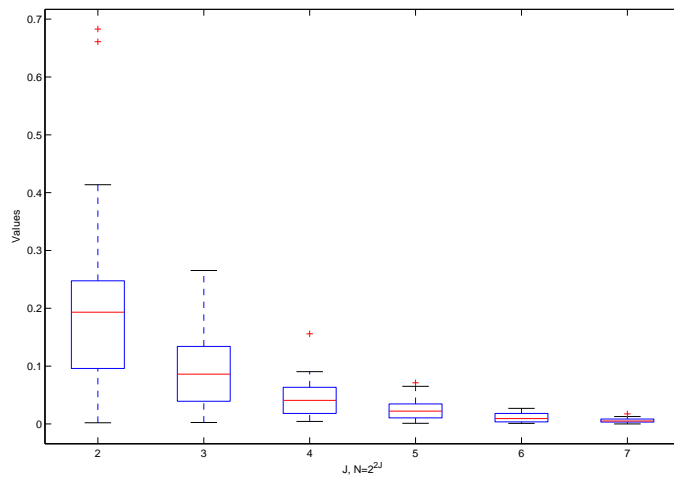met.

FIGURE 5.2. Raw and Noisy Data with $H = .7$

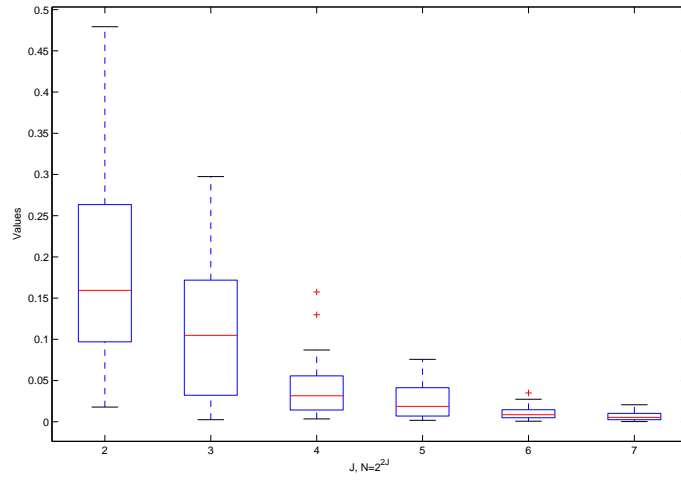FIGURE 5.3. Absolute Estimation Error for noisy data with $H = .4$

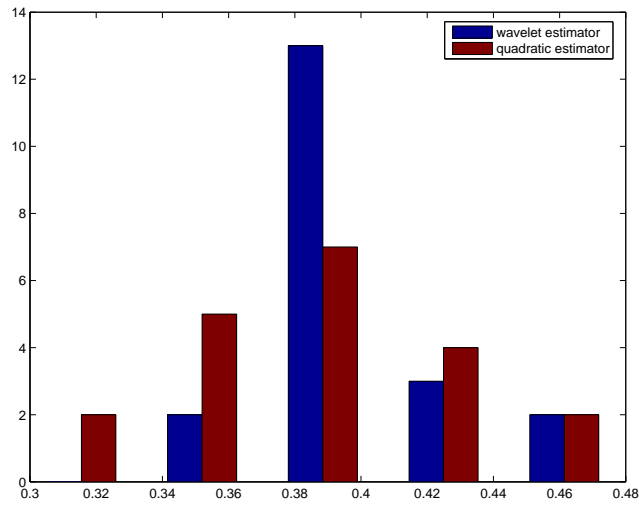FIGURE 5.4. Absolute Estimation Error for noisy data with $H = .7$

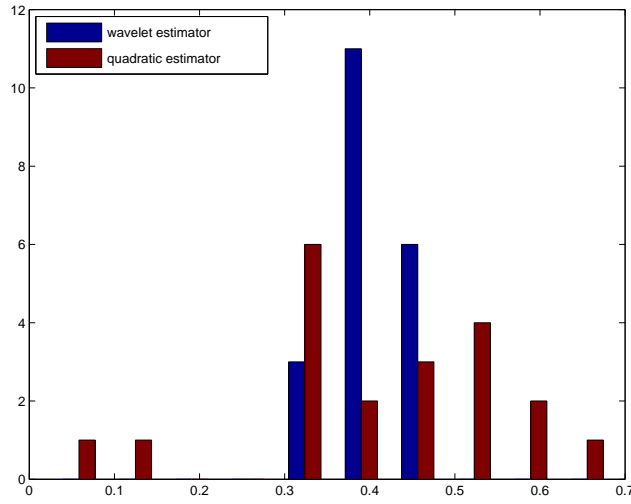FIGURE 5.5. Comparison of Estimators Distributions with true observations $(H = .4)$

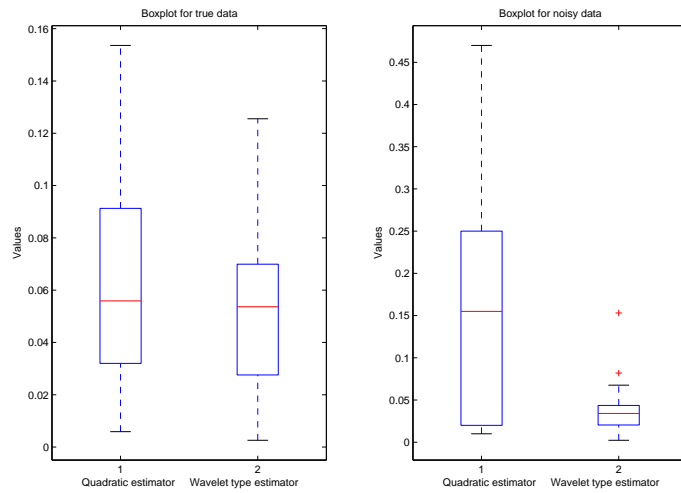FIGURE 5.6. Comparison of Estimators Distributions with noisy observations $(H = .4)$

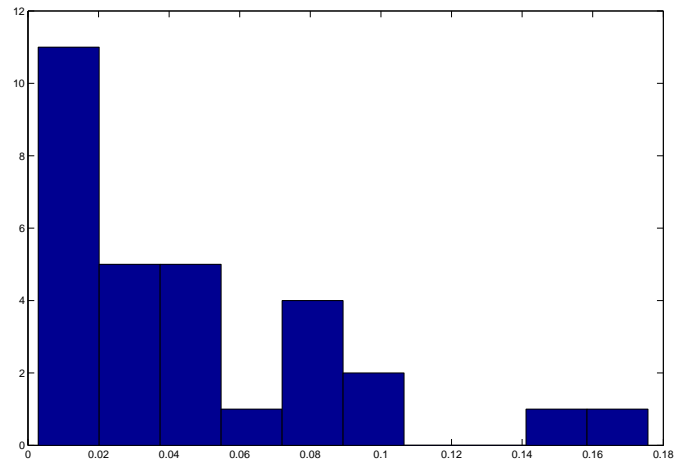FIGURE 5.7. Boxplot for quadratic variation estimator and wavelet type estimator $(H = .4)$

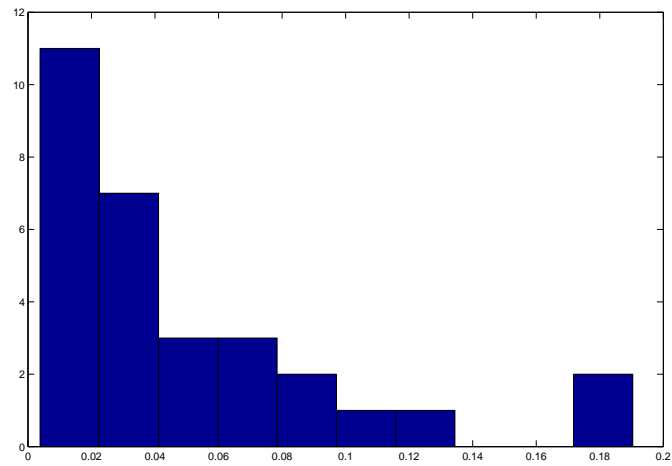FIGURE 5.8. Distribution of Absolute Error for noisy data with
$H = .4$ using Db(4)

FIGURE 5.9. Distribution of Absolute Error for noisy data with
$H = .4$ using Db(2)

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