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# FINE SPECTRA OF LACUNARY MATRICES

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**Abstract**

We examine the spectra and fine spectra of lacunary matrices over the sequence spaces $c_0, c$ and $\ell_\infty$.

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**Keywords**: Spectrum of an operator, lacunary matrices, the sequence spaces $c_0, c$ and $\ell_\infty$.

## 1 Introduction

Let $X$ and $Y$ be Banach spaces and $T : X \to Y$ be a bounded linear operator. By $R(T)$, we denote the range of $T$, i.e.,

$$R(T) = \{y \in Y : y = Tx; x \in X\}.$$

By $B(X)$, we denote the set of all bounded linear operators on $X$ into itself. If X is any Banach space and $T \in B(X)$ then the *adjoint* $T^*$ of $T$ is a bounded linear operator on the dual $X^*$ of $X$ defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$ with $||T|| = ||T^*||$. Let $X \neq \{\theta\}$ be a complex normed space and $T : \mathcal{D}(T) \to X$ be a linear operator with domain $\mathcal{D}(T) \subset X$. With $T$ , we associate the operator $T_\lambda = T - \lambda I$, where $\lambda$ is a complex number and $I$ is the identity operator on $\mathcal{D}(T)$. If $T_\lambda$ has an inverse, which is linear, we denote it by $T_\lambda^{-1}$, that is $T_\lambda^{-1} = (T - \lambda I)^{-1}$ and call it the *resolvent operator* of $T$ . Many properties of $T_\lambda$ and $T_\lambda^{-1}$ depend on $\lambda$, and spectral theory is concerned with those properties. For

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instance, we shall be interested in the set of all $\lambda$ in the complex plane such that $T_\lambda^{-1}$ exists. Boundedness of $T_\lambda^{-1}$ is another property that will be essential. We shall also ask for what $\lambda$'s the domain of $T_\lambda^{-1}$ is dense in $X$. For our investigation of $T$ , $T_\lambda$ and $T_\lambda^{-1}$ , we need some basic concepts in spectral theory which are given as follows (see [12], pp. 370-371):

Let $X \neq \{\theta\}$ be a complex normed space and $T : \mathcal{D}(T) \to X$ be a linear operator with domain $\mathcal{D}(T) \subset X$. A *regular value* $\lambda$ of $T$ is a complex number such that
(**R1**) $T_\lambda^{-1}$ exists,
(**R2**) $T_\lambda^{-1}$ is bounded,
(**R3**) $T_\lambda^{-1}$ is defined on a set which is dense in $X$.

The *resolvent set* $\rho(T)$ of $T$ is the set of all regular values $\lambda$ of $T$ . Its complement $\sigma(T) = \mathbb{C}\backslash\rho(T)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of $T$ . Furthermore, the spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows: The *point spectrum* $\sigma_p(T,X)$ is the set such that $T_\lambda^{-1}$ does not exist. A complex number $\lambda \in \sigma_p(T,X)$ is called an *eigenvalue* of $T$. The *continuous spectrum* $\sigma_c(T,X)$ is the set such that $T_\lambda^{-1}$ exists and satisfies (R3) but not (R2). The *residual spectrum* $\sigma_r(T,X)$ is the set of complex numbers such that $T_\lambda^{-1}$ exists but does not satisfy (R3).

A triangle is a lower triangular matrix with nonzero entries on the main diagonal. We shall write $\ell_\infty$, $c$ and $c_0$ for the spaces of all bounded, convergent and null sequences, respectively. Let $\mathbb{N}$ denote the set of positive integers. Let $\mu$ and $\gamma$ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers $a_{nk}$ , where $n, k \in \mathbb{N}$. Then, we say that $A$ defines a matrix mapping from $\mu$ into $\gamma$, and we denote it by writing $A : \mu \to \gamma$, if for every sequence $x = (x_k) \in \mu$ the sequence $Ax = \{(Ax)_n\}$, the $A$-transform of $x$, is in $\gamma$; where

$$(Ax)_n = \sum_k a_{nk}x_k \qquad (n \in \mathbb{N}). \tag{1.1}$$

By $(\mu : \gamma)$, we denote the class of all matrices $A$ such that $A : \mu \to \gamma$. Thus, $A \in (\mu : \gamma)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \mu$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \gamma$ for all $x \in \mu$.

By a *lacunary* $\theta = (k_r)$; $r = 0,1,2,\ldots$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$ and the length of the intervals will be $h_r = k_r - k_{r-1}$. For any lacunary sequence $\theta$ let

$$N_\theta = \{x = (x_k) : \text{ there exists } L \text{ such that } y_r = \frac{1}{h_r}\sum_{I_r}|x_k - L| \to \infty\}.$$

If $x \in N_\theta$ we let

$$||x||_\theta = \sup_r\Big(\frac{1}{h_r}\sum_{k\in I_r}|x_k|\Big).$$

Then $(N_\theta, ||\cdot||_\theta)$ is a BK-space (see, Freedman et. al. [7]). Several authors including Karakaya [17], Savaş, Patterson [18], Orhan and Fridy [9] defined some new sequence spaces using lacunary sequences.

Let $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$, then the *lacunary operator* is defined by the relation $L(x) = y$ where

$$y_r = \frac{1}{h_r} \sum_{k \in I_r} x_k$$

and so $L$ is represented by the matrix

$$L = \begin{bmatrix} \frac{1}{h_1} & \frac{1}{h_1} & \cdots & \frac{1}{h_1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{h_2} & \frac{1}{h_2} & \cdots & \frac{1}{h_2} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{h_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We summarize the knowledge in the existing literature concerning the spectrum and the fine spectrum of the linear operators defined by some particular limitation matrices over some sequence spaces. Wenger [19] examined the fine spectrum of the integer power of the Cesàro operator over $c$ and Rhoades [16] generalized this result to the weighted mean methods. Reade [15] worked the spectrum of the Cesàro operator over the sequence space $c_0$. Gonzales [11] studied the fine spectrum of the Cesàro operator over the sequence space $\ell_p$. Okutoyi [14] computed the spectrum of the Cesàro operator over the sequence space $bv$. Recently, Yıldırım [20] worked the fine spectrum of the Rhally operators over the sequence spaces $c_0$ and $c$. Next, Coşkun [8] studied the spectrum and fine spectrum for the p-Cesàro operator acting over the space $c_0$. Akhmedov and Başar [1, 2] have determined the fine spectrum of the Cesàro operator over the sequence spaces $c_0$, $\ell_\infty$ and $\ell_p$. Quite recently, de Malafosse [13], Altay and Başar [4] have, respectively, studied the spectrum and the fine spectrum of the difference operator over the sequence spaces $s_r, c_0$ and $c$ where $s_r$ denotes the Banach space of all sequences $x = (x_k)$ normed by

$$||x||_{s_r} = \sup_{k \in \mathbb{N}} \frac{|x_k|}{r^k}, \qquad (r > 0).$$

Akhmedov and Başar [3], Altay and Başar [5] have determined the fine spectrum of the difference operator $\Delta$ and generalised difference operator $B(r,s)$ over the sequence spaces $\ell_p$ and $c_0, c$, respectively. Later Bilgiç and Furkan [6] worked on the spectrum of the operator $B(r,s,t)$, defined by a triple-band lower triangular matrix, over the sequence spaces $\ell_1$ and $bv$.

In this work, our purpose is to determine the fine spectra of the lacunary operator $L$ over the sequence spaces $c_0$ and $c$ and $\ell_\infty$.

**Lemma 1.1.** *Let $\mu$ denote one of the sequence spaces $c_0, c$ or $\ell_\infty$. Then the lacunary operator $L : \mu \to \mu$ is bounded and $||L||_{(\mu:\mu)} = 1$.*

*Proof.* Let us do the proof for $c_0$. The proof for $\mu = c$ or $\ell_\infty$ is similar. Let $x = (x_1, x_2, \ldots) \in c_0$ and recall that $c_0$ is normed by $||x|| = \sup_n |x_n|$. Let $Lx = y = (y_1, y_2, \ldots)$. Then for each $r \in \mathbb{N}$ we have

$$|y_r| \leq \frac{1}{h_r} \sum_{k \in I_r} |x_k| \leq ||x||.$$

Hence $||Lx|| \leq ||x||$ and so $||L|| \leq 1$.

Now, take $x=(x_1,x_2,\ldots)\in c_0$ with the entries

$$x_k=\begin{cases}1 & \text{if } k\in I_1\\ 0 & \text{otherwise}\end{cases}.$$

Then $Lx=(1,0,0,\ldots)$ and $||Lx||/||x||=1$. Hence $||L||=1$. □

## 2 The fine spectra of the lacunary operators

**Lemma 2.1.** $\sigma_p(L,c_0)\supset\{\lambda\in\mathbb{C}:|\lambda|<1\}$.

*Proof.* Suppose $\lambda\in\mathbb{C}$ be such that $|\lambda|<1$. Let $s=\min\{r\in\mathbb{N}:h_r\geq 2\}$ and define the sequence of positive integers $(n_r)$; $r=s,s+1,\ldots$ by the recurrence relation $n_s=k_s$ and $n_r=k_{n_{r-1}}$ for $r>s$. In this way we get to a new partition of the set $(0,\infty)$, i.e.

$$(0,\infty)=\bigcup_{r\leq s}I_r\cup\bigcup_{t=1}^{\infty}(n_{s+t-1},n_{s+t}].$$

Now define the sequence $x=(x_1,x_2,\ldots)$ so that

$$x_q=\begin{cases}0 & \text{if } q<s\\ 1 & \text{if } q=s\\ \frac{h_s\lambda-1}{h_s-1} & \text{if } q\neq s \text{ and } q\in I_s\\ \frac{h_s\lambda-1}{h_s-1}\lambda^t & \text{if } q\in(n_{s+t-1},n_{s+t}]\end{cases}.$$

Let us first show that $x\in c_0$. The fraction $\frac{h_s\lambda-1}{h_s-1}$ is a constant, so let $A=|\frac{h_s\lambda-1}{h_s-1}|$. Let $\varepsilon>0$ be given. Choose $t_0\in\mathbb{N}$ so that $|\lambda|^{t_0}A<\varepsilon$. Then for $N=n_{s+t_0}$ we have

$$|x_q|<\varepsilon \text{ for } q>N.$$

Hence $x\in c_0$. Now, let us show that $x$ is an eigenvector of the operator $L$ corresponding to the value $\lambda$. We need to show that $Lx=\lambda x$, i.e. $(Lx)_q=\lambda x_q$ for all $q\in\mathbb{N}$.
Case 1: $q<s$
$x_q=0$. Since $h_1=h_2=\cdots=h_{s-1}=1$, the $q-th$ row of the matrix $L$ is $(0,0,\ldots,0,1,0,0,\ldots)$ where 1 is in the $q-th$ place. Hence

$$(Lx)_q=1x_q=0.$$

Case 2: $q=s$
$x_q=1$. The $q-th$ row of the matrix $L$ is $(0,0,\ldots,0,\frac{1}{h_s},\frac{1}{h_s},\ldots,\frac{1}{h_s},0,0,\ldots)$ where the terms $\frac{1}{h_s}$ appear $h_s$ times and the first $\frac{1}{h_s}$ appears in the $s-th$ place. We can see that $s\in I_s$ and $x_s=1$. If $r\in I_s\backslash\{s\}$ then $x_r=\frac{h_s\lambda-1}{h_s-1}$. So we have

$$(Lx)_q=\frac{1}{h_s}\sum_{q\in I_s}x_q=\frac{1}{h_s}[1+(h_s-1)\frac{h_s\lambda-1}{h_s-1}]=\lambda=\lambda x_q.$$

Case 3: $q\neq s$ and $q\in I_s$
$x_q=\frac{h_s\lambda-1}{h_s-1}$. The $q-th$ row of the matrix $L$ is $(0,0,\ldots,0,\frac{1}{h_q},\frac{1}{h_q},\ldots,\frac{1}{h_q},0,0,\ldots)$ where the

terms $\frac{1}{h_q}$ appear $h_q$ times and they appear in all the $r-th$ places where $r \in I_q$. Then $r \in (k_{q-1}, k_q]$. Since $q \in I_s \backslash \{s\}$, we have $q \in (s, k_s]$, in other words $q \in [s+1, k_s]$. Hence $r \in (k_s, k_{k_s}] = (n_s, n_{s+1}]$ and this means $x_r = \frac{h_s\lambda - 1}{h_s - 1}\lambda$ for all $r \in I_q$. Then

$$(Lx)_q = \frac{1}{h_q}\sum_{r\in I_q} x_r = \frac{1}{h_q}[h_q\frac{h_s\lambda-1}{h_s-1}\lambda] = \frac{h_s\lambda-1}{h_s-1}\lambda = \lambda x_q.$$

<u>Case 4:</u> $t$ is fixed and $q \in (n_{s+t-1}, n_{s+t}]$
$x_q = \frac{h_s\lambda-1}{h_s-1}\lambda^t$. The $q-th$ row of the matrix $L$ is $(0,0,\ldots,0,\frac{1}{h_q},\frac{1}{h_q},\ldots,\frac{1}{h_q},0,0,\ldots)$ where the terms $\frac{1}{h_q}$ appear $h_q$ times and they appear in all the $r-th$ places where $r \in I_q$. Then $r \in (k_{q-1}, k_q]$. $q \in (n_{s+t-1}, n_{s+t}]$, in other words $q \in [n_{s+t-1}+1, n_{s+t}]$. Hence $r \in (k_{n_{s+t-1}}, k_{n_{s+t}}] = (n_{s+t}, n_{s+t+1}]$ and this means $x_r = \frac{h_s\lambda-1}{h_s-1}\lambda^{t+1}$ for all $r \in I_q$. Then

$$(Lx)_q = \frac{1}{h_q}\sum_{r\in I_q} x_r = \frac{1}{h_q}[h_q\frac{h_s\lambda-1}{h_s-1}\lambda^{t+1}] = \frac{h_s\lambda-1}{h_s-1}\lambda^{t+1} = \lambda x_q.$$

Since $t$ is arbitrary the equality above holds for any $t$ with $q \in (n_{s+t-1}, n_{s+t}]$.

Hence $x$ is an eigenvector corresponding to the value $\lambda$ with $|\lambda| < 1$. So we have $\sigma_p(L, c_0) \supset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. □

**Corollary 2.2.** $\sigma_p(L, \ell_\infty) \supset \sigma_p(L, c) \supset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

**Theorem 2.3.** *For $\mu$ one of the sequence spaces $c_0, c$ or $\ell_\infty$, we have*

$$\sigma(L, \mu) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

*Proof.* Since the lacunary operator is bounded over each of the spaces $c_0, c$ and $\ell_\infty$, the spectra of the operator over these spaces are compact. We also know that for any spectral value $\lambda$ we have $|\lambda| \leq ||T||$ when $T$ is a bounded linear operator (see e.g. [12]). This means that for $\lambda \in \sigma(L, \mu)$ we have $\lambda \leq ||L||_{(\mu:\mu)} = 1$ when $\mu = c_0, c$ or $\ell_\infty$. Combining this with the results of Lemma 2.1 and Corollary 2.2 we have

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_p(L, \mu) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$$

when $\mu = c_0, c$ or $\ell_\infty$. Now, using the compactness of the set $\sigma_p(L, \mu)$ we get to the result we need. □

**Theorem 2.4.** *The lacunary operator $L : \mu \to \mu$ is not compact, when $\mu$ is one of the sequence spaces $c_0, c$ or $\ell_\infty$.*

*Proof.* By Theorem 8.3-1 of [12], the set of eigenvalues of any compact linear operator on a normed space is countable. But, the set of eigenvalues of $L$ is uncountable by Lemma 2.1 and Corollary 2.2. □

**Theorem 2.5.** $\sigma_p(L, \ell_\infty) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

*Proof.* Let $|\lambda| \leq 1$ be given. The sequence $x$ which was given in the proof of Lemma 2.1, is bounded since $\lambda^t$ is bounded for $|\lambda| \leq 1$ and the fraction $\frac{h_s\lambda-1}{h_s-1}$ is a constant. $x$ is also an eigenvector for $\lambda$ with $|\lambda| \leq 1$. The proof of the equality $Lx = \lambda x$, is the same as in the proof of Lemma 2.1. □

Since $\sigma(L,\ell_\infty)$ is partitioned into with $\sigma_p(L,\ell_\infty), \sigma_c(L,\ell_\infty)$ and $\sigma_r(L,\ell_\infty)$ we deduce the next result.

**Corollary 2.6.** $\sigma_c(L,\ell_\infty) = \sigma_r(L,\ell_\infty) = \emptyset$.

**Theorem 2.7.**

$$\sigma_p(L,c_0) = \begin{cases} \{\lambda \in \mathbb{C} : |\lambda| < 1\} \cup \{1\} & \text{if } h_1 = 1 \\ \{\lambda \in \mathbb{C} : |\lambda| < 1\} & \text{if } h_1 \neq 1 \end{cases}.$$

*Proof.* By Lemma 2.1 and Theorem 2.3 we have

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_p(L,c_0) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

So we only need to examine the values $\lambda$ with $|\lambda| = 1$.

Let us first show that any value $\lambda$ with $|\lambda| = 1$ and $\lambda \neq 1$ cannot be an eigenvalue of the lacunary operator $L$. Let $x \in c_0$ be an eigenvector for a given $\lambda$ with $|\lambda| = 1$ and $\lambda \neq 1$. Then the linear system of equations

$$\begin{aligned} \lambda x_1 &= \frac{1}{h_1}\sum_{k\in I_1} x_k \\ \lambda x_2 &= \frac{1}{h_2}\sum_{k\in I_2} x_k \\ &\vdots \end{aligned} \tag{2.1}$$

hold. Let $s$ be defined as in the proof of lemma 2.1. If $s > 1$, the first $s-1$ linear equations of (2.1) will be of the form

$$\lambda x_k = x_k$$

$k = 1,2,\ldots,s-1$. Hence $x_k = 0$ for $k = 1,2,\ldots,s-1$. So without loss of generality let $s = 1$. Let $x_m$ be the first nonzero entry of $x$.

Case 1: $m = 1$

The first equation of (2.1) will be

$$(\lambda - \frac{1}{h_1})x_1 = \frac{1}{h_1}(x_2 + x_3 + \cdots + x_{k_1}).$$

Since $x_1 \neq 0$ and $\lambda - \frac{1}{h_1} \neq 0$, there is at least one nonzero entry in $\{x_2,x_3,\ldots,x_{k_1}\}$. Let us choose one of these nonzero entries and let it be $x_{t_1}$. Let $a = |x_{t_1}| > 0$. The $t_1 - th$ equation of (2.1) will be

$$\lambda x_{t_1} = \frac{1}{h_{t_1}}(x_{k_{(t_1-1)}+1} + x_{k_{(t_1-1)}+2} + \cdots + x_{k_{t_1}}).$$

The right side of above equality is the arithmetic mean of the entries $x_{k_{(t_1-1)}+1}, x_{k_{(t_1-1)}+2}, \ldots, x_{k_{t_1}}$. Since absolute value of the left side is $|\lambda x_{t_1}| = |\lambda|\ |x_{t_1}| = a$, at least one of the entries $x_{k_{(t_1-1)}+1}, x_{k_{(t_1-1)}+2}, \ldots, x_{k_{t_1}}$ has absolute value greater or equal to $a$. Let $x_{t_2}$ be an entry with $|x_{t_2}| \geq a$.

The $t_2 - th$ equation of (2.1) will be

$$\lambda x_{t_2} = \frac{1}{h_{t_2}}(x_{k_{(t_2-1)}+1} + x_{k_{(t_2-1)}+2} + \cdots + x_{k_{t_2}}).$$

The right side of above equality is again the arithmetic mean of the entries $x_{k_{(t_2-1)}+1}, x_{k_{(t_2-1)}+2}, \ldots, x_{k_{t_2}}$. Since absolute value of the left side is $|\lambda x_{t_2}| = |\lambda|\,|x_{t_2}| \geq a$, at least one of the entries $x_{k_{(t_2-1)}+1}, x_{k_{(t_2-1)}+2}, \ldots, x_{k_{t_2}}$ has absolute value greater or equal to $a$. Let $x_{t_3}$ be an entry with $|x_{t_3}| \geq a$. In the same way we can find $x_{t_4}, x_{t_5}, \ldots$. By the way of construction $t_i > t_{i-1}$; $i = 2, 3, \ldots$ and hence $(x_{t_1}, x_{t_2}, \ldots)$ is a subsequence of $x$ which does not converge to zero. This is a contradiction to the fact that $x \in c_0$. So there is no eigenvector corresponding to the given $\lambda$ with $m = 1$.

<u>Case 2:</u> $m > 1$

The $m - th$ equation of (2.1) will be

$$\lambda x_m = \frac{1}{h_m}(x_{k_{(m-1)}+1} + x_{k_{(m-1)}+2} + \cdots + x_{k_m}).$$

$x_m \neq 0$ and as in Case 1, the right side of above equality is the arithmetic mean of the entries $x_{k_{(m-1)}+1}, x_{k_{(m-1)}+2}, \ldots, x_{k_m}$. So at least one of the entries $x_{k_{(m-1)}+1}, x_{k_{(m-1)}+2}, \ldots, x_{k_m}$ has absolute value greater or equal to $|x_m|$. Using this procedure recursively as in Case 1 we can find a subsequence of $x$ which does not converge to zero, contradicting to the fact that $x \in c_0$. So there is no eigenvector corresponding to the given $\lambda$ with $m > 1$.

$\lambda$ was arbitrary with $|\lambda| = 1$ and $\lambda \neq 1$. Case 1 and Case 2 show that there is no eigenvector in $c_0$ that corresponds to any $\lambda$ with $|\lambda| = 1$ and $\lambda \neq 1$.

If $h_1 \neq 1$, then $s = 1$ and we can prove that $\lambda = 1$ is not an eigenvalue of the operator $L$ in the same way as above.

If $h_1 = 1$, $x = (1, 0, 0, \ldots)$ is an eigenvector of $L$ corresponding to the value $\lambda = 1$. □

**Theorem 2.8.** $\sigma_p(L, c) = \{\lambda \in \mathbb{C} : |\lambda| < 1\} \cup \{1\}$.

*Proof.* By Corollary 2.2 and Theorem 2.3 we have

$$\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_p(L, c) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

We only need to examine the values $\lambda$ with $|\lambda| = 1$.

$(1, 1, \ldots)$ is an eigenvector for the value $\lambda = 1$. Hence $1 \in \sigma_p(L, c)$. Now, suppose $x \in c$ is an eigenvector which corresponds to a value $\lambda$ with $|\lambda| = 1$ and $\lambda \neq 1$. And suppose $x$ converges to a value $a$. By Silverman-Toeplitz Theorem the operator L preserves limits. This means $Lx$ converges to $a$ and $(L - \lambda I)x$ converges to $a(1 - \lambda)$. Since $x$ is an eigenvector $(L - \lambda I)x = (0, 0, \ldots)$, and so $a(1 - \lambda) = 0$. Hence $a = 0$. So $x \in c_0$. This means that $x$ is also an eigenvector of the operator $L$ over $c_0$ corresponding to the value $\lambda$. But by Theorem 2.7 $x$ cannot be such an eigenvector. Therefore any $\lambda$ with $|\lambda| = 1$ and $\lambda \neq 1$ is not an eigenvalue of the operator $L$ over $c$. □

**Lemma 2.9 ([10], p. 59).** *$T$ has a dense range if and only if $T^*$ is one to one.*

If $T : c_0 \to c_0$ is a bounded linear operator represented by the matrix $A$, then it is known that the adjoint operator $T^* : c_0^* \to c_0^*$ is defined by the transpose $A^t$ of the matrix $A$. It should be noted that the dual space $c_0^*$ of $c_0$ is isometrically isomorphic to the Banach space $\ell_1$.

**Theorem 2.10.** $\sigma_r(L, c_0) = \emptyset$.

*Proof.* Suppose $h_1 \neq 1$, then by Theorem 2.7 $\sigma_r(L, c_0) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. So let $|\lambda| = 1$. In this case the adjoint operator $L_\lambda^*$, which is the transpose of $L_\lambda$, is a triangle, so it is one to one. Hence, by Lemma 2.9 $\lambda \notin \sigma_r(L, c_0)$. So $\sigma_r(L, c_0) = \emptyset$ when $h_1 \neq 1$.

Now, suppose $h_1 = 1$, then by Theorem 2.7 $\sigma_r(L, c_0) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\} \setminus \{1\}$. So let $|\lambda| = 1$ and $\lambda \neq 1$. In this case the transpose $L_\lambda^*$ is again a triangle, so $L_\lambda^*$ is one to one. Hence, using Lemma 2.9 again we can say $\lambda \notin \sigma_r(L, c_0)$. We also have $\sigma_r(L, c_0) = \emptyset$ for $h_1 = 1$. □

**Corollary 2.11.**

$$\sigma_c(L, c_0) = \begin{cases} \{\lambda \in \mathbb{C} : |\lambda| = 1\} \setminus \{1\} & \text{if } h_1 = 1 \\ \{\lambda \in \mathbb{C} : |\lambda| = 1\} & \text{if } h_1 \neq 1 \end{cases}.$$

If $T : c \to c$ is a bounded matrix operator represented by the matrix A, then $T^* : c^* \to c^*$ acting on $\mathbb{C} \oplus \ell_1$ has a matrix representation of the form

$$\begin{bmatrix} \chi & 0 \\ b & A^t \end{bmatrix},$$

where $\chi$ is the limit of the sequence of row sums of $A$ minus the sum of the limit of the columns of $A$, and $b$ is the column vector whose $k^{\text{th}}$ entry is the limit of the $k^{\text{th}}$ column of $A$ for each $k \in \mathbb{N}$. For $L_\lambda : c \to c$, the matrix $L_\lambda^*$ is of the form

$$\begin{bmatrix} 1-\lambda & 0 \\ 0 & L_\lambda^t \end{bmatrix}.$$

**Theorem 2.12.** $\sigma_r(L, c) = \emptyset$.

*Proof.* By Theorem 2.8 $\sigma_r(L, c) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\} \setminus \{1\}$. So let $|\lambda| = 1$ and $\lambda \neq 1$. In this case $L_\lambda^*$ is a triangle, hence it is one to one. Hence, by Lemma 2.9 we have $\lambda \notin \sigma_r(L, c)$. So $\sigma_r(L, c) = \emptyset$. □

**Corollary 2.13.** $\sigma_c(L, c) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \setminus \{1\}$.

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