Communications in **M**athematical **A**nalysis
Volume 7, Number 1, pp. 12–20 (2009)
ISSN 1938-9787

www.commun-math-anal.org

CONVERGENCE THEOREMS OF NEW ISHIKAWA ITERATIVE PROCEDURES WITH ERRORS FOR MULTI-VALUED Φ-HEMICONTRACTIVE MAPPINGS

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(Communicated by Gaston M. N'Guérékata)

Abstract

In this paper, we introduce and study some new Ishikawa iterative procedures with errors for multi-valued mappings in real normed linear spaces. General results on the convergence of the new Ishikawa iterative procedures with errors for multi-valued Φ-hemicontractive mappings without Lipschitz assumption and without boundedness conditions are given in real normed linear spaces.

AMS Subject Classification: 47H06; 47H09; 47H10

Keywords:Multi-valued Φ-hemicontractive mapping; Multi-valued Φ-strongly pseudocontractive mapping; Ishikawa iterative procedures with errors; Multi-valued Φ-hemiaccretive mapping equation; Real normed linear spaces; Fixed point.

1 Introduction

Let X be a real normed linear space, and X^* its dual space. Let $\langle *, * \rangle$ denote the generalized duality pairing between X and X^*, and J stand for the generalized normalized duality

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mapping from X to 2^{X^*}given by $J(x)$

$$J(x) := \{f \in X^* : \langle x, f\rangle = ||f|| \cdot ||x||, \quad ||f|| = ||x||\}, \quad x \in X.$$

Definition 1.1 ([4],[5]). Let X be a real normed linear space, D a nonempty convex subset of X, $CB(D)$ the family of all nonempty bounded closed subsets of D. A multi-valued mapping $T : D \to CB(D)$ is said to be uniformly continuous on D if for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that $H(Tx, Ty) < \varepsilon$ for any given $x, y \in D$ with $||x-y|| < \delta$, where $H(.,.)$ denotes the Hausdorff metric on $CB(D)$ defined by

$$H(A,B) := \max\{\sup_{y\in B}\inf_{x\in A}||x-y||, \sup_{x\in A}\inf_{y\in B}||x-y||\}, \quad A, B \in CB(D).$$

Definition 1.2 ([4],[5]). Let D be a nonempty subset of X. $T : D \to 2^D$ is said to be a multi-valued Φ-strongly pseudocontractive mapping if there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that for each $x, y \in D$, there exists a $j(x-y) \in J(x-y)$ such that

$$\langle u-v, j(x-y)\rangle \leq ||x-y||^2 - \Phi(||x-y||) \cdot ||x-y||,$$

for all $u \in Tx$ and $v \in Ty$.
T is said to be a multi-valued Φ-strongly pseudoaccretive mapping if $I-T$ is a multi-valued Φ-strongly pseudocontractive mapping.

Definition 1.3 ([4]). Let D be a nonempty subset of X. $T : D \to 2^D$ is said to be a multi-valued Φ-hemicontractive mapping if the fixed-point set $F(T)$ of T is nonempty, and there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that for each $x \in D$ and $x^* \in F(T)$, there exists a $j(x-x^*) \in J(x-x^*)$ such that

$$\langle u-x^*, j(x-x^*)\rangle \leq ||x-x^*||^2 - \Phi(||x-x^*||) \cdot ||x-x^*||,$$

for all $u \in Tx$.
T is said to be a multi-valued Φ-hemiaccretive mapping if $I-T$ is a multi-valued Φ-hemicontractive mapping.

If T is a single-valued mapping, then the example in [1] shows that the class of Φ-strongly pseudo-contractive mappings is a proper subset of the class of Φ-hemicontractive mappings.

The iterative approximation of fixed points for the class of pseudocontractive mappings has been studied extensively by various authors (see [1-11] and the references therein). For single-valued cases, Osilike[11, Theorem 2] obtained the convergence of the Ishikwa iterative sequence for Lipschitz Φ-hemicontractive mappings T in real Banach spaces. Under the boundedness condition of the range of T, Liu and Kang [7, Theorem 3.3] got the convergence of the Ishikwa iterative sequence for T without Lipschitz assumption.

For multi-valued cases, Huang et al [4] defined the Ishikwa iterative sequence$\{x_n\}$ with errors for T, and any given $x_0 \in X$,

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \beta_n e_n + \gamma_n u_n, \quad \exists e_n \in Ty_n, \\ y_n &= \hat{\alpha}_n x_n + \hat{\beta}_n f_n + \hat{\gamma}_n v_n, \quad \exists f_n \in Tx_n, \end{aligned} \qquad n = 0, 1, 2, \cdots, \tag{1.1}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\},\{\hat{\alpha}_n\},\{\hat{\beta}_n\}$ *and* $\{\hat{\gamma}_n\}$ are six sequences in $[0,1]$, and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in X satisfying the following conditions:

$$\begin{aligned} \alpha_n+\beta_n+\gamma_n=\hat{\alpha}_n+\hat{\beta}_n+\hat{\gamma}_n=1; \\ \lim_{n\to\infty}\beta_n=\lim_{n\to\infty}\hat{\beta}_n=\lim_{n\to\infty}\hat{\gamma}_n=0; \\ \sum_{n=0}^{\infty}\beta_n<\infty,\ \sum_{n=0}^{\infty}\gamma_n<\infty. \end{aligned} \tag{1.2}$$

Huang et al also proved that if $\{e_n\}$ and $\{f_n\}$ are both bounded, then the sequence $\{x_n\}$ defined in (1.1) converges strongly to the unique fixed point of T ([4, Theorem 3.2]). The following problem arises naturally then: *Can we relax the boundedness condition of* $\{e_n\}$ *or* $\{f_n\}$? In other words, if either $\{e_n\}$ or $\{f_n\}$ in (1.1) is unbounded, or more general, if both $\{e_n\}$ and $\{f_n\}$ in (1.1) are unbounded, then the $\{x_n\}$ defined by (1.1) still converges strongly to the unique fixed point of T? Actually, in many cases, the boundedness conditions of the sequences $\{e_n\}$ and $\{f_n\}$ in (1.1) are not easy to examine in advance. Hence, this is an interesting and important problem.

In order to solve the problem, in this paper we introduce and study some new Ishikawa iterative procedures with errors for multi-valued mappings in real normed linear spaces, see (2.1)-(2.3) in Theorem 2.2 and (2.15)-(2.17) in Theorem 2.4 below. General results on the convergence of the new Ishikawa iterative procedures with errors for multi-valued Φ-hemicontractive mappings without Lipschitz assumption and without boundedness conditions are obtained in real normed linear spaces. Our results improve essentially the corresponding results of [1, 3-7, 10-11].

Lemma 1.4. ([6]) *Let X be a real normed linear space and J be the normalized duality mapping. Then, for any given $x,y\in X$,*

$$||x+y||^2\le||x||^2+2<y,j(x+y)>,\quad for\ all\ j(x+y)\in J(x+y).$$

Lemma 1.5. ([6],[9]) *Let $A,B\in CB(X)$ and $a\in A$. If $\lambda>0$, then there exists a $b\in B$ such that* $\mathrm{dist}(a,b)\le(1+\lambda)H(A,B)$.

2 Main Results

Lemma 2.1. *Let $\{a_n\},\{b_n\}$ and $\{c_n\}$ be nonnegative real sequences such that*

$$a_{n+1}^2\le(1+b_n)a_n^2+c_na_{n+1},\quad \forall\, n\ge n_0\ (some\ integer),$$

with $\sum_{n=1}^{\infty}b_n<\infty$, $\sum_{n=1}^{\infty}c_n<\infty$. Then $\{a_n\}$ is bounded.

Proof. By the assumption, we have

$$0\le a_{n+1}\le\frac{c_n+\sqrt{c_n^2+4(1+b_n)a_n^2}}{2}\le c_n+(1+b_n)a_n\quad\forall\, n\ge n_0\ .$$

Letting $d_n := \max\{b_n, c_n\}$, we get $\sum_{n=n_0}^{\infty} d_n < \infty$. Further, we obtain

$$a_{n+1}+1 \le (1+d_n)(a_n+1), \quad \forall\, n \ge n_0,$$

$$\prod_{n=n_0}^{k}(a_{n+1}+1) \le \prod_{n=n_0}^{k}(1+d_n)\prod_{n=n_0}^{k}(a_n+1), \quad \forall\, k \ge n_0,$$

$$a_{k+1}+1 \le (a_{n_0}+1)\prod_{n=n_0}^{k}(1+d_n) \le (a_{n_0}+1)\exp\Big(\sum_{n=n_0}^{\infty} d_n\Big), \quad \forall\, k \ge n_0.$$

Hence, $\{a_n\}$ is bounded.

Theorem 2.2. *Let X be a real normed linear space and D a nonempty convex subset of X. Let $T : D \to CB(D)$ be uniformly continuous on D and Φ-hemicontractive. For any given $x_0, v_0, u_0 \in D$, let $\{x_n\}$ be the modified Ishikawa iterative sequence with errors, defined by*

$$\begin{cases} y_n = \hat{a}_n x_n + \hat{b}_n \xi_n + \hat{c}_n v_n, & \exists \xi_n \in Tx_n, \\ x_{n+1} = \hat{\alpha}_n x_n + \hat{\beta}_n \eta_n + \hat{\gamma}_n u_n, & \exists \eta_n \in Ty_n, \end{cases} \qquad n = 0,1,2,\cdots, \tag{2.1}$$

where $\{u_n\}, \{v_n\}$ are both bounded sequences in D(Let their bound be $L \ge 2$),

$$\begin{aligned} &\hat{a}_n = 1-\hat{b}_n-\hat{c}_n,\ \hat{b}_n = \frac{b_n}{L+||x_n||+||\xi_n||},\ \hat{c}_n = \frac{c_n}{L+||x_n||+||\xi_n||}, \\ &\hat{\alpha}_n = 1-\hat{\beta}_n-\hat{\gamma}_n,\ \hat{\beta}_n = \frac{\beta_n}{L+||x_n||+||\eta_n||},\ \hat{\gamma}_n = \frac{\gamma_n}{L+||x_n||+||\eta_n||}, \end{aligned} \tag{2.2}$$

$\{\beta_n\}, \{\gamma_n\}, \{b_n\}$ and $\{c_n\}$ are four sequences in $[0,1]$ satisfying the following conditions:

$$\sum_{n=0}^{\infty}\beta_n = \infty,\ \sum_{n=0}^{\infty}\beta_n^2 < \infty,\ \sum_{n=0}^{\infty}\gamma_n < \infty,\ \lim_{n\to\infty} b_n = 0,\ \lim_{n\to\infty} c_n = 0. \tag{2.3}$$

Then $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. Since T is Φ-hemicontractive, it follows that the fixed-point set $F(T)$ of T is nonempty, and there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that for each $x \in D$ and $x^* \in F(T)$, there exists a $j(x-x^*) \in J(x-x^*)$ such that

$$\langle \xi - x^*, j(x-x^*)\rangle \le ||x-x^*||^2 - \Phi(||x-x^*||)\cdot||x-x^*||, \quad \forall\, \xi \in Tx. \tag{2.4}$$

If $q \in F(T)$, i.e. $q \in Tq$, then, by (2.4), there exists a $j(q-x^*) \in J(q-x^*)$ such that

$$||q-x^*||^2 = \langle q-x^*, j(q-x^*)\rangle \le ||q-x^*||^2 - \Phi(||q-x^*||)\cdot||q-x^*||.$$

So T has a unique fixed point x^*.

Since D is a convex subset of X and $T : D \to CB(D)$, it follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned} ||x_{n+1}-y_n|| &= (\hat{b}_n + \hat{c}_n - \hat{\beta}_n - \hat{\gamma}_n)x_n - \hat{b}_n\xi_n - \hat{c}_n v_n + \hat{\beta}_n\eta_n + \hat{\gamma}_n u_n|| \\ &\le 2(b_n + c_n + \beta_n + \gamma_n) \to 0 \quad (n \to \infty) \end{aligned} \tag{2.5}$$

Noting $\eta_n \in Ty_n$, we see by Lemma 1.5 that there exists a point $\tilde{\xi}_{n+1} \in Tx_{n+1}$ such that

$$||\eta_n - \tilde{\xi}_{n+1}|| \leq (1+\frac{1}{n})H(Ty_n, Tx_{n+1}). \tag{2.6}$$

From (2.1), (2.4), (2.6) and Lemma 1.4, it can be concluded that

$$\begin{aligned}
||x_{n+1}-x^*||^2 =& ||\hat{\alpha}_n(x_n-x^*)+\hat{\beta}_n(\eta_n-x^*)+\hat{\gamma}_n(u_n-x^*)||^2 \\
\leq& \hat{\alpha}_n^2||x_n-x^*||^2+2\hat{\beta}_n\langle \eta_n-\hat{\xi}_{n+1}, j(x_{n+1}-x^*)\rangle \\
&+2\hat{\beta}_n\langle \hat{\xi}_{n+1}-x^*, j(x_{n+1}-x^*)\rangle + 2\hat{\gamma}_n\langle u_n-x^*, j(x_{n+1}-x^*)\rangle \\
\leq& (1-\hat{\beta}_n-\hat{\gamma}_n)^2||x_n-x^*||^2+2\hat{\beta}_n g_n\cdot||x_{n+1}-x^*||+2\hat{\beta}_n\big(||x_{n+1}-x^*||^2 \\
&-\Phi(||x_{n+1}-x^*||)\cdot||x_{n+1}-x^*||\big)+2\hat{\gamma}_n||u_n-x^*||\cdot||x_{n+1}-x^*||,
\end{aligned}$$

where $g_n := ||\eta_n - \hat{\xi}_{n+1}||$. By (2.5), (2.6) and the uniform continuity of T, we have $g_n \to 0$ as $n \to 0$. Then, further, we obtain

$$\begin{aligned}
||x_{n+1}-x^*||^2 \leq& ||x_n-x^*||^2+\frac{(\hat{\beta}_n+\hat{\gamma}_n)^2-2\hat{\gamma}_n}{1-2\hat{\beta}_n}||x_n-x^*||^2-\frac{2\hat{\beta}_n}{1-2\hat{\beta}_n}||x_{n+1}-x^*|| \\
&\cdot(\Phi(||x_{n+1}-x^*||)-g_n)+\frac{2\hat{\gamma}_n||u_n-x^*||}{1-2\hat{\beta}_n}||x_{n+1}-x^*|| \\
\leq& ||x_n-x^*||^2+\frac{2(\hat{\beta}_n^2+\hat{\gamma}_n^2)}{1-2\hat{\beta}_n}||x_n-x^*||^2-\frac{2\hat{\beta}_n}{1-2\hat{\beta}_n}||x_{n+1}-x^*|| \\
&\cdot(\Phi(||x_{n+1}-x^*||)-g_n)+\frac{2\hat{\gamma}_n(||u_n||+||x^*||)}{1-2\hat{\beta}_n}||x_{n+1}-x^*||.
\end{aligned} \tag{2.7}$$

Next, we will show

$$\liminf_{n\to\infty}\Phi(||x_{n+1}-x^*||)=0.$$

If it is not true, then there exists a positive constant m_0 such that

$$\liminf_{n\to\infty}\Phi(||x_{n+1}-x^*||)=m_0>0.$$

Combining the definition of Φ, (2.3) with the fact that $g_n \to 0$ as $n \to \infty$, there exist a $n_0 \in \mathbb{N}$ (positive integer set) and a positive constant s_0 such that for $n \geq n_0$,

$$\Phi(||x_{n+1}-x^*||)\geq\frac{m_0}{2},\quad ||x_{n+1}-x^*||\geq s_0,\quad g_n\leq\frac{m_0}{4},\quad \hat{\beta}_n\leq\beta_n\leq\frac{1}{4}.$$

By (2.2) and (2.7), for any $n \geq n_0$ we have

$$\begin{aligned}
||x_{n+1}-x^*||^2 \leq& ||x_n-x^*||^2+\frac{2(\hat{\beta}_n^2+\hat{\gamma}_n^2)}{1-2\hat{\beta}_n}||x_n-x^*||^2-\frac{2\hat{\beta}_n}{1-2\hat{\beta}_n}||x_{n+1}-x^*||\cdot\frac{m_0}{4} \\
&+\frac{2\hat{\gamma}_n(||u_n||+||x^*||)}{1-2\hat{\beta}_n}||x_{n+1}-x^*|| \\
\leq& ||x_n-x^*||^2+4(\beta_n^2+\gamma_n^2)||x_n-x^*||^2-\frac{2\hat{\beta}_n}{1-2\hat{\beta}_n}||x_{n+1}-x^*||\cdot\frac{m_0}{4} \\
&+4\gamma_n(L+||x^*||)||x_{n+1}-x^*||.
\end{aligned} \tag{2.8}$$

Denoting

$$\tilde{a}_n := ||x_n - x^*||, \quad \tilde{b}_n := 4(\beta_n^2 + \gamma_n^2), \quad \tilde{c}_n := 4\gamma_n(L + ||x^*||),$$

from (2.8) it follows that

$$\tilde{a}_{n+1}^2 \le (1 + \tilde{b}_n)\tilde{a}_n^2 + \tilde{c}_n \tilde{a}_{n+1}, \quad \forall\, n \ge n_0. \tag{2.9}$$

By (2.2) and (2.3), we obtain

$$\sum_{n=1}^{\infty} \gamma_n^2 < \infty, \quad \sum_{n=1}^{\infty} \tilde{b}_n < \infty, \quad \sum_{n=1}^{\infty} \tilde{c}_n < \infty.$$

Using Lemma 2.1 and (2.9), we get $\{\tilde{a}_n\}$ is bounded (Let its bound be $M_0 > 0$). It implies $\{x_n\}$ is bounded. By (2.5) and the uniform continuity of T, we know $\{y_n\}$ and $\{Ty_n\}$ are both bounded. Since $\eta_n \in Ty_n$, $\{L + ||x_n|| + ||\eta_n||\}$ is bounded. Let its bound be $M > 0$. Denoting $\dfrac{2\hat{\beta}_n}{1 - 2\hat{\beta}_n}$ by t_n, it follows from (2.3) that

$$t_n \ge 2\hat{\beta}_n \ge \frac{2\beta_n}{M}, \quad \sum_{n=n_0}^{\infty} t_n = \infty.$$

By (2.8), we have that

$$\tilde{a}_{n+1}^2 \le \tilde{a}_n^2 + \tilde{b}_n \tilde{a}_n^2 - t_n s_0 \frac{m_0}{4} + \tilde{c}_n \tilde{a}_{n+1}, \quad \forall\, n \ge n_0 \tag{2.10}$$

Taking $n = n_0, n_0 + 1, \cdots, r$ in (2.10) above, we obtain

$$\tilde{a}_{r+1}^2 \le \tilde{a}_{n_0}^2 + \sum_{n=n_0}^{r} \tilde{b}_n \tilde{a}_n^2 - \frac{s_0 m_0}{4} \sum_{n=n_0}^{r} t_n + \sum_{n=n_0}^{r} \tilde{c}_n \tilde{a}_{n+1},$$

$$\frac{s_0 m_0}{4} \sum_{n=n_0}^{r} t_n \le \tilde{a}_{n_0}^2 + M_0^2 \sum_{n=n_0}^{r} \tilde{b}_n + M_0 \sum_{n=n_0}^{r} \tilde{c}_n$$

This leads to a contradiction as $r \to \infty$. Hence, $\liminf\limits_{n\to\infty} \Phi(||x_{n+1} - x^*||) = 0$.

Thus there exists a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that $x_{n_i+1} \to x^*$ as $i \to \infty$. Combining the definition of Φ, (2.3) with the fact that $g_n \to 0$ as $n \to \infty$, for any given ε in $(0, 1)$, there exists a $n_j \in \mathbb{N}$ such that the followings hold

$$\begin{cases} ||x_{n_j+1} - x^*|| < \dfrac{\varepsilon}{2}, \quad \displaystyle\sum_{n=n_j}^{\infty} \beta_n^2 < \frac{\varepsilon}{64}, \quad \sum_{n=n_j}^{\infty} \gamma_n^2 < \sum_{n=n_j}^{\infty} \gamma_n < \frac{\varepsilon}{64(L + ||x^*||)}, \\ \qquad g_n \le \dfrac{1}{2}\Phi(\dfrac{\varepsilon}{2}), \quad \hat{\beta}_n \le \beta_n < \dfrac{1}{4}, \quad \forall\, n \ge n_j. \end{cases} \tag{2.11}$$

Next, we will show that for every $m \in \mathbb{N}$, the following inequality holds

$$||x_{n_j+m} - x^*|| < \varepsilon. \tag{2.12}$$

For $m=1$, by (2.11) we have $||x_{n_j+1}-x^*||<\frac{\varepsilon}{2}<\varepsilon$, So (2.12) holds for $m=1$. Now suppose

$$||x_{n_j+r}-x^*||<\varepsilon, \quad r=1,2,\cdots m. \tag{2.13}$$

Set

$$p:=\max\{r\in\mathbb{N}, ||x_{n_j+r}-x^*||<\frac{\varepsilon}{2}, 1\le r\le m+1\}.$$

Considering (2.11), p is well defined. If $p=m+1$, then

$$||x_{n_j+m+1}-x^*||<\frac{\varepsilon}{2}<\varepsilon.$$

By induction, (2.12) holds for any $m\in\mathbb{N}$. If $1\le p\le m$, then, by the definition of p , we have

$$||x_{n_j+r}-x^*||\ge\frac{\varepsilon}{2}, \quad p+1\le r\le m+1. \tag{2.14}$$

Combining (2.7), (2.11), (2.14) with the definition of Φ, we obtain, for $r=p,p+1,\cdots,m$

$$\begin{aligned}||x_{n_j+r+1}-x^*||^2 \le{}& ||x_{n_j+r}-x^*||^2+\frac{2(\hat{\beta}^2_{n_j+r}+\hat{\gamma}^2_{n_j+r})}{1-2\hat{\beta}_{n_j+r}}||x_{n_j+r}-x^*||^2-||x_{n_j+r+1}-x^*||\\ &\cdot(\Phi(\frac{\varepsilon}{2})-\frac{1}{2}\Phi(\frac{\varepsilon}{2}))\frac{2\hat{\beta}_{n_j+r}}{1-2\hat{\beta}_{n_j+r}}+\frac{2\hat{\gamma}_{n_j+r}(||u_{n_j+r}||+||x^*||)}{1-2\hat{\beta}_{n_j+r}}||x_{n_j+r+1}-x^*||\\ \le{}& ||x_{n_j+r}-x^*||^2+\frac{2(\hat{\beta}^2_{n_j+r}+\hat{\gamma}^2_{n_j+r})}{1-2\hat{\beta}_{n_j+r}}||x_{n_j+r}-x^*||^2\\ &+\frac{2\hat{\gamma}_{n_j+r}(||u_{n_j+r}||+||x^*||)}{1-2\hat{\beta}_{n_j+r}}||x_{n_j+r+1}-x^*||.\end{aligned}$$

Further, from (2.2), (2.11) and (2.13), we deduce that

$$\begin{aligned}||x_{n_j+m+1}-x^*||^2 \le{}& ||x_{n_j+p}-x^*||^2+\sum_{r=p}^{m}\frac{2(\hat{\beta}^2_{n_j+r}+\hat{\gamma}^2_{n_j+r})}{1-2\hat{\beta}_{n_j+r}}||x_{n_j+r}-x^*||^2\\ &+\sum_{r=p}^{m}\frac{2\hat{\gamma}_{n_j+r}(||u_{n_j+r}||+||x^*||)}{1-2\hat{\beta}_{n_j+r}}||x_{n_j+r+1}-x^*||\\ \le{}& ||x_{n_j+p}-x^*||^2+\sum_{r=p}^{m}4(\beta^2_{n_j+r}+\gamma^2_{n_j+r})||x_{n_j+r}-x^*||^2\\ &+4\gamma_{n_j+m}(L+||x^*||)||x_{n_j+m+1}-x^*||\\ &+\sum_{r=p}^{m-1}4\gamma_{n_j+r}(L+||x^*||)||x_{n_j+r+1}-x^*||\\ <{}&(\frac{\varepsilon}{2})^2+\frac{1}{8}\varepsilon^2+\frac{\varepsilon}{8}||x_{n_j+m+1}-x^*||+\frac{\varepsilon}{8}\cdot\varepsilon\\ \le{}&\frac{\varepsilon^2}{2}+\frac{\varepsilon}{8}||x_{n_j+m+1}-x^*||.\end{aligned}$$

Hence, we obtain

$$||x_{n_j+m+1}-x^*|| \le \frac{1}{2}\left\{\frac{\varepsilon}{8}+\sqrt{(\frac{\varepsilon}{8})^2+4(\frac{\varepsilon^2}{2})}\right\} < \varepsilon.$$

By induction, (2.12) holds for any $m \in \mathbb{N}$.

Therefore, for any given ε in $(0,1)$, there exists a $n_j \in \mathbb{N}$ such that

$$||x_{n_j+m}-x^*|| < \varepsilon, \quad \forall\, m \in \mathbb{N}.$$

It means that $\{x_n\}$ converges strongly to the unique fixed point of T. The proof is finished. □

Remark 2.3. Theorem 2.2 improves [4, Theorem 3.2], [5, Theorem 3.2], [7, Theorem 3.3] [10, Theorem 1] and [11, Theorem 2] since

(1) Banach spaces in these articles are extended to arbitrary normed linear spaces.

(2) the requirement in Theorem 3.2 of [4] that the sequences $\{e_n\}$ and $\{f_n\}$ are bounded is dispensed with. So is the similar requirement in Theorem 3.2 of [5].

(3) Theorem 3.3 of [7] is extended to the case in which T is a multi-valued mapping and the boundedness requirement of the range of the mapping T in Theorem 3.3 of [7] is dropped.

(4) Theorem 1 of [10] and Theorem 2 of [11] are extended to the case in which T is a multi-valued mapping without the requirement that the mapping T is Lipschitz continuous.

Theorem 2.4. *Let X be a real normed linear space, $T : X \to CB(X)$ be uniformly continuous and Φ-hemiaccretive. For any given $f \in X$, define $S : X \to CB(X)$ by $Sx := x - Tx + f$ for all $x \in X$. For any given $x_0, v_0, u_0 \in X$, let $\{x_n\}$ be the modified Ishikawa iterative sequence with errors, defined by*

$$\begin{cases} y_n = \hat{a}_n x_n + \hat{b}_n \xi_n + \hat{c}_n v_n, \quad \exists\, \xi_n \in Sx_n, \\ x_{n+1} = \hat{\alpha}_n x_n + \hat{\beta}_n \eta_n + \hat{\gamma}_n u_n, \quad \exists\, \eta_n \in Sy_n, \end{cases} \quad n = 0,1,2,\cdots, \tag{2.15}$$

where $\{u_n\}, \{v_n\}$ are both bounded sequences in X(Let their bound be $L \ge 2$),

$$\begin{aligned} \hat{a}_n &= 1-\hat{b}_n-\hat{c}_n,\ \hat{b}_n = \frac{b_n}{L+||x_n||+||\xi_n||},\ \hat{c}_n = \frac{c_n}{L+||x_n||+||\xi_n||}, \\ \hat{\alpha}_n &= 1-\hat{\beta}_n-\hat{\gamma}_n,\ \hat{\beta}_n = \frac{\beta_n}{L+||x_n||+||\eta_n||},\ \hat{\gamma}_n = \frac{\gamma_n}{L+||x_n||+||\eta_n||}, \end{aligned} \tag{2.16}$$

are four sequences in $[0,1]$ satisfying the following conditions:

$$\sum_{n=0}^{\infty}\beta_n = \infty, \sum_{n=0}^{\infty}\beta_n^2 < \infty, \sum_{n=0}^{\infty}\gamma_n < \infty, \lim_{n\to\infty} b_n = 0, \lim_{n\to\infty} c_n = 0. \tag{2.17}$$

Then $\{x_n\}$ converges strongly to the unique solution of the Φ-hemiaccretive mapping equation $f \in Tx$.

Remark 2.5. Theorem 2.4 is a new and general result on the iterative approximation of the solution of the multi-valued Φ-hemiaccretive mappings equation without Lipschitz assumption and without boundedness conditions in real normed linear spaces, which improves [4, Theorem 4.2], [7, Theorem 3.1], [10, Theorem 2] and [11, Theorem 1].

Acknowledgments

The authors thank the referees for their careful reading and helpful comments.

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