Geophysica (1990), 26, 1, 1-28

A Study of Large-Scale Atmospheric Waves and the Response to External Forcing

A. Wiin-Nielsen

Geophysical Institute
University of Copenhagen
Haraldsgade 6, 2200 Copenhagen N, Denmark

Abstract

A nonlinear study of forced atmospheric waves is carried out. The nonlinear aspect is the interaction between the waves and the zonal current. The investigation consists of a baroclinic and a barotropic part.

Using a two-level, quasi-geostrophic model with boundary layer friction and internal friction depending on the vertical windshear it is demonstrated how the model behaves under varying intensity of the zonal differential heating.

A zonal steady state may exist. It is stable for a sufficiently small differential heating, but becomes unstable, if the heating exceeds a certain critical value, which depends on the wavelength except for sufficiently short waves, where the zonal steady state is stable in all cases.

Steady states containing finite amplitude waves may exist. They may be stable or unstable, and they may describe physical or unphysical states. The latter is a state with negative values for quantities, which in reality are positive definite. The physical steady states exist and are stable in exactly the region where the zonal steady states are unstable. The main conclusion is that for each wavelength and each value of the differential heating only one stable steady state exists.

The barotropic model contains six components to describe the flow. In addition there is external forcing and dissipation. Two components describe the zonal flow and four amplitude components describe the waves in such a way that nonlinear interactions take pace. The investigation uses the same methodology as in the baroclinic case. A complete analogy is obtained between the two cases in the sense that a zonal steady state may exist for a small forcing. For sufficiently large forcing the zonal state becomes unstable, but a stable steady state with waves may then exist.

Unphysical steady states appear due to the mathematical treatment, which replaces the original low-order system with a set of equations of higher order. False solutions are introduced in this way, and they would not appear in the original system.

1. *Introduction*

The study of atmospheric waves from a theoretical point of view has for several decades been dominated by a very large number of essentially linear stability studies. These investigations are to a large degree to be considered as a continuation of the original studies

Published by the Finnish Geophysical Society, Helsinki (1991)

by *Charney* (1947) and *Eady* (1949) who considered the stability of zonal currents to infinitesimal wave-type disturbances. While the most general cases, treated by the two authors, require knowledge of special functions and the incorporation of specific boundary conditions, it is true that these analytical studies in many later cases are being replaced by numerical studies, in which the eigenvalues of the proper matrix are determined by numerical methods. A recent example of the latter procedure is the investigation by *Kasahara* and *Tanaka* (1989).

Some nonlinear studies of the baroclinic stability problem have been made, for example by *Yang* (1967) and *Pedlosky* (1970). These investigations were concerned with the interaction between the zonal flow and the eddies for finite amplitude disturbances. The same problem has recently been considered by *Thompson* (1987) whose approach is quite different. He used a low-order model to investigate certain basic aspects of the response of a baroclinic flow to external heating and frictional dissipation. The nonlinear aspect of this study is the interaction between the waves and the zonal flow although the model is reduced to such a level that only the transport of sensible heat, but not the momentum transport can influence the zonal current. The investigation is, however, an example of the type of study in which the nonlinearity permits the determination of multiple steady states. The stability of these states can then be investigated. Similar nonlinear studies, dealing with the problem of blocking, were first carried out by *Wiin-Nielsen* (1979) and *Charney* and *DeVore* (1979), followed by numerous other studies using essentially the same technique. Examples are the vacillation study by *Lorenz* (1963), the orography studies by *Roads* (1980) and *Källén* (1982), the study of structural determinism by *Reinhold* (1986) and the blocking study by *Wiin-Nielsen* (1984).

The influence of heating and friction of the growth of baroclinic waves has among others been studied by *Wiin-Nielsen et al.* (1967), but the investigation was linear. However, it is mentioned here because it uses a procedure which is extended by *Thompson* (1987) to the nonlinear problem mentioned above. This procedure consists of replacing the basic equations for the problem with a new set of equations describing the development of certain zonally averaged quantities such as the heat transport by the waves, the eddy kinetic energy and others. While such a procedure does not change the physical processes incorporated in the model, it can introduce 'false solutions' which do not make sense from a physical point of view. However, the procedure makes the solution of the problem easier, and the dependent variables describe directly physical processes of interest.

Thompson (1987) prefers to incorporate frictional processes by using lateral diffusion for both momentum and heat. We shall in the following employ a description using a surface stress and a stress in the so-called free atmosphere proportional to the vertical windshear as was done by *Wiin-Nielsen et al.* (1967). The two different ways of including friction in the model have been discussed by *Charney* (1959) who definitely warns against using lateral diffusion. The treatment of frictional processes by lateral diffusion has some mathematical properties which makes the problem to be solved more symmetrical, and due to this symmetry one can go further using analytical methods.

The baroclinic model to be uses in this study will have a single baroclinic wave superimposed on a zonal current in a beta-plane channel. Restricting further the study to a two-level, quasi-geostrophic model we shall need four amplitudes in real numbers to describe the wave. The zonal currents at the two levels will be described by a single component, and we shall thus investigate the behaviour of a six component system, for which we shall formulate a consistent set of low-order equation. With the additional differences in the formulation of the frictional processes there are sufficient differences between our model and the one used by *Thompson* (1987) to make a comparative study interesting.

The most essential difference turns out to be that for sufficiently low values of the external heating we find only one steady state solution which is purely zonal and stable with respect to small disturbances, while *Thompson* (1987) finds three solution of which the two non-zonal solutions are unstable. For sufficiently high values of the heating both models have three steady state solution, but of these only one is stable. This solution is in both models characterized by a westerly flow in the middle of the channel, a northward transport of sensible heat, and positive values of the eddy kinetic energies. The possible zonal steady state is unstable in both models. The same is true for the unphysical steady state. However, the critical level of the differential heating which permits the non-zonal stable steady state is different in the two models which are difficult to compare in details due to the fact that *Thompson* (*loc.cit*) gives only non-dimensional results without specifying the numerical value used for the various parameters, including the value of the lateral diffusion coefficient.

The baroclinic model is constructed in such a way that it contains an eddy heat transport, but no eddy momentum transport. Both processes need of course to be present, but as a step toward this goal it is worthwhile to treat the purely barotropic case following essentially the same procedure as in the first case. For this purpose we have adopted a low order model formulated by *Wiin-Nielsen* (1961), but adding forcing and boundary layer friction to it.

While there is some reality in the dissipation mechanism, it is always difficult to formulate a realistic forcing for the barotropic flow, since in reality the forcing comes, at least in part, from the baroclinic flow, which is excluded in the model. We restrict our case to a constant forcing on each of the zonal components, but many other cases could have been selected. One could for example have used an equivalent barotropic model in which one makes use of the thermodynamic equation neglecting the horizontal advection. This choice would, however, lead to a model with heating and at the same time requiring that the direction of the horizontal wind should not change with height. Such an assumption is obviously unrealistic. As with all barotropic considerations the results are of interest more in principle than in practice.

2. *The baroclinic model*

The model will be the same as the one employed by *Charney* (1959). This means a standard two-level, quasi-geostrophic model with the curl of the surface stress proportional to the relative vorticity at 100 *kPa*. The curl of the internal stress becomes proportional to the thermal vorticity. Since the two information levels are at 25 and 75 *kPa* we shall for simplicity take the vorticity at 100 *kPa* to be 1/2 of the vorticity at 75 *kPa*. We may then proceed to give these well-known equations:

$$\frac{\partial \zeta_*}{\partial t} + \vec{V}_* \cdot \nabla \zeta_* + \vec{V}_T \cdot \nabla \zeta_T + \beta \upsilon_* = -\frac{\varepsilon}{4}(\zeta_* - \zeta_T) \tag{2.1}$$

$$\frac{\partial(\zeta_T - q^2\psi_T)}{\partial t} + \vec{V}_* \cdot \nabla(\zeta_T - q^2\psi_T) + \vec{V}_T \cdot \nabla \zeta_* + \beta \upsilon_T = \frac{\varepsilon}{4}(\zeta_* - \zeta_T) - \varepsilon_T \zeta_T - q^2 \frac{\kappa}{2f_0} Q$$

We shall adopt the following numerical values:

$$q^2 = \frac{2f_0^2}{\sigma p^2} = 4 \times 10^{-12}\, m^{-2}$$

$$\varepsilon = 4 \times 10^{-6}\, s^{-1}$$

$$\varepsilon_T = 1.2 \times 10^{-6}\, s^{-1} \tag{2.2}$$

$$\kappa = R/c_p = 0.286$$

$$f_0 = 10^{-4}\, s^{-1}$$

$$\beta = 1.6 \times 10^{-11}\, m^{-1}\, s^{-1}$$

ζ is the vorticity, $\vec{V}$ the horizontal wind, Q the heating per unit mass and unit time, and subscripts * and T are defined by the symbolic equations:

$$(\ )_* = 1/2\{(\ )_1 + (\ )_3\} \tag{2.3}$$

$$(\ )_T = 1/2\{(\ )_1 - (\ )_3\} \tag{2.4}$$

where the subscripts *1* and *3* refer to the 25 and 75 *kPa* levels, respectively.

For the low order model we define the two streamfunctions by the following expressions:

$$\psi_* = \frac{B_*}{2\lambda}\sin(2\lambda y) + \frac{E_*}{k}\sin(\lambda y)\sin(kx) + \frac{F_*}{k}\sin(\lambda y)\cos(kx) \tag{2.5}$$

with an analogous expression for the thermal streamfunction in which the subscript * is replaced by the subscript T. For the heating we adopt $Q = \hat{Q}\sin(2\lambda y)$ indicating that the heating is a function of 'latitude' only. In these expression $k = 2\pi/L$, where L is the wavelength, $\lambda = \pi/W$ where W is the width of the channel $W = 10^7\ m$, and the coefficients B, E and F have the dimension of velocity.

The specification (2.5) has been selected for simplicity. It is easy to see that the momentum transport vanishes because it requires additional scales in the zonal or the meridional directions. On the other hand, it is the intention to vary the wave number k and thereby gain some insight in the scale dependence.

(2.5) is introduced in (2.1), and we use the orthogonality of the trigonometric functions to derive equations for the time dependent amplitudes. They are:

$$\frac{dB_*}{dt} = -\frac{\varepsilon}{4}(B_*-B_T)$$

$$\frac{dB_T}{dt} = \frac{1}{1+q^2/4\lambda^2}\left\{-\frac{k}{4}\frac{q^2}{k^2}(E_*F_T - E_TF_*) + \frac{\varepsilon}{4}(B_* - B_T) - \varepsilon_T B_T + \frac{\lambda}{4}\frac{q^2}{\lambda^2}\frac{\kappa}{f_0}\hat{Q}\right\}$$

$$\frac{dE_*}{dt} = (a_*B_*-b_*)F_* + a_*B_TF_T - \frac{\varepsilon}{4}(E_*-E_T) \tag{2.6}$$

$$\frac{dF_*}{dt} = -(a_*B_*-b_*)E_* - a_*B_TE_T - \frac{\varepsilon}{4}(F_*-F_T)$$

$$\frac{dE_T}{dt} = (a_TB_*-b_T)F_T + c_TB_TF_* + \frac{\varepsilon}{4}\frac{b_T}{b_*}(E_*-E_T) - \varepsilon_T\frac{b_T}{b_*}E_T$$

$$\frac{dF_T}{dt} = -(a_TB_*-b_T)E_T - c_TB_TE_* + \frac{\varepsilon}{4}\frac{b_T}{b_*}(F_*-F_T) - \varepsilon_T\frac{b_T}{b_*}F_T$$

in which we have introduced the following notations:

$$a_* = k\frac{5\lambda^2+k^2}{2(\lambda^2+k^2)} \qquad b_* = \frac{\beta k}{\lambda^2+k^2}$$

$$a_T = k\frac{k^2+q^2-3\lambda^2}{2(\lambda^2+k^2+q^2)} \qquad b_T = \frac{\beta k}{\lambda^2+k^2+q^2} \qquad c_T = k\frac{k^2-q^2-3\lambda^2}{2(\lambda^2+k^2+q^2)} \tag{2.7}$$

The quantity $(E_*F_T - E_TF_*)$, appearing in the second equation above is proportional to the transport of sensible heat. We see this by calculating the quantity

$$\overline{\psi_{TE}\upsilon_{*E}} = \frac{E_*F_T - E_TF_*}{2k}\sin^2(\lambda y) \tag{2.8}$$

in which the subscript E denotes the eddy part of the quantity, while the overbar is the zonal average. Similarly, we may note from (5) that

$$\overline{\upsilon_T\upsilon_*} = 1/2(E_*E_T + F_*F_T)\sin^2(\lambda y)$$

$$\overline{\upsilon_*^2} = 1/2(E_*^2+F_*^2)\sin^2(\lambda y)$$

$$\overline{\upsilon_T^2} = 1/2(E_T^2 + F_T^2)\sin^2(\lambda y) \tag{2.9}$$

It is thus natural to seek an equation for $(E_*F_T - E_TF_*)$, and, as it turns out, it is exactly the quantities appearing in the three parentheses above which turn up in the equation. To ease notations we introduce:

$$T = E_*F_T - E_TF_*$$

$$S = E_*E_T + F_*F_T \tag{2.10}$$

$$K_* = E_*^2 + F_*^2$$

$$K_T = E_T^2 + F_T^2$$

and replace the first four equations in (2.6) by:

$$\frac{dT}{dt} = ((a_* - a_T)B_* - (b_* - b_T))S + a_*B_TK_T - c_TB_TK_* - \left(\frac{\varepsilon}{4}\left(1 + \frac{b_T}{b_*}\right) + \varepsilon_T\frac{b_T}{b_*}\right)T$$

$$\frac{dS}{dt} = ((a_* - a_T)B_* - (b_* - b_T))T + \frac{\varepsilon}{4}K_T + \frac{\varepsilon b_T}{4b_*}K_* - \left(\frac{\varepsilon}{4}\left(1 + \frac{b_T}{b_*}\right) + \varepsilon_T\frac{b_T}{b_*}\right)S$$

$$\frac{dK_T}{dt} = -\,2c_TB_TT + \frac{\varepsilon b_T}{2b_*}(S - K_T) - 2\varepsilon_T\frac{b_T}{b_*}K_T$$

$$\frac{dK_*}{dt} = 2a_*B_TT + \frac{\varepsilon}{2}(S - K_*) \tag{2.11}$$

The first two equations in (2.6) and the four equations in (2.11) form a closed, nonlinear system in the six variables: B_*, B_T, T, S, K_T and K_*.

3. *Steady states and their stability*

The major task in this section is to determine the steady states of the system of six equations derived in section 2. We observe immediately that in any steady state we must have $B_* = B_T$. It is also obvious from (2.11) that a zonal steady state exists because the last four equations are satisfied if the four dependent variables which relate to the waves vanish. A zonal steady state is therefore:

$$B_* = B_T$$

$$B_T = \frac{\lambda}{4\varepsilon_T}\frac{q^2}{\lambda^2}\frac{\kappa}{f_0}\hat{Q} \tag{3.1}$$

$$T = S = K_* = K_T = 0$$

In the zonal steady state the windshear B_T is related directly to the intensity of the heating through the middle equation in (3.1) which, using the selected numerical values, is:

$$B_T = 7.54 \times 10^3\,\hat{Q} \tag{3.2}$$

It is seen from (3.2) that for a value of $\hat{Q} = 1 \times 10^{-3} kJt^{-1}\, s^{-1}$ we obtain a value of $B_T = 7.54\ m\, s^{-1}$ which is of a reasonable magnitude and independent of wave number.

We proceed to find additional steady states. From the last two equations in (2.11) we notice that K_* and K_T can be expressed in terms of S and T. These expressions are:

$$K_* = S + \frac{4a_*}{\varepsilon} B_T T$$

$$K_T = \frac{\varepsilon}{\varepsilon + 4\varepsilon_T} S - \frac{4c_T b_*}{b_T(\varepsilon + 4\varepsilon_T)} B_T T \tag{3.3}$$

The two expressions in (3.3) are substituted in the first two equations in (2.11). Since we assumed from the beginning that the heating is purely zonal we get two homogeneous equations in S and T. Using $B_* = B_T$ they may be written in the following form:

$$\{d_1 B_T - (b_* - b_T)\}\, S - \{d_3 b_T^2 + d_5\}\, T = 0$$

$$d_4 S + \{d_2 b_T - (b_* - b_T)\}\, T = 0 \tag{3.4}$$

with the following definitions of the coefficients

$$d_1 = \frac{2\varepsilon + 4\varepsilon_T}{\varepsilon + 4\varepsilon_T} a_* - a_T - c_T$$

$$d_2 = a_*\left(1 - \frac{b_T}{b_*}\right) - a_T + \frac{\varepsilon}{\varepsilon + 4\varepsilon_T} c_T \frac{b_*}{B_T}$$

$$d_3 = 4a_* c_T \left\{\frac{1}{\varepsilon} + \frac{b_*}{b_T(\varepsilon + 4\varepsilon_T)}\right\} \qquad (3.5)$$

$$d_4 = \varepsilon_T \left\{\frac{b_T}{B_*} + \frac{\varepsilon}{\varepsilon + 4\varepsilon_t}\right\}$$

$$d_5 = \frac{\varepsilon}{4}\left(1 + \frac{b_T}{b_*}\right) + \varepsilon_T \frac{b_T}{b_*}$$

The system (3.4) will have nontrivial solutions only if the determinant is zero. This condition gives an equation for the determination of B_T. We note therefore that in the present steady state the thermal wind is determined independent of the heating. This fact is solely due to the assumption that the heating is assumed to be of a zonal form. The equation for the thermal wind is:

$$e_2 B_T^2 - e_1 B_T + e_0 = 0 \qquad (3.6)$$

with

$$e_2 = d_1 d_2 + d_3 d_4$$

$$e_1 = (b_* - b_T)(d_1 + d_2) \qquad (3.7)$$

$$e_0 = (b_* - b_T)^2 + d_4 d_5$$

Having obtained these formal solutions we are now interested in the regions in which the various solutions exist, and in whether of not the solutions are stable. It is obvious that the zonal steady state exists everywhere. To find the region in which the non-zonal solution exists we proceed as follows. In a coordinate system with L, the wavelength, as abscissa and the heating intensity $\hat{Q}$ as ordinate we cover the first quadrant with gridpoints and proceed to calculate the solutions of (3.6). These solutions have to have a real value of B_T and, in addition, the computed values of K_* and K_T have to be positive. If the latter conditions are fulfilled we have obtained a physically meaningful solution. If the condition is unfulfilled we have obtained a 'false' solution. As we shall see such solutions do exist. However, we need not pay much attention to them, because they appear due to the fact that we have replaced the original system of equations with a higher order system where the dependent variables are of a higher order than the original amplitudes. Imagine for example that we calculated a time-dependent solution by numerical integration of the system containing the variables B_*, B_T, E_*, F_*, E_T, F_T. We could then at any time, and also in the asymptotic limit, calculate K_* and K_T and would of course obtain positive values.

The result of the repeated determination of the solution in the grid mentioned above is that the quadrant is divided in two parts as separated by the curve shown in Fig. 1. All the points to the left of the vertical branch and those below the other branch of the curve do not produce a physically acceptable solution. In other words, only the points above the curve have physically meaningful solution of which one has a positive value of B_T, the other a negative value. For each steady state value of the thermal wind we proceed to calculate the steady state values of T, S, K_* and K_T.

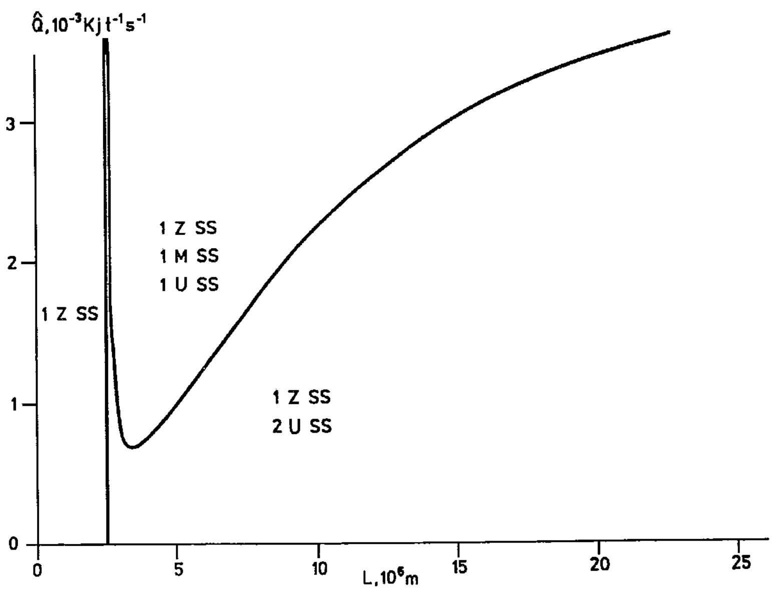

Fig. 1. The existence of various steady states in a diagram with wavelength as abscissa and heating intensity as ordinate. ZSS means a zonal steady state, MSS a meaningful steady state, and USS an unphysical steady state.

We may illustrate these steady states by plotting the variables as a function of wavelength for a given value of the intensity of the heating. We have selected $\hat{Q} = 2 \times 10^{-3}\ kJt^{-1}\ s^{-1}$. For the physically meaningful solution Fig. 2 contains curves of B_T, T, K_*, K_T as functions of the wavelength. We notice that the baroclinic variables T and

K_T go to zero as we reach the boundaries of the wavelength interval within which solutions exist. This means that the solutions at the boundaries (i.e. on the curve in Fig. 1) are of an equivalent barotropic nature. The other meaningless solution has not been plotted, but the calculations show that all these solutions have negative values of K_* and K_T.

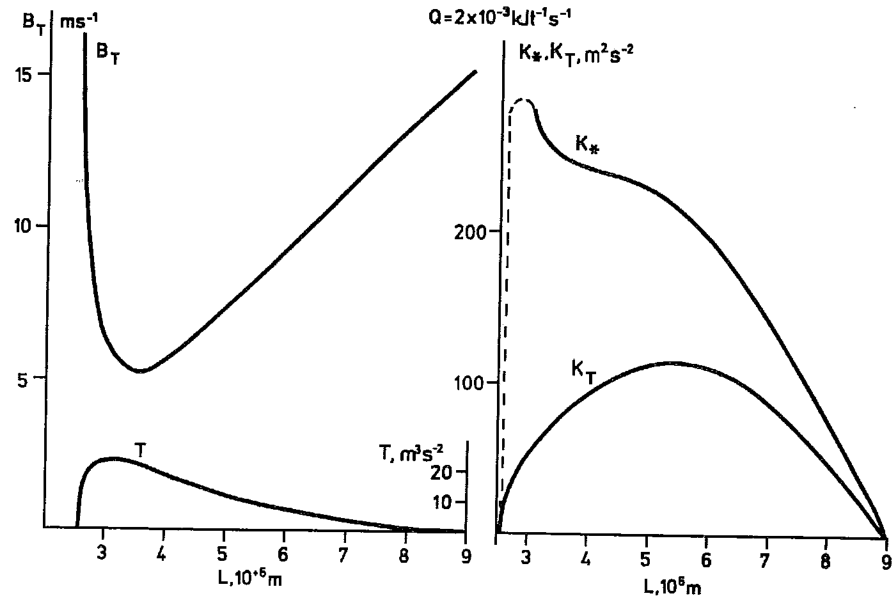

Fig. 2. The windshear, B_T (left scale) and the heat transport (right scale) as a function of wavelength. On the right side: K_* and K_T as functions of the wavelength.

We may summarize the steady state findings by saying that below the curve in Fig. 1 only one physically meaningful steady state exists, i.e. the zonal steady state. Above the curve in Fig. 1 two meaningful solutions exist, a zonal and a non-zonal steady state. The various possibilities, including the non-physical solutions, are marked in Fig. 1 in the regions.

If a steady state solution shall be interesting, it should also be stable to small disturbances. It is thus necessary to conduct a stability investigation of the steady states. It would be desirable, if the unacceptable solutions were also unstable, but it is not a necessity due to the arguments presented above. For a given steady state we have derived the linear perturbation equations using standard perturbation theory. The perturbations have been assumed of the form:

$$T' = \hat{T}\exp(rt) \tag{3.8}$$

with analogous expressions for the other five dependent variables. One obtains then a standard eigenvalue problem with a six by six matrix. Due to the model used in this study it is very unlikely that the eigenvalue problem can be solved by analytical methods, because it would in any case lead to a sixth degree equation for the six eigenvalues. Our model is incidentally in this regard much more cumbersome than the one employed by *Thompson* (1987) because his use of only lateral diffusion gives more symmetrical equations. It was therefore decided from the beginning to use numerical methods. For this purpose we adopted a computer program which first normalized the columns without changing the eigenvalues. This program was followed by another which brought the matrix into Hessenberg form, and the final program calculated the eigenvalues from the latter matrix. For each type of steady state the eigenvalues were determined in a grid covering the region of interest.

Referring once again to Fig. 1 we may summarize the results of this extensive determination of the eigenvalues as follows:

A. The zonal steady states (marked ZSS) are stable for short wavelength for all values of the heating and, in addition, everywhere below the curve. Everywhere else they are unstable.

B. The unphysical steady states (marked USS) are unstable whenever they exist.

C. The physically meaningful, non-zonal steady states (marked MSS) are stable whenever they exist, i.e. above the curve in Fig. 1.

We may also summarize these statements by saying that in each point of the diagram in Fig. 1 only one stable steady state exists. This state is a baroclinic wave on a zonal current above the curve and a zonal current without waves below the curve. The baroclinic wave transports sensible heat northward, and it has positive values of K_T and K_*.

The numerical results may also be used to calculate the phase difference between the temperature field and the streamfunction at 50 *kPa*. The wave in the x-direction was specified in the form:

$$\frac{1}{k}\left(E\sin(kx) + F\cos(kx)\right) = \frac{A}{k}\sin k\,(x-\Theta) \tag{3.9}$$

where E and F can have subscripts * or T. It follows therefore that

$$T = A_* A_T \sin(k(\Theta_* - \Theta_T))$$

$$S = A_* A_T \cos(k(\Theta_* - \Theta_T))$$

$$K_T = A_T^2$$

$$K_* = A_*^2 \tag{3.10}$$

From the first two expressions we obtain:

$$\tan(k(\Theta_* - \Theta_T)) = T / S \tag{3.11}$$

It is thus seen that we may compute the amplitude at 50 *kPa*, the amplitude in the thermal wave, and the phase difference between the two waves. This has been done for $\hat{Q} = 2 \times 10^{-3}\, kJt^{-1}\, s^{-1}$, and the results are given in Fig. 3 which contains the phase difference $\Theta_3 - \Theta_1$ as well. The latter quantity can be calculated from the formula

$$\tan(k(\Theta_3 - \Theta_1)) = \frac{2A_* A_T}{A_*^2 - A_T^2} \sin(k(\Theta_* - \Theta_T)) \tag{3.12}$$

Fig. 3 shows how the baroclinic waves become more and more equivalent barotropic as we approach the wavelengths at which the non-zonal steady state ceases to exist. This result is in complete agreement with the main conclusions obtained by *Wiin-Nielsen* (1989) for the same model without heating and friction.

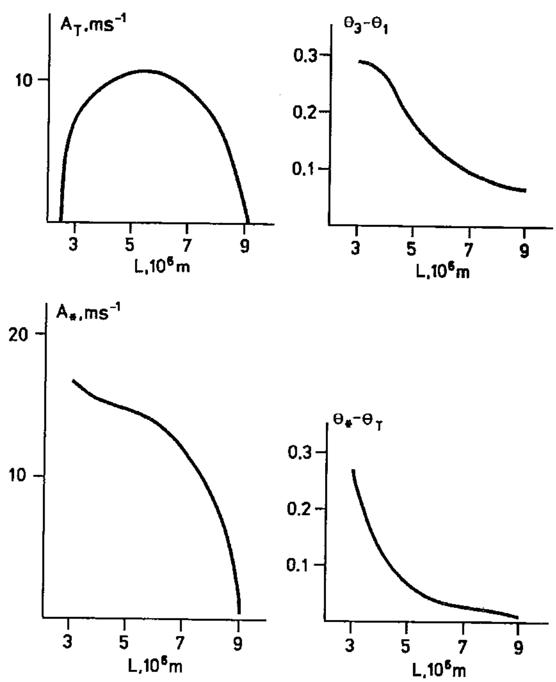

Fig. 3. The magnitude of the meridional wind at 50 *kPa* (lower left), the thermal meridional wind (upper left), the phase difference Θ_*-Θ_T (lower right) and Θ_3-Θ_1 (upper right) as function of wavelength. Note that the phase differences are given as fractions of a wavelength.

We note also that we need only three of the relations (3.10) to produce results such as those given in Fig. 3. This is due to the fact that the four quantities are interrelated. From (3.9) it is easy to show that

$$S^2 + T^2 = K_* K_T \tag{3.13}$$

(3.13) is of course an integral of the system of equations and could be used to eliminate one of the variables, but this is without any great use unless we were to find other integrals. It can, however, be used to check the rather complicated algebraic manipulations necessary to derive the basic system, and it can naturally also be used to check numerical results.

4. *The barotropic case*

It is essential in a nonlinear barotropic model to have sufficient resolution to permit the interaction between the zonal current and the waves. To obtain this in a minimal system we may consider a single wavelength in the zonal direction with a time-dependent amplitude, which is also a function of the meridional coordinate. The dependence of the amplitude on the south-north direction can be given by a series expansion in trigonometric functions as long as the considerations are restricted to a beta-plane channel. Such a treatment has been used by *Eliasen* (1954) to obtain solutions of the barotropic linear stability problem by numerical methods. Using the same methods, but restricting the number of trigonometric functions to the minimal number of two, *Wiin-Nielsen* (1961) formulated a six component barotropic system, which was analysed for stability in its linear form, but also studied in its nonlinear form with respect to momentum transport and interaction between the zonal current and the waves including some numerical integrations. This system had no forcing and no dissipation.

Some nonlinear barotropic low order systems dealing with possible explanations of blocking have been treated by *Charney* and *DeVore* (1979), *Wiin-Nielsen* (1979) and *Wiin-Nielsen* (1984). Similar studies related to blocking are described among the papers in *Benzi, Saltzman* and *Wiin-Nielsen* (1986).

We shall adopt the six component system adding forcing and dissipation to it. The forcing, which in a barotropic system is somewhat artificial, will be restricted to zonal forcing as in the baroclinic case. Since the derivation of the basic system of equations has been described in detail in *Wiin-Nielsen* (1961) it will not be necessary to repeat it here. It suffices to summarize the system by specifying the components which are included. The streamfunction is:

$$\psi(x,y,t) = D(B+C)\,(1-y\,/\,D) + \frac{B}{2\lambda}\sin(2\lambda y) + \frac{C}{4\lambda}\sin(4\lambda y)$$

$$+\frac{E_1}{K}\sin\lambda y\cos kx+\frac{E_3}{k}\sin 3\lambda y\cos kx \tag{4.1}$$

$$+\frac{F_1}{k}\sin\lambda y\sin kx+\frac{F_3}{k}\sin 3\lambda y\sin kx$$

The specification (4.1) is obviously the most simple interacting, low-order system for a barotropic flow. The two meridional scales in the eddy part of the streamfunction will interact with each other to produce changes in the two components of the zonal flow. Similarly, a given eddy meridional scale may interact with one or both scales represented in the zonal flow to create changes in one or more eddy components. With respect to the width D of the channel we have preferred to use a value equal to the equator-pole distance as the natural distance, but it is known from other studies that the magnitude of the results are sensitive to this parameter.

The first two terms in (4.1) contain the zonal part of the streamfunction. The zonal wind, corresponding to these terms, is:

$$u_z = B\,(1-\cos 2\lambda y) + C\,(1-\cos 4\lambda y) \tag{4.2}$$

The mean zonal wind is thus zero at the boundaries $y = o$ and $y = D$, where D is the width of the channel. Furthermore, $\lambda = \pi / D$. It is seen that the mean zonal wind is symmetrical around the center of the channel, and it may have one or two maxima, depending on the values of the time-dependent coefficients B and C.

The last four terms in (4.1) describe the wave. $k = 2\pi / L$ where L is the wave length, and E_1, F_1, E_3 and F_3 are the time-dependent amplitude components. The momentum transport by the wave is:

$$(u_E\,V_E)_z = -\,(E_3F_1 - E_1F_3)\sin 2\lambda y\sin 2\lambda y \tag{4.3}$$

and it is seen that if $M = E_3F_1 - E_1F_3 < o$ then the momentum transport is to the north in the southern half of the channel and to the south in the northern half, giving convergence of the momentum transport in the middle of the channel. The reverse situation appears when $M > o$.

(4.1) is substituted in the barotropic vorticity equation to which has been added constant zonal forcing and boundary layer friction. In this equation we introduce a non-dimensional time

$$\tau = \frac{t}{T_d} \tag{4.4}$$

in which T_d is one day. We introduce furthermore the notations:

$$\gamma_{11} = T_d k \frac{q-3}{2(1+q)} ; \gamma_{13} = T_d k \frac{3q-1}{2(1+q)} ; \gamma_{15} = T_d k \frac{5q+1}{2(1+q)} ; \gamma_{17} = T_d k \frac{7q-1}{2(1+q)}$$

$$\gamma_{33} = T_d k \cdot \frac{3q-1}{2(1+9q)} ; \gamma_{35} = T_d k \frac{15q-1}{2(1+9q)} \tag{4.5}$$

$$\beta_1 = T_d k \frac{r}{1+q} \; \beta_3 = T_d k \frac{r}{1+9q}$$

in which

$$q = \frac{\lambda^2}{k_2} ; r = \frac{\beta}{k^2} \tag{4.6}$$

The six component system becomes then:

$$\frac{dB}{d\tau} = 2T_d kq\, M - T_d \varepsilon\, B + T_d \varepsilon\, B_*$$

$$\frac{dC}{d\tau} = -\, 2T_d kq\, M - \varepsilon T_d\, C + \varepsilon T_d\, C_*$$

$$\frac{dE_1}{d\tau} = -\left[T_d kCF_1 - \gamma_{11}\, BF_1 - \gamma_{15}\, BF_3 - \gamma_{17}\, CF_3 - \beta_1\, F_1\right] - \varepsilon T_d E_1$$

$$\frac{dF_1}{d\tau} = \left[T_d kCE_1 - \gamma_{11}\, BE_1 - \gamma_{15}\, BE_3 - \gamma_{17}\, CE_3 - \beta_1 E_1\right] - \varepsilon T_d F_1 \tag{4.7}$$

$$\frac{dE_3}{d\tau} = -\left[T_d k(B+C)F_3 + \gamma_{33}\, BF_1 - \gamma_{35}\, CF_1 - \beta_3 F_3\right] - \varepsilon T_d E_3$$

$$\frac{dF_3}{d\tau} = \left[T_d k(B+C)E_3 + \gamma_{33}\, BE_1 - \gamma_{35}\, CE_1 - \beta_3 E_3\right] - \varepsilon T_d F_3$$

The system (4.7) can be compared with the system (6.7) in *Wiin-Nielsen* (1961).

Apart from the addition of zonal forcing and dissipation the two systems are exactly the same when we note that *sin(kx)* and *cos(kx)* has been exchanged in (4.1) causing the changes in sign in the definition of *M* and in the last four equations in the system (4.7).

We could naturally work with the system (4.7) as it stands. It is, however, quite cumbersome to do so when we want to find steady states and to investigate the stability of these states. In agreement with the procedure used in the baroclinic case it is in certain respects an advantage to replace the last four equations in (4.7) by four new equations in the variables

$$M = E_3F_1 - E_1F_3$$

$$S = E_1E_3 + F_1F_3 \tag{4.8}$$

$$K_1 = E_1^{\ 2} + F_1^{\ 2}$$

$$K_3 = E_3^{\ 2} + F_3^{\ 2}$$

The derivation of the four equations for the rate of change of M,S,K_1 and K_3 is completely straightforward considering the derivatives of the four quantities in (4.8). We find:

$$\frac{dM}{d\tau} = (\beta_3 - \beta_1 - \gamma_{13}B)S - (\gamma_{33}B - \gamma_{35}C)K_1 - (\gamma_{15}B + \gamma_{17}C)K_3 - 2\varepsilon T_d M$$

$$\frac{dS}{d\tau} = -(\beta_3 - \beta_1 - \gamma_{13}B)\,M - 2\varepsilon T_d S \tag{4.9}$$

$$\frac{dK_1}{d\tau} = -2\,(\gamma_{15}B + \gamma_{17}C)\,M - 2\varepsilon T_d K_1$$

$$\frac{dK_3}{d\tau} = -2\,(\gamma_{33}B - \gamma_{35}C)\,M - 2\varepsilon T_d K_3$$

In (4.3) we have indicated how the parameter M enters in the momentum transport. The quantity S may be related to another transport. Let us evaluate the transport

$$\overline{v_1 v_3} = \frac{1}{L}\int_0^L v_1\, v_3\, dx \tag{4.10}$$

using the basic specification in (4.1). We find

$$\overline{v_1 v_3} = \tfrac{1}{2}\,(E_1E_3 + F_1F_3)\,\sin^2\lambda y\,\cos(2\lambda y) \tag{4.11}$$

We note also that the two "kinetic energies" enter the expressions:

$$\overline{v_1^{\ 2}} = \tfrac{1}{3}\,(E_1^{\ 2} + F_1^{\ 2})\,\sin^2\lambda y$$

$$\overline{v_3^{\,2}} = \frac{1}{2}(E_3^{\,2} + F_3^{\,2})\sin^2\lambda y \tag{4.12}$$

and are thus the kinetic energies in the middle of the channel.

The parameter S may also be called the phase function. We may see this by introducing an amplitude and a phase angle for each of the two waves.
Let

$$E_1 = A_1\cos\Theta_1,\ F_1 = A_1\sin\Theta_1,\ E_3 = A_3\cos\Theta_3,\ F_3 = A_3\sin\Theta_3 \tag{4.13}$$

It is then seen that

$$S = A_1A_3\cos(\Theta_1-\Theta_3) \tag{4.14}$$

where $\Theta_1 - \Theta_3$ is the phase difference between the two waves. For M we get

$$M = A_1A_3\sin(\Theta_1-\Theta_3) \tag{4.15}$$

The phase is thus most easily computed from the expression

$$\Theta_1 - \Theta_3 = \arctan(M/S) \tag{4.16}$$

5. *Steady states*

Considering the system consisting of the first two equations in (4.7) and the four equations in (4.9) it is seen by inspection that one steady state is purely zonal and is given by

$$\overline{B} = B_*,\ \overline{C} = C_*,\ \overline{M} = \overline{S} = \overline{K_1} = \overline{K_3} = 0 \tag{5.1}$$

Additional steady states exist. They are obtained by noting that the last three equations in (4.9) may be solved for $\overline{S}$, $\overline{K_1}$ and $\overline{K_3}$ in terms of $\overline{M}$ with the results that

$$\overline{S} = -\frac{(\beta_3-\beta_1-\gamma_3\overline{B})}{\delta}\overline{M}$$

$$\overline{K_1} = -\frac{2(\gamma_{15}\overline{B}+\gamma_{17}\overline{C})}{\delta}\overline{M} \tag{5.2}$$

$$\overline{K_3} = -\frac{2(\gamma_{33}\overline{B}-\gamma_{35}\overline{C})}{\delta}\overline{M}$$

with $\delta = 2\varepsilon T_d$.

The three expressions in (5.2) are substituted in the first equation of (4.9). The resulting equation will have the common factor $\overline{M}$ which we may assume is different from zero. In this way we obtain a single equation in $\overline{B}$ and $\overline{C}$. We may also note that if we multiply the first equation in (4.9) by $2M$, the second by $2S$, the third by $-K_3$, and the fourth by $-K_1$ and add the resulting equations we obtain the result that

$$\frac{d\Gamma}{d\tau} = -2\delta\Gamma \tag{5.3}$$

with

$$\Gamma = M^2 + S^2 - K_1K_3 \tag{5.4}$$

The solution of (5.3) is

$$\Gamma = \Gamma_0 \exp(-2\delta\tau) \tag{5.5}$$

and it follows therefore that in the asymptotic limit of τ going to infinity we have

$$\overline{\Gamma} = \overline{M}^2 + \overline{S}^2 - \overline{K}_1\overline{K}_3 = 0 \tag{5.6}$$

This result can of course also be obtained directly from the four steady state equations resulting from setting the four time-derivatives equal to zero in (4.9). We may therefore just as well substitute the three expressions in (5.2) in (5.6).

To complete the determination of the non-zonal steady state we need an additional relation between $\overline{B}$ and $\overline{C}$. This is obtained by going back to the first two equations in (4.7). By adding these equations we obtain

$$\frac{d(B+C)}{d\tau} = -\tfrac{1}{2}\delta(B+C) + \tfrac{1}{2}\delta(B_*+C_*) \tag{5.7}$$

eliminating the momentum transport M. The solution of (5.7) is

$$B + C = (B_*+C_*)(1 - \exp(-\tfrac{1}{2}\delta\tau)) + (B_0+C_0)\exp(-\tfrac{1}{2}\delta\tau) \tag{5.8}$$

In the asymptotic limit or from the steady state form of (5.7) we obtain

$$\overline{B} + \overline{C} = B_* + C_* = F \tag{5.9}$$

(5.9) is used to eliminate one of the variables, say $\overline{C}$, and a single equation in $\overline{B}$ remains. This equation, obtained then from (5.2), (5.6) and (5.9), is

$$A_2\overline{B}^2 + A_1\overline{B} - A_0 = 0 \tag{5.10}$$

where

$$A_2 = 4\,(\gamma_{15}-\gamma_{17})\,(\gamma_{33}+\gamma_{35}) - {\gamma_{13}}^2$$

$$A_1 = 2\left[2(\gamma_{17}(\gamma_{33}+\gamma_{35}) \cdot \gamma_{35}(\gamma_{15}-\gamma_{17}))F + \gamma_{13}(\beta_3-\beta_1)\right]$$

$$A_0 = 4\,\gamma_{17}\gamma_{35}F^2 + (\beta_3-\beta_2)^2 + \delta^2 \tag{5.11}$$

To complete the determination of the non-zonal steady state we find the steady state momentum transport $\overline{M}$ from the first equation in (4.7) with the result that

$$\overline{M} = -\frac{\frac{1}{2}\delta(B_*-\overline{B})}{\gamma_{00}}\;;\; \gamma_{00} = 2T_d\,kq \tag{5.12}$$

whereafter $\overline{S}$, $\overline{K}_1$ and $\overline{K}_3$ are obtained from (5.2)

The formal solution given above will not always lead to acceptable solutions. The first requirement is that $\overline{B}$, determined from (5.10), shall be real. This requires that the discriminant

$$D = {A_1}^2 + 4\,A_0A_2 > 0 \tag{5.13}$$

The second requirement is that $\overline{K}_1 \geq 0$ and $\overline{K}_3 \geq 0$, according to their definition. Solution in which (5.13) is satisfied, but in which negative energies are obtained, are indeed found. Such solutions are "false" solutions resulting from the fact that we are working with the system (4.9) which is of the second order compared with (4.7). Such solutions are excluded using the same argument as in the baroclinic case.

6. *Stability of the barotropic steady states*

The stability to the steady states could also in this case be obtained by numerical methods using the standard technique employed in the baroclinic case. However, due to the simpler nature of the barotropic case it has been possible to reduce the stability investigation significantly by analytical methods. Such a procedure leads necessarily to rather cumbersome, albeit elementary, algebraic manipulations, ant it has therefore been decided to disregard the details and reproduce only the major results.

We consider first the zonal steady state. Due to its simplicity it is possible to evaluate the determinant directly. Of the six eigenvalues one can immediately see that four of them are two double roots $\nu = -\delta/2$ and $\nu = -\delta$, both corresponding to stability. δ is given just after (5.2). The remaining two eigenvalues will lead to stability only if

$$A_2 B_*^2 + A_1 B_* - A_0 > 0 \tag{6.1}$$

where the coefficients are given in (5.11). The neutral curve will depend on the value of the total forcing $F_* = B_* + C_*$. Examples of these neutral curves are given in Fig. 4 for $F_* = -15$, 0 and 15 ms^{-1}. For each curve the zonal steady state is stable below the curve and unstable above it. In the barotropic case the situation is thus analogous to the baroclinic case in the sense that a sufficiently strong forcing will produce instability of the zonal steady state. In the baroclinic case it is the intensity of the heating, which is the deciding factor, and in the barotropic case we must consider B_* and C_* as the forcing coming from orography or baroclinic processes, which feed energy into the barotropic system.

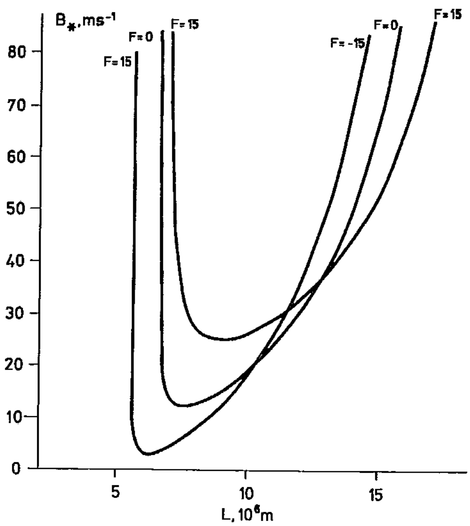

Fig. 4. Neutral curves for the zonal steady state, which is stable below the curves and unstable above them. The three values of F are given in ms^{-1}.

The case of the steady states containing waves can also be reduced considerably by a direct evaluation of the eigenvalue determinant. It turns out that the same eigenvalues $\nu = -\delta/2$ and $\nu = -\delta$ appear again, but the first of them is this time a single root only. A cubic equation remains then for the last three eigenvalues. After considerable manipulations it is possible to show that a fourth eigenvalue is $\nu = -2\delta$, and a quadratic equation remains. A standard analysis of the roots of this final equation gives the important results that the wavy steady state is stable exactly where the zonal state is unstable and vice versa. Referring to Fig. 1 we may say that the wavy steady state is stable above the curve and unstable below it.

The detailed proof of the statement made above is quite laborious. So far it has not been possible to construct a more elegant proof. Others may be able to do so. It is, however, important to note that a complete analogy with the result obtained in the baroclinic case has been established.

Because of the lengthly algebraic calculations we have checked the results reported above by a numerical evaluation of the eigenvalues directly from the basic formulation of the problem. Full agreement was found in all cases.

The interpretation of the results is eased by considering the energetics of the model used here with forcing on the zonal components only, but dissipation on both the zonal and the eddy components. In a wavy steady state the eddies will experience a certain dissipation. It is thus necessary that energy is transferred from the zonal flow to the eddies. The zonal kinetic energy is in turn maintained by a generation due to the external forcing. The generation must balance the sum of the dissipation on the zonal scale and the transfer of energy to the eddies. A schematic energydiagram is given in Fig. 5.

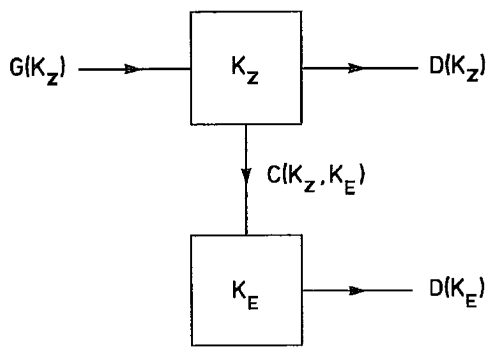

Fig. 5. Energy diagram for the low order, barotropic model.

The arguments above are based on the assumption that both dissipations are positive. This is indeed that case in this model since it is straightforward to calculate that

$$D(K_z) = 2\varepsilon K_z$$

$$D(K_E) = \varepsilon((1+q)K_1 + (1+9q)K_3) \tag{6.2}$$

As discussed previously we consider only physical solutions with positive energies. The most informative quantity is the energy conversion from zonal to eddy kinetic energy. It is evaluated to be

$$C(K_z,K_E) = -kq(B-C)M > 0 \tag{6.3}$$

Since the conversion in (6.3) should be positive in a steady state as discussed above it follows that if M is positive then $B < C$. According to (4.3) a positive value of M means a transport of momentum away from the middle of the channel and to the north in the northern part. Fig. 6 shows a family of windprofiles for $B = 30\ ms^{-1}$ and for various values of C. We notice that large positive values of C give a zonal current with two maxima. The profile with $C = 30\ ms^{-1}$ will according to (6.3) have a vanishing energy conversion. Since $M > 0$ requires $B < C$ we can thus conclude that a system of two strong jetstreams, sometimes called a split jetstream, need a convergence of the momentum transport close to the maxima to be maintained in a steady state.

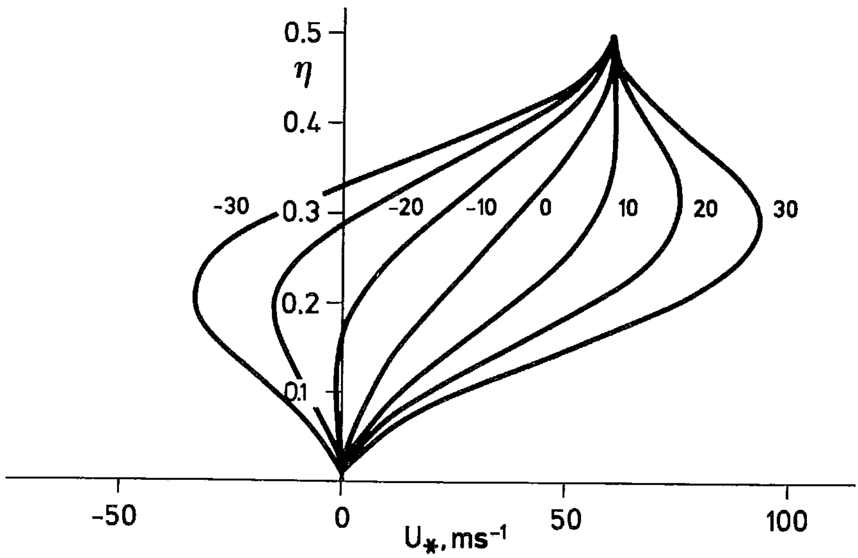

Fig. 6. Various wind profiles for $B = 30\ ms^{-1}$ and C as indicated on the curves, which are drawn for the interval $0 \leq \eta \leq 1$. They are symmetrical around $\eta = 1/2$.

On the other hand, if $M < 0$ then $B > C$ according to (6.3). Fig. 6 contains windprofiles satisfying the conditions. To maintain these windprofiles we must have a convergence of the momentum transport in the middle of the channel.

Fig. 7 shows for the case of $B_* = 30\ ms^{-1}$ the forcing C_* as a function of the wavelength. The region above the curve contains wavy steady states, which are stable. The remaining part of the diagram has only stable, zonal steady states.

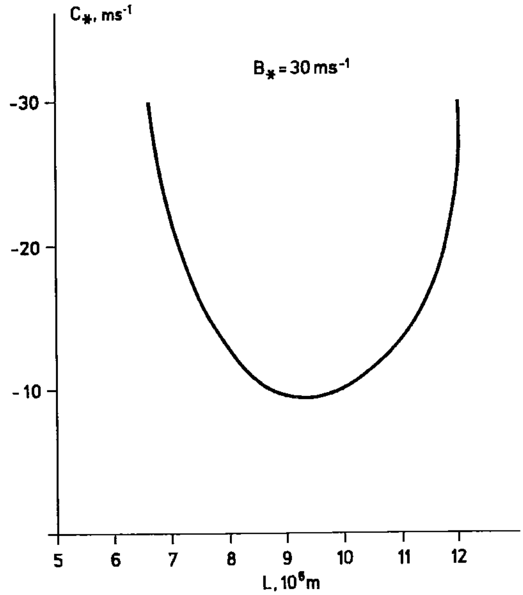

Fig. 7. The region above the curve contains non-zonal steady states for the values of C_* on the ordinate and L on the abscissa. ($B_* = 30\ ms^{-1}$).

Various aspects of the steady states are displayed in Figs. 8-11. All the figures belong to the case $B_* = 30\ ms^{-1}$, $C_* = -15\ ms^{-1}$. Fig. 8 shows B and C. In the non-zonal steady state B is somewhat smaller and C equally larger than the values in the zonal steady state, where $B = B_*$, $C = C_*$. The kinetic energies are shown in Fig. 9 together with the momentum transport, while Fig. 10 shows the zonal and eddy energies in the same diagram. In Fig. 10 the energies are shown in the unit kJm^{-2}. Compared to observations it is clear that we have selected an example, where the zonal kinetic energy is somewhat too large and the eddy kinetic energy too small. Another selection could be made to obtain better agreement with observations. Finally, in Fig. 11 we show $G(K_Z)$ and $D(K_Z)$ as a function of wavelength. These quantities are equal in the zonal steady state. In the wavy regime the difference between the two curves is equal to $C(K_Z,K_E)$, which in turn is the same as $D(K_E)$ (see Fig. 5).

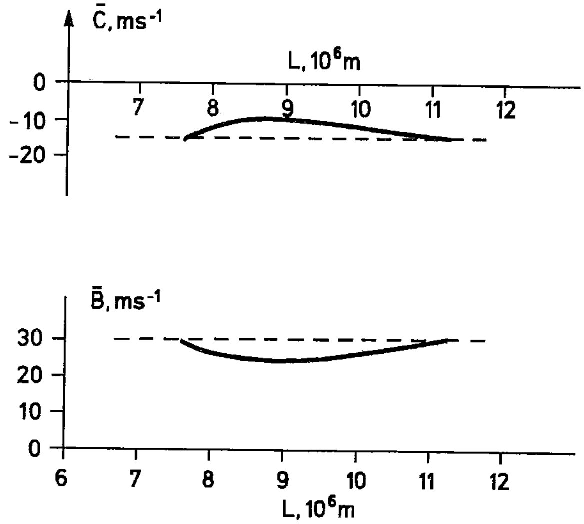

Fig. 8. The steady state values of B and C for the zonal and non-zonal cases as a function of wavelength for $B_* = 30\ ms^{-1}$ and $C_* = -15\ ms^{-1}$.

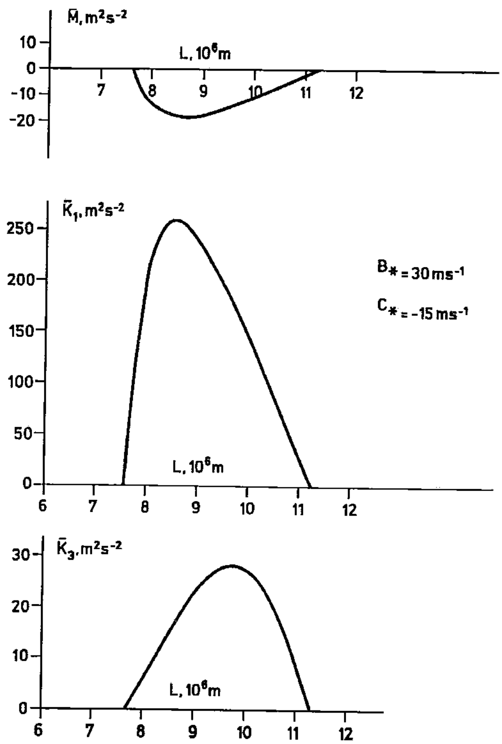

Fig. 9.- The momentum transport M and the two kinetic energies, K_1 and K_3, as functions of wavelength ($B_* = 30\ ms^{-1}$, $C_* = -15\ ms^{-1}$).

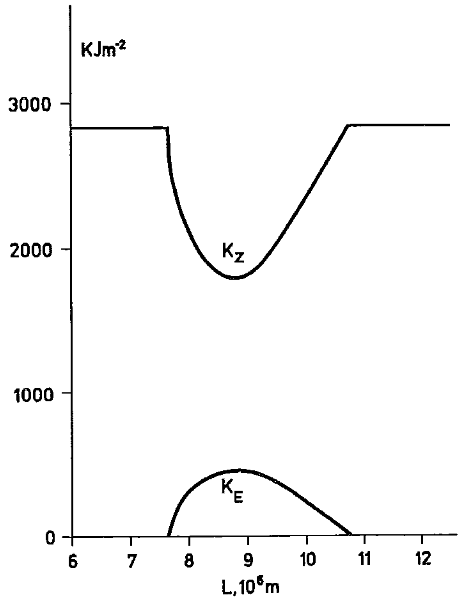

Fig. 10. The zonal and eddy kinetic energies for the steady states in the zonal and the non-zonal cases ($B_* = 30\ ms^{-1}$, $C_* = -15\ ms^{-1}$).

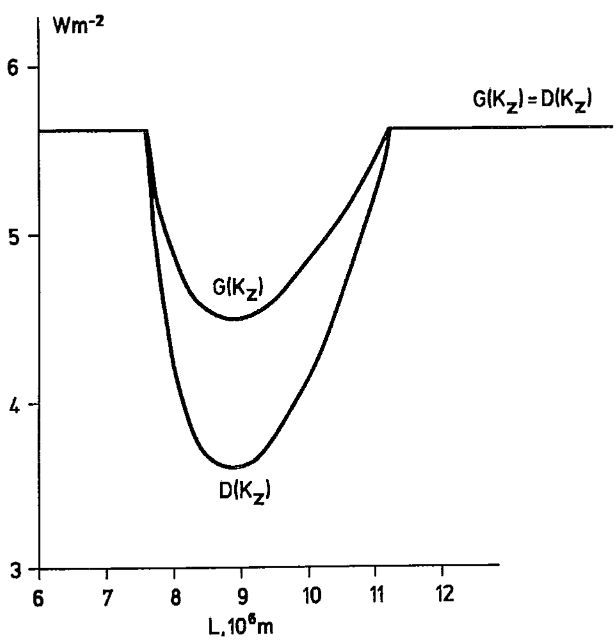

Fig. 11. The generation $G(K_Z)$ and the dissipation $D(K_Z)$ as functions of wavelength for the zonal and non-zonal states ($B_* = 30\ ms^{-1}$, $C_* = -15\ ms^{-1}$).

7. *Conclusions*

The major purpose of the investigations reported in this paper has been to generalize some linear studies to the nonlinear domain. While some studies of a similar nature have been made in the last decade or so, mainly in connection with the phenomenon of blocking, we have in this paper preferred to be rather unspecific with respect to the physical nature of the forcing. We have, on the other hand, tried to illustrate, in a form as simple as possible, the typical behaviour of the system under various intensities of the forcing.

These studies try to make the investigations of the behaviour of atmospheric waves more realistic. The situation in the linear, classical investigations of the stability is that one assumes a zonal current which is constant in time. Superimposing small amplitude waves one determines the condition under which the waves will grow in amplitude. The validity of the results is very limited, because in the domain of growth the waves will soon attain amplitudes, which brings the developments into the nonlinear domain and thus violating the basic assumption. In the nonlinear stage we expect interactions between the waves and the zonal current. There should also be both heat and momentum transports, and there will be energy conversions between the zonal flow and the eddies.

In the first part of this study we have considered the purely baroclinic problem, i.e. the problem with vertical, but no horizontal, windshear. The motion is forced by external heating and a dissipation mechanism containing the stress at the surface of the earth and an internal stress depending on the vertical windshear. Considering the dependence on the intensity of the heating we have shown that sufficiently small values of the heating permit zonal stable steady states only. The zonal wind increases with height in the steady states containing no waves, and the heat transport is carried out by a mean meridional circulation.

When the heating exceeds a critical value (well below typical heating values in the atmosphere) we find non-zonal stable steady states in a certain wavelength interval, and within this interval the zonal steady states are unstable. In other words, a wave regime is created, when the intensity of the heating becomes supercritical. As the intensity of the heating increases further the wavelength interval containing stable wave solutions will also increase. The zonal current, which exists in the wave regime increases with height, has westerly winds in the center of the channel and easterly winds to the north and the south of the westerly current. The waves on the zonal current slope to the west with height, transport sensible heat northwards and have positive kinetic energies. Any unphysical solution with negative kinetic energy is found to be unstable.

With this simple, nonlinear, baroclinic model we are thus able to account for some, but not all, aspects of the atmospheric general circulation in a qualitative, schematic sense. The missing features may be ascribed to the simplicity of the model, designed in such a way that the momentum transport is excluded. consequently, we cannot expect to obtain good surface winds or wind structures in the upper atmosphere, which depend strongly on the momentum transport, as we know from observational studies. We note also that the model has forcing on the zonal modes only. As a result of this assumption the zonal motion is independent of the forcing in the stable wave regime. An improved model should include heating as a function of longitude.

As a first step in investigating the momentum transport we have selected a low order barotropic model with forcing and dissipation, where the forcing once again is restricted to the zonal components. A major result of the analysis of this model is that the behaviour is similar in certain respects to the baroclinic model just described. The similarity is found in the dependence on the intensity of the forcing. Stable zonal steady states exist for sufficiently low values of the forcing. Forcing above a critical level makes the zonal steady states unstable in a wavelength interval, but in the same interval stable steady states containing waves can now exist. The interval increases in size as the intensity of the forcing increases.

As long as we know that stable steady states containing waves exist, we may in the barotropic model determine the energy flow from simple reasoning. Since there is no forcing on the waves, but a dissipation, it follows that the waves in the model must receive their energy from the zonal flow. The zonal flow, having also a dissipation, must then be maintained by generation of zonal kinetic energy by the external forcing. In the steady state the generation of zonal kinetic energy must balance the loss, which is the sum of the energy conversion to the eddy kinetic energy and the dissipation of the zonal kinetic energy.

The low order barotropic model used here is capable of describing the first bifurcation from a zonal flow to a flow containing waves. Further bifurcations cannot be obtained in the present model due to the small number of components. However, barotropic model with a somewhat larger number of wave components can show additional bifurcations as shown by numerical integrations of such models.

Acknowledgements

This investigation has been much influenced by the paper by *Thompson* (1987). It may as a matter of fact be considered as an expansion of the just cited paper to include a different dissipation mechanism and to cover the broad spectrum of waves, while *Thompson* concentrated on a single wave number.

The author wants to thank Denmarks Meteorological Institute for assistance in preparing the figures for this paper.

References

Charney, J.G., 1947: The dynamics of long waves in a baroclinic, westerly current. *J. of Meteor.*, **4**, 135-162.

Charney, J.G., 1959: On the general circulation of the atmosphere. Rossby Memorial Volume, Rockefeller. Institute Press, 178-193.

Charney, J.G. and J.G. DeVore, 1979: Multiple flow equilibria in the atmosphere and blocking. *J Atm. Sci*, **36**, 1205-1216.

Eady, E.T., 1949: Long waves and cyclone waves. *Tellus*, 1, no. 3, 33-52.

Eliasen, E., 1954: Numerical solution of the perturbation equations for a linear flow. *Tellus*, **6**, 183-191.

Kasahara, A. and H.L. Tanaka, 1989: Application of vertical mode expansion to problems of baroclinic instability. *J. Atm. Sci.*, **46**, 489-510.

Källén, E., 1982: A note on orographically induced instabilities in a two-level model. *J. Atm. Sci.*, **39**, 500-505.

Lorenz, E.N., 1963: Mechanics of vacillations. *J. Atm. Sci.*, **20**, 448-464.

Pedlosky, J., 1970: Finite amplitude baroclinic waves. *J. Atm. Sci.*, **27**, 15-30.

Phillips, N.A., 1956: The general circulation of the atmosphere: a numerical experiment. *Q. J. Roy. Met. Soc.*, **82**, 123-164.

Reinhold, B., 1986: Structural determinism of linear baroclinic waves and simple nonlinear equilibration. *J. Atm. Sci.*, **43**, 1484-1504.

Roads, J., 1980: Stable near-resonant states forced by orography in a simple baroclinic model. *J. Atm. Sci.*, **37**, 1958-1967.

Thompson, P.D., 1987: Large-scale dynamical response to differential heating: Statistical equilibrium states and amplitude vacillations. *J. Atm. Sci.*, **44**, 1237-1248.

Wiin-Nielsen, A., A. Vernekar and C.H. Yang, 1967: On the development of baroclinic waves influenced by friction and heating. *Pure and Applied Geophys.*, **68**, 131-161.

Wiin-Nielsen, A., 1979: Steady states and stability properties of a low-order model barotropic system with forcing and dissipation. *Tellus*, **31**, 375-386.

Wiin-Nielsen, A., 1984: Low- and high-index steady states in a low order model with vorticity forcing. *Contr. Atm. Phys.*, **57**, 291-306.

Wiin-Nielsen, A., 1989: On the structure of transient atmospheric waves, I. *Atmosphere*, **2**, 3-15

Yang, C.H., 1967: Nonlinear aspects of the large-scale motion in the atmosphere. Ph.D. dissertation, Univ. of Michigan, 173 pp. (Available from the Library of Congress, U.S.A.).