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The Magnetic Anomaly of Two-Dimensional Sources Having Arbitrary Shape and Magnetization

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Abstract

Magnetic anomalies of sources with arbitrary two-dimensional boundary surfaces and magnetization distribution can be calculated using a combinaton of closed form and numerical integration. The surfaces and the magnetization may be functions of either the horizontal or the vertical coordinate. Layered sources can be modeled by summation algorithms of homogeneous models, or by using vertical magnetization polynomes and surface functions of the horizontal coordinate.

Key words: applied geophysics, magnetic modeling

1. Introduction

Calculation of magnetic and gravity anomalies generated by sources having arbitrary shapes and arbitrary distributions of physical properties is in general a complicated task. In fact, this problem can only be solved in a closed form for very simple combinations of geometric and physical parameters.

Of the publications concerning more complicated gravity sources reference should be made to *Murthy et al.* (1979) and *Guspi* (1990), who considered two-dimensional sources having depth-dependent densities and which were bounded by polygonal surfaces. *Talwani et al.* (1960) solved the anomalies of three-dimensional bodies by using a combination of numerical integration and closed form algorithms for horizontal lamina with variable density. *Murthy et al.* (1989) considered anomalies of three-dimensional sources whose densities vary as quadratic functions of depth and whose cross-sections are represented by series of polygons.

Algorithms for calculating magnetic anomalies, which are generally more complex than corresponding gravity anomalies, have been considered for example by *Hongisto* (1988) who calculated the anomaly of a dipping prism with linearly varying susceptibility in the vertical direction. *Shuey et al.* (1972) gave formulae for calculating the anomalies due to two-dimensional polygonal bodies whose magnetizations vary linearly both in the horizontal and vertical directions. The method given by *Talwani et al.* (1960) for gravity

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modeling is in principle also applicable to magnetic modeling. In three other papers (*Ruotoistenmäki* 1992, 1993 and 1994) I have considered methods for calculating anomalies generated by two- and three-dimensional gravity sources and three-dimensional magnetic sources using a combination of numerical integration and closed form formulas. The method can be considered to be a generalization of that given by *Talwani et al.* (1960). In this paper I shall present the analogous algorithms for complicated two-dimensional magnetic sources. The advantage of these algorithms, compared to those of three dimensional sources represented in *Ruotoistenmäki* (1993) is that they are much faster. Of course, the geometry of the sources must then be able to be approximated by two-dimensional models.

The algorithms given have been derived by applying the symbol mathematical "tool box" program Derive (*Rich et al.* 1989). With this program, the algorithms can also be easily worked up for the numerical calculation programs.

2. *Magnetic anomaly of a two-dimensional line double distribution*

The formula for the anomaly at point (x_c, y_c, $z_c = 0$) generated by a two-dimensional line double source can be integrated from the anomaly of three dimensional double source element *dx dy dz* located at point (*x, y, z*). Coordinates *x* and *y* are horizontal, and *z* is positive downwards.

The distance from the source point to the calculation point is:

$$R = \sqrt{(\propto^2 + \beta^2 + z^2)}\ ,$$

where:

$$\eth = x - x_c,\ \beta = y - y_c$$

The anomaly formula for the source element can now be expressed in the form (e.g. *Kunaratnam*, 1981):

$$\mathrm{dF}(x_c, y_c, 0) = I\,(f_1 + f_2 + f_3 + f_4 + f_5)\, d\!\propto d\beta\, dz, \tag{1}$$

where:

$$f_1 = A_1 \frac{\delta}{\delta\beta}\frac{\delta}{\delta z}\frac{1}{R} = A_1 \frac{3\,\beta\, z}{R^5}\ ,$$

$$f_2 = A_2 \frac{\delta}{\delta\!\propto}\frac{\delta}{\delta z}\frac{1}{R} = A_2 \frac{3 \propto z}{R^5}\ ,$$

$$f_3 = A_3 \frac{\delta}{\delta\alpha}\frac{\delta}{\delta\beta}\frac{1}{R} = A_3 \frac{3 \alpha \beta}{R^5},$$

$$f_4 = -A_4 \frac{\delta}{\delta\beta}\frac{\delta}{\delta\beta}\frac{1}{R} = A_4 \frac{z^2 + \alpha^2 - 2\beta^2}{R^5},$$

$$f_5 = -A_5 \frac{\delta}{\delta\alpha}\frac{\delta}{\delta\alpha}\frac{1}{R} = A_5 \frac{z^2 - 2\alpha^2 + \beta^2}{R^5},$$

where the magnetization is characterized by the parameters:
I = anomalous intensity of magnetization, and
$A_1 = M r + N q,$
$A_2 = L r + N p,$
$A_3 = L q + M p,$
$A_4 = N r - M q,$
$A_5 = N r - L p,$

L, M, N are the direction cosines of the magnetization, and p, q, r the direction cosines of the unperturbed field.

As in three-dimensional modeling, the effective susceptibility is assumed to be lower than 0.1 SI so that the magnetostatic interaction in the source can be neglected (see *Eskola et al.*, 1977, 1980).

Next, we choose the y-axis as being parallel to the strike and integrate Eq. (1) with respect to y:

$$dF(x_c,0) = \left[\lim_{Y \to \infty} I \int_{-Y}^{Y} (f_1 + f_2 + f_3 + f_4 + f_5)\, dy \right] d\alpha\, dz,$$

which expands to:

$$dF(x_c,0) = I \left[\frac{4 A_2 \alpha z}{(\alpha^2 + z^2)^2} + \frac{4 A_5 z^2}{(\alpha^2 + z^2)^2} - \frac{2 A_5}{\alpha^2 + z^2} \right] d\alpha\, dz. \qquad (2)$$

The anomaly algorithm for a two-dimensional line double source can now be obtained by integrating Eq. (2) with respect to x and z. This integral has a closed form solution if the integration limits (i.e. the source boundaries) and the magnetization distribution are of low degree. Otherwise, the integration must be performed numerically.

In the following equations the anomalies are calculated to a constant level of $z_c = 0$. In case of rugged topography the origo must be moved to each calculation point separately (i.e. all coordinates x_c, y_c, z_c are zero and the coordinates of the source points must be calculated relative to this point).

3. *Horizontally dependent surface and magnetization functions*

When the magnetization and the upper and lower surfaces of the source are arbitrary functions of the x-coordinate we get from Eq. (2):

$$F(x_c,0) = \int_{x_1}^{x_2} \int_{S_{h1}(x)}^{S_{h2}(x)} I_x(x) \left[\frac{4\,A_2 \propto z}{(\propto^2 + z^2)^2} + \frac{4\,A_5\, z^2}{(\propto^2 + z^2)^2} - \frac{2\,A_5}{\propto^2 + z^2} \right] dz\, dx, \tag{3}$$

where x_1 and x_2 are the vertical planes bounding the source, $I_x(x)$ is the function defining the source magnetization, and $S_{h1}(x)$ and $S_{h2}(x)$ are the functions defining the upper and lower surfaces of the source.

Integrating Eq. (3) with respect to z we get:

$$F(x_c,0) = 2 \int_{x_1}^{x_2} I_x\,(f_{sh2} - f_{sh1})\, dx, \tag{4}$$

where:

$$f_{shi}(i{=}1,2) = -\frac{A_5\, S_{hi} + A_2 \propto}{S_{hi}^2 + \propto^2}$$

In a general case this integral does not have a closed form solution and the integration must be made numerically.

The anomaly due to a layered model can be obtained from Eq. (4) as:

$$F(x_c,0) = 2 \int_{x_1}^{x_2} \sum_{n=1}^{m} I_n\ (f_{n-1} - f_n)\, dx, \tag{5}$$

where

$$f_n(x) = \frac{A_5\, S_n + A_2 \propto}{S_n^2 + \propto^2}$$

and $I_n(x)$ is the function defining the horizontal magnetization distribution of the n'th sub-layer, $S_0(x)$ is the uppermost surface and $S_n(x)$ ($n>0$) is the bottom of the n'th layer. From Eq. (3) it can be seen that also the direction cosine parameters A_5 and A_2 can be defined as arbitrary functions of the x-coordinate.

4. *Horizontally dependent surfaces and vertically dependent magnetization functions*

When the surfaces are functions of the horizontal x-coordinate and the intensity of magnetization is a fifth degree polynome of depth, we first define the magnetization polynome, which describes the variation of magnetization from zero level:

$$I_z(z) = A\,z^5 + B\,z^4 + C\,z^3 + D\,z^2 + E\,z + F \tag{6}$$

Equation (3) can now be written in the form:

$$F(x_c,0) = \int_{x_1}^{x_2} \int_{S_{h1}(x)}^{S_{h2}(x)} I_z(z) \left[\frac{4\,A_2 \propto z}{(\propto^2 + z^2)^2} + \frac{4\,A_5\,z^2}{(\propto^2 + z^2)^2} - \frac{2\,A_5}{\propto^2 + z^2} \right] dz\,dx, \tag{7}$$

from which, after integration with respect to z, we get the anomaly algorithm:

$$F(x_c,0) = \int_{x_1}^{x_2} (f_{sh2} - f_{sh1})\,dx, \tag{8}$$

where:

$$f_{shi}\,(i=1,2) = f_{1i}\,(f_{2i} + f_{3i} + f_{4i} + f_{5i} + f_{6i}),$$

and:

$$f_{1i} = \frac{1}{12 \propto^3 (\propto^2 + S_{hi}^2)},$$

$$f_{2i} = 6\,(2 \propto^2 m_4 + m_2)\,(\propto^2 + s_{hi}^2)\,\mathrm{ATAN}\left[\frac{S_{hi}}{\propto}\right],$$

$$f_{3i} = \propto (\propto^2 m_3\,(\propto^2 + S_{hi}^2)\,\mathrm{LN}\,((\propto^2 + S_{hi}^2)^6)),$$

$$f_{4i} = \propto (\propto^4 S_{hi}\,(3\,m_5\,S_{hi}^3 + 2\,(2\,m_6\,S_{hi}^2 + 3\,(m_7\,S_{hi} + 2\,m_8)))),$$

$$f_{5i} = \propto (\propto^2 (S_{hi}^3\,(3\,m_5\,S_{hi}^3 + 2\,(2\,m_6\,S_{hi}^2 + 3\,(m_7\,S_{hi} + 2\,m_8))) - 6\,m_1)),$$

$$f_{6i} = \propto (6\,m_2\,S_{hi}),$$

with:

$$m_1 = -4 \propto (A\, A_5 \propto^5 - A_2 (\propto^4 B - \propto^2 D + F) - A_5 \propto (\propto^2 C - E)) \,,$$

$$m_2 = -4 \propto^2 (A\, A_2 \propto^5 - A_2 \propto (\propto^2 C - E) + A_5 (\propto^4 B - \propto^2 D + F)) \,,$$

$$m_3 = 2\,(5\, A\, A_5 \propto^4 - 2\, A_2 \propto (2 \propto^2 B - D) + A_5 (E - 3 \propto^2 C)) \,,$$

$$m_4 = 2\,(6\, A\, A_2 \propto^5 - 2\, A_2 \propto (2 \propto^2 C - E) + A_5 (5 \propto^4 B - 3 \propto^2 D + F)) \,,$$

$$m_5 = 2\, A\, A_5 \,,$$

$$m_6 = 2\,(2\, A\, A_2 \propto + A_5 B) \,,$$

$$m_7 = -2\,(3\, A\, A_5 \propto^2 - 2\, A_2 \propto B - A_5 C) \,,$$

$$m_8 = -2\,(4\, A\, A_2 \propto^3 - 2\, A_2 \propto C + A_5 (3 \propto^2 B - D)) \,.$$

Eq. (8) must also be integrated numerically in x-direction.

When the magnetization distribution is dependent on the depth to the upper surface of a source which may also be folded in the vertical direction, we proceed as described in *Ruotoistenmäki* (1992, 1993 and 1994):

To represent the vertical distribution of magnetization in the folded strata, we first define the magnetization intensity function $I_{z2}(z')$ between normalized depth values z', which describes the magnetization of the strata between the upper ($z' = 0$) and the lower ($z' = 1$) surfaces in the nonfolded state:

$$I_{z2}(z') = A'\, z'^5 + B'\, z'^4 + C'\, z'^3 + D'\, z'^2 + E'\, z' + F' \qquad (9)$$

Assuming that during the folding of the strata, the ratio of vertical thicknesses of the various sub-layers remains constant (the total thickness can be variable) and substituting the normalized depth values z', we obtain from Eq. (9) the magnetization function $I_{z3}(z)$, describing the source magnetization at the real depth z:

$$I_{z3}(z) = A' \left[\frac{z - S_{h1}}{T_x}\right]^5 + B' \left[\frac{z - S_{h1}}{T_x}\right]^4 + C' \left[\frac{z - S_{h1}}{T_x}\right]^3 + D' \left[\frac{z - S_{h1}}{T_x}\right]^2 + E' \frac{z - S_{h1}}{T_x} + F' \qquad (10)$$

where $T_x(x) = S_{h2}(x) - S_{h1}(x)$ is the vertical thickness of the source at point x.

Equation (10) can be rearranged in the form:

$$I_{z3}(z) = A''\, z^5 + B''\, z^4 + C''\, z^3 + D''\, z^2 + E''\, z + F'' \qquad (11)$$

where:

$A'' = T_c A'$

$B'' = - T_c (5 A' S_{h1} - B' T_x)$

$C'' = T_c (10 A' S_{h1}^2 - 4 B' S_{h1} T_x + C' T_x^2)$

$D'' = - T_c (10 A' S_{h1}^3 - T_x (6 B' S_{h1}^2 - 3 C' S_{h1} T_x + D' T_x^2))$

$E'' = T_c (5 A' S_{h1}^4 - T_x (4 B' S_{h1}^3 - T_x (3 C' S_{h1}^2 - 2 D' S_{h1} T_x + E' T_x^2)))$

$F'' = - T_c (A' S_{h1}^5 - T_x (B' S_{h1}^4 - T_x (C' S_{h1}^3 - T_x (D' S_{h1}^2 - E' S_{h1} T_x + F' T_x^2))))$

and

$$T_c = \frac{1}{T_x^5}$$

The magnetic anomaly of the source can now be calculated from Eq. (8) by substituting the coefficients A through F by A'' through F''. If S_{h1} is negative (or null) and S_{h2} is positive we must substitute the value of S_{h1} by a small positive number after the coefficients A'' through F'' have been defined. If S_{h2} is also negative, all coefficients must be set to zero.

In the above equations, the magnetization polynomes may also be of higher degree, which naturally results in more complicated anomaly formulas.

It is also possible to define a two-dimensional magnetization function by multiplying Eq. (8) with a pre-defined function of the x-coordinate. This type of model can be used to represent e.g. folded sediments, whose magnetization distribution has been modified by regional metamorphism.

Also in Eq. (7) the direction cosine parameters A_5 and A_2 can be defined as arbitrary functions of the x-coordinate. Moreover, also they can be polynomes of z. In such a case the equations must be re-derived in same manner as described with I_{z2} above (resulting to relative complex equations, however).

5. *Vertically dependent surface and magnetization functions*

When the surfaces and magnetization are functions of the z-coordinate we obtain from eq (2):

$$F(x_c,0) = \int_{z_1}^{z_2} I_z(z) \int_{S_{e1}(z)}^{S_{e2}(z)} \left[\frac{4 A_2 \propto z}{(\propto^2 + z^2)^2} + \frac{4 A_5 z^2}{(\propto^2 + z^2)^2} - \frac{2 A_5}{\propto^2 + z^2} \right] dx\, dz\,, \qquad (12)$$

where z_1 and z_2 are the horizontal planes bounding the source, $I_z(z)$ is the function defining the magnetization distribution of the source, and $S_{e1}(z)$ and $S_{e2}(z)$ are the functions defining the left and right edges of the source.

After integration of Eq. (12) with respect to x the anomaly algorithm can be written in the form:

$$F(x_c,0) = 2 \int_{z_1}^{z_2} I_z \, (f_{se2} - f_{se1}) \, dz \, , \qquad (13)$$

where:

$$f_{sei}(i{=}1,2) = \frac{A_5 \, S_{ei} - A_2 \, z - A_5 \, x_c}{(S_{ei} - x_c)^2 + z^2}$$

For a general case, Eq. (13) must be calculated numerically.

Preliminary computations have shown that if the magnetization is a polynome of the x-coordinate, the algorithms resulting from Eq. (12) are so complicated that it is more practical to use Eq. (8) or the summation algorithm when calculating anomalies for dipping layered sources. The advantage of the summation model is that it allows the possibility of calculating anomalies due to sources that are folded in the x-direction.

The summation algorithm of the layered model can be obtained directly from Eq. (13):

$$F(x_c,0) = 2 \int_{z_1}^{z_2} \sum_{n=1}^{m} I_n \, (f_n - f_{n-1}) \, dz; \qquad (14)$$

where

$$f_n(z) = \frac{(A_5 \, S_n - A_2 \, z - A_5 \, x_c)}{(S_n - x_c)^2 + z^2}$$

and $I_n(z)$ is the function defining the vertical magnetization distribution of the n'th sub-layer, $S_0(z)$ is the leftmost edge and $S_n(z)$ (n>0) is the rightmost edge of the n'th layer.

6. An example

Figure 1 (a) shows the geometry of a five-layer source and the anomalies calculated using the multilayer sum algorithm from Eq. (5). The ratio of the vertical thicknesses of the sub-layers 1-5 (numbered in the figure) is 2:2:3:1:2. The functional form of the horizontal susceptibility distribution in the separate layers was kept invariant, as shown in Fig. 2. The amplitudes of the susceptibility function of the layers 1-5 have been multiplied by the factors 0.5, 2., 1., 3. and 1.5 respectively.

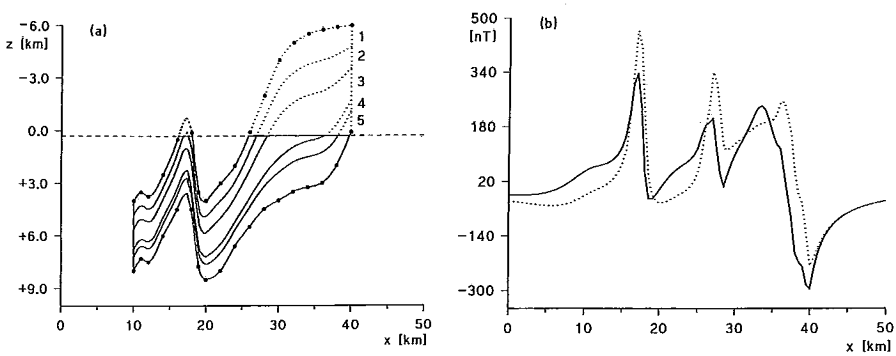

Fig. 1. The geometry (a) and magnetic anomalies (b) of the layer model. In (a) the outcropping parts of the source above the erosion level (dashed line) are shown with dotted lines. In (b) the solid line shows the anomaly of horizontally unhomogeneous source and the dotted line depicts that of the horizontally homogeneous source.

The upper and lower surfaces of the strata and the horizontal susceptibility distribution have been defined by spline interpolation (see e.g. *Press et al.*, 1990) fitted to the dotted points in figures 1(a) and 2. At $x =$ 19 km the source has been cut by a right-handed vertical fault. At depth values less than 0.3 km (the dashed line in the figure) the surfaces have been cut horizontally and the susceptibilities have been set to zero. The magnetic north is in the direction of the x-axis, and the intensity of the Earth's field was set at 50000 nT with an inclination of 75 degrees north. The remanence of the source is zero.

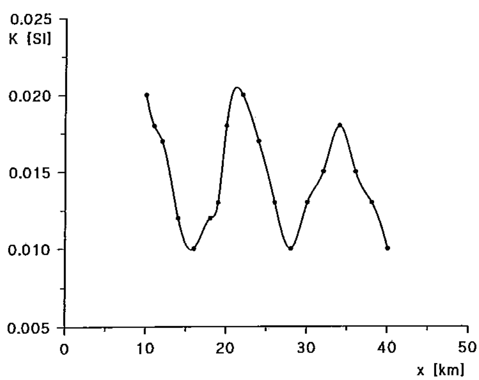

Fig. 2. The horizontal susceptibility distribution along the profile direction of the unhomogeneous layer model in Fig. 1 (a).

The anomalies were calculated by Simpson's extended numerical integration algorithm (*Press et al.* (1990). The solid line shows the anomaly for a 75 degree inclination and a variable horizontal susceptibility distribution. The dotted line shows the anomaly for a constant horizontal susceptibility distribution (0.015 SI) and a 90 degree inclination of the Earth's field.

From Fig. 1 it is apparent that the vertical fault and some of the layers are clearly indicated in the anomaly. Their form and position are strongly affected by the amount of inclination and the regional susceptibility distribution.

The calculation time of this model with a 99 MHz 486 PC was less than ten seconds. Thus, it is apparent that with modern and even faster PC processors and with sophisticated integration algorithms these two-dimensional models can be calculated in real time (i.e. within few seconds).

7. *Conclusions*

The magnetic anomalies of two-dimensional sources with arbitrary boundary surfaces and magnetization distribution can be calculated using a combination of closed form and numerical integration. The algorithms are simple and easy to apply for calculation programs. The surfaces and the magnetization parameters can be continuous functions, such as trigonometric functions or polynomes, in the source area, or they can be piecewise functions such as polygones or spline functions fitted to values defined by the user. The source types can be, for example, intrusions or sediments having horizontal and/or vertical magnetization distribution. In layered sources the magnetization can vary continuously or discontinuously between separate layers. Layered sources can be modeled by summation algorithms of homogeneous models, or by using vertical magnetization polynomes and surface functions of the horizontal coordinate.

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