Communications in **M**athematical **A**nalysis
Volume 7, Number 1, pp. 21–36 (2009)
ISSN 1938-9787
www.commun-math-anal.org

LIPSCHITZ SPACES ASSOCIATED WITH REFLECTION GROUP $\mathbb{Z}_2^d$

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(Communicated by Toka Diagana)

Abstract

In this paper we introduce the Lipschitz-Dunkl spaces associated with reflection group $\mathbb{Z}_2^d$. We provide characterizations of these spaces involving Bochner-Riesz means and we obtain an approximation result that involves partial Dunkl integrals.

AMS Subject Classification: MSC(2000): 44A15, 44A35, 46E30.

Keywords: Dunkl operator, Dunkl transform, Dunkl translation operator, Bochner-Riesz means, partial Dunkl integrals, Lipschitz-Dunkl spaces.

1 Introduction

The Dunkl theory is built around Dunkl operators. They are differential-difference operators, associated to a finite reflection group, introduced by C.F. Dunkl [3] in 1989. They can be regarded as a generalization of the usual partial derivatives by additional reflection terms. Dunkl operators lead to generalizations of various analytic structures like the Fourier transform and the convolution product. By introducing Dunkl's intertwining operator and Dunkl kernel [4] and thereafter Dunkl transform [5], C.F. Dunkl built up a framework for an harmonic analysis theory with reflection groups. During the last years, the Dunkl theory have gained considerable interest in various fields of mathematics.

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This paper deals with Lipschitz-Dunkl spaces. Assuming that the reflection group involving in definition of the Dunkl operator is $\mathbb{Z}_2^d$, we consider the function space ΛD_β that we call Lipschitz-Dunkl and which is the set of functions $f \in L^\infty_\kappa(\mathbb{R}^d)$ satisfying

$$\sup_{z\in\mathbb{R}^d\setminus\{0\}} (\|z\|^{-\beta}\|\tau_z f - f\|_{\infty,\kappa}) < \infty,$$

where τ_z is the Dunkl translation operator (see [16, 18]) and $0 < \beta \le 1$.

We obtain a characterization of the spaces ΛD_β that involves Bochner-Riesz means, (for the definition of Bochner-Riesz means for the classical Fourier transform, one can see [15, p.170])

$$\sigma_T^\delta f(x) = c_\kappa^2 \int_{\|y\|\le T} \left(1 - \frac{\|y\|^2}{T^2}\right)^\delta \mathcal{F}_\kappa f(y)\, E_\kappa(ix,y)\, w_\kappa(y) dy, \quad x \in \mathbb{R}^d,$$

when $0 < \beta < 1$.
Also, using Bochner-Riesz means, we establish a necessary condition for a function $f \in L^2_\kappa(\mathbb{R}^d)$ to be in ΛD_1. Finally we prove an approximation result for the functions belonging to ΛD_β, where $0 < \beta < 1$, involving the partial Dunkl integrals,

$$s_T f(x) = c_\kappa^2 \int_{\|y\|\le T} \mathcal{F}_\kappa f(y) E_\kappa(ix,y) w_\kappa(y) dy, \quad x \in \mathbb{R}^d.$$

Similar results have been obtained by Betancor and Rodríguez-Mesa in [1] in the framework of Hankel transfom. Their work was motivated by one of D.V. Giang [7], where Lipschitz spaces, on $\mathbb{R}$, are studied through the classical Fourier transform and Hilbert transform.

The authors are supported by the DGRST research project 04/UR/15-02.

2 Dunkl harmonic analysis on $\mathbb{R}^d$

For $\alpha \in \mathbb{R}^d \setminus \{0\}$, we denote by σ_α the reflection in the hyperplane H_α orthogonal to α :

$$\sigma_\alpha(x) = x - \frac{2\langle\alpha,x\rangle}{\|\alpha\|^2}\alpha, \qquad x \in \mathbb{R}^d.$$

A finite subset $\mathcal{R}$ of $\mathbb{R}^d \setminus \{0\}$ is called a root system if it verifies :

(1) $\mathcal{R} \cap \mathbb{R}.\alpha = \{-\alpha, \alpha\}$, for all $\alpha \in \mathcal{R}$.

(2) $\sigma_\alpha(\mathcal{R}) = \mathcal{R}$, for all $\alpha \in \mathcal{R}$.

Let $\mathcal{R}$ be a root system. The subgroup $W = W(\mathcal{R})$ of the orthogonal group $\mathbf{O}(d)$, which is generated by the reflections $\{\sigma_\alpha \mid \alpha \in \mathcal{R}\}$ is called the reflection group associated with $\mathcal{R}$. It is shown in [6] that W is finite. For a given $\alpha \in \mathbb{R}^d \setminus \bigcup_{\beta\in\mathcal{R}} H_\beta$, we consider the following subsystem $\mathcal{R}_+$ of $\mathcal{R}$:

$$\mathcal{R}_+ = \{\beta \in \mathcal{R} \mid \langle\alpha,\beta\rangle > 0\},$$

that we call positive subsystem of $\mathcal{R}$. Notice that for each $\beta \in \mathcal{R}$, either $\beta \in \mathcal{R}_+$ or $-\beta \in \mathcal{R}_+$. A function $\kappa : \mathcal{R} \longrightarrow \mathbb{C}$ is called a multiplicity function on $\mathcal{R}$ if it is $W-$invariant. This means that κ is constant on each orbit of $\mathcal{R}$ under the action of W. In this paper, we assume that the multiplicity function κ is nonnegative. We associate with κ the index

$$\gamma = \gamma(\mathcal{R}) = \sum_{\alpha \in \mathcal{R}_+} \kappa(\alpha) \geq 0,$$

and the weight function w_κ defined by

$$w_\kappa(x) = \prod_{\alpha \in \mathcal{R}_+} |\langle \alpha, x \rangle|^{2\kappa(\alpha)}, \qquad x \in \mathbb{R}^d.$$

Notice that, since κ is $W-$invariant, the definition of the weight function w_κ does not depend of the special choice of $\mathcal{R}_+$. On the other hand, w_κ is $W-$invariant and homogeneous of degree 2γ.

Further, we introduce the Mehta-type constant c_κ by

$$c_\kappa = \left(\int_{\mathbb{R}^d} e^{-\|x\|^2/2} w_\kappa(x) dx \right)^{-1}.$$

In the case where $d = 1$ and $\alpha = 1$, we have $c_\kappa = \dfrac{1}{2^{\gamma+\frac{1}{2}}\, \Gamma\left(\gamma + \frac{1}{2}\right)}$.

We denote by $L^p_\kappa(\mathbb{R}^d)$, $1 \leq p \leq +\infty$, the space of complex-valued functions f, measurable on $\mathbb{R}^d$ such that

$$\|f\|_{p,\kappa} = \left[\int_{\mathbb{R}^d} |f(x)|^p w_\kappa(x) dx \right]^{\frac{1}{p}} < +\infty, \text{ if } p < +\infty,$$

and

$$\|f\|_{\infty,\kappa} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f(x)| < +\infty.$$

$\mathcal{E}(\mathbb{R}^d)$ designates the space of infinitely differentiable functions on $\mathbb{R}^d$.

Definition 2.1. Let $\mathcal{R}$ be a root system. We fix a positive subsystem $\mathcal{R}_+$ of $\mathcal{R}$ and we consider a multiplicity function κ on $\mathcal{R}$. The Dunkl operators T_j; $1 \leq j \leq d$, on $\mathbb{R}^d$ associated with the reflection group W and the multiplicity function κ are the first-order differential- difference operators given by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in \mathcal{R}_+} \kappa(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}, \quad f \in \mathcal{E}(\mathbb{R}^d), \quad x \in \mathbb{R}^d,$$

where $\alpha_j = \langle \alpha, e_j \rangle$, $(e_1, e_2, \ldots, e_d)$ being the canonical basis of $\mathbb{R}^d$.

Notice that this definition does not depend on the special choice of $\mathcal{R}_+$. Furthermore, in the case $\kappa = 0$ the operators T_j, $1 \leq j \leq d$, reduce to the corresponding partial derivatives.
Examples : 1) In $\mathbb{R}$, any root system takes the form $\mathcal{R} = \{-\alpha, \alpha\}$ where $\alpha > 0$. The

corresponding reflection group is $\mathbb{Z}_2 = \{id, \sigma\}$ acting on $\mathbb{R}$ by $\sigma(x) = -x$. Therefore, the Dunkl operator on $\mathbb{R}$ associated with the multiplicity parameter γ is given by

$$T f(x) = \frac{df}{dx}(x) + \gamma \frac{f(x) - f(-x)}{x}, \quad f \in \mathcal{E}(\mathbb{R}), \quad x \in \mathbb{R}.$$

2) In $\mathbb{R}^d$, $\mathcal{R} = \{\pm e_1, \pm e_2, \ldots, \pm e_d\}$ is a root system. The reflection group associated with $\mathcal{R}$ is $W = \mathbb{Z}_2^d$ and the Dunkl operators related to W are given, for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{i=1}^{d} \kappa_i \frac{f(x) - f(\widehat{x}^i)}{x_i}, \quad 1 \le j \le d, \quad f \in \mathcal{E}(\mathbb{R}),$$

where $\widehat{x}^i = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1} \ldots, x_d)$ and $\kappa_1, \kappa_2, \ldots, \kappa_d$ are nonnegative real numbers.

According to [4], there exists a linear isomorphism V_κ of the algebra of polynomials on $\mathbb{R}^d$, $\mathcal{P} = \mathbb{C}[\mathbb{R}^d]$, determined uniquely by

$$V_\kappa(\mathcal{P}_n) = \mathcal{P}_n, \; V_{\kappa|_{\mathcal{P}_0}} = id \text{ and } T_j V_\kappa = V_\kappa \frac{\partial}{\partial x_j}, \quad n \in \mathbb{N} \cup \{0\}, \; 1 \le j \le d,$$

where $\mathcal{P}_n \subset \mathcal{P}$ is the subspace of homogeneous polynomials of degree n.
It is known that V_κ, which is called the Dunkl's intertwining operator, is a positive operator (see [12].) More precisely, M. Rösler has proved in [12] the following integral representation of V_κ:

Theorem 2.2. *For each $x \in \mathbb{R}^d$, there exists a unique probability measure μ_x^κ on the Borel $\sigma-$algebra of $\mathbb{R}^d$ such that*

$$V_\kappa(p)(x) = \int_{\mathbb{R}^d} p(y) d\mu_x^\kappa(y), \qquad p \in \mathcal{P}. \tag{2.1}$$

The representing measures μ_x^κ are compactly supported and theirs supports verify $\mathrm{supp}\mu_x^\kappa \subseteq \{y \in \mathbb{R}^d \;\; | \;\; \|y\| \le \|x\|\}$. *Moreover, they satisfy*

$$\mu_{rx}^\kappa(B) = \mu_x^\kappa(r^{-1}B), \quad \mu_{w(x)}^\kappa(B) = \mu_x^\kappa(w^{-1}(B)), \qquad r > 0, \quad w \in W,$$

for each Borel set $B \subseteq \mathbb{R}^d$.

By means of the identity (2.1), the Dunkl's intertwining operator V_κ, may be extended to $\mathcal{E}(\mathbb{R}^d)$. In fact, V_κ establishes a topological isomorphism from $\mathcal{E}(\mathbb{R}^d)$ onto itself (see [18, Theorem 3.1])

The Dunkl kernel $E_\kappa(x,y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ has been introduced by C.F. Dunkl in [4] by means of the Dunkl intertwining V_κ as follows

$$E_\kappa(x,y) = V_\kappa\left(e^{\langle \cdot, y\rangle}\right)(x), \qquad x,y \in \mathbb{R}^d.$$

For $y \in \mathbb{R}^d$, the function $x \longmapsto E_\kappa(x,y)$ can be viewed as the unique solution on $\mathbb{R}^d$ of the following initial problem

$$\begin{cases} T_j u(x,y) &= y_j u(x,y), \qquad 1 \le j \le d, \\ u(0,y) &= 1. \end{cases}$$

Examples : 1) If $d = 1$ then, for $\gamma > 0$, the Dunkl kernel takes the form

$$E_\gamma(x,y) = j_{\gamma-\frac{1}{2}}(ixy) + \frac{xy}{2\gamma+1} j_{\gamma+\frac{1}{2}}(ixy), \qquad x,y \in \mathbb{R},$$

where, for $\alpha > -1/2$,

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^{+\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\Gamma(n+\alpha+1)}, \quad z \in \mathbb{C},$$

with J_α is the Bessel function of the first kind and order α (see [19])
2) In the case of the reflection group $W = \mathbb{Z}_2^d$, the multiplicity function κ is represented by d positive real numbers $\kappa_1, \kappa_2, \ldots, \kappa_d$ (we suppose that $\kappa > 0$.) The expression of Dunkl kernel E_κ and [20, Theorem 4.3] yield

$$E_\kappa(x,y) = \prod_{m=1}^{d} \left(\frac{\Gamma\left(\kappa_m + \frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(\kappa_m)} \int_{-1}^{1} (1-t^2)^{\kappa_m - 1}(1+t) e^{t x_m y_m}\, dt \right), \ \ x,y \in \mathbb{R}^d.$$

Finally, by using [11, Lemma 2.1], we get

$$E_\kappa(x,y) = \prod_{m=1}^{d} \left(j_{\kappa_m - \frac{1}{2}}(ix_m y_m) + \frac{x_m y_m}{2\kappa_m + 1} j_{\kappa_m + \frac{1}{2}}(ix_m y_m) \right), \qquad x,y \in \mathbb{R}^d.$$

Below, let us collect some known properties of Dunkl kernel which we can find in many references, for instance, [4, 12, 14, 16, 17].
Properties : (i) The Dunkl kernel has the Bochner-type representation

$$E_\kappa(x,z) = \int_{\mathbb{R}^d} e^{\langle y,z \rangle}\, d\mu_x^\kappa(y), \qquad x \in \mathbb{R}^d, \quad z \in \mathbb{C}^d,$$

where μ_x^κ are the representing measures from Theorem 2.2 .
(ii) For all $x,y \in \mathbb{R}^d$, $\qquad |E_\kappa(x,iy)| \le 1$.
(iii) For all $x,y \in \mathbb{C}^d$, $\qquad E_\kappa(x,y) = E_\kappa(y,x)$.
(iv) For all $x,y \in \mathbb{C}^d$ and $\lambda \in \mathbb{C}$, $\quad E_\kappa(\lambda x,y) = E_\kappa(x,\lambda y)$.

The function $E_\kappa(x,iy)$ plays the role of the exponential function $e^{i\langle x,y \rangle}$ in the classical Fourier analysis. In fact, C.F. Dunkl [5] has introduced the Dunkl transform, in terms of Dunkl kernel, by

$$\mathcal{F}_\kappa f(x) = \int_{\mathbb{R}^d} f(y) E_\kappa(-ix,y) w_\kappa(y) dy, \qquad f \in L_\kappa^1(\mathbb{R}^d), \ \ x \in \mathbb{R}^d.$$

Further results concerning the Dunkl transform were later provided by M.F.E. de Jeu in [9].

The results listed in the below theorem are proved in [5, 9]

Theorem 2.3. *(i) The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is invariant under $\mathcal{F}_\kappa$.*

(ii) (Inversion formula) For all $f \in L_\kappa^1(\mathbb{R}^d)$ such that $\mathcal{F}_\kappa f$ belongs to $L_\kappa^1(\mathbb{R}^d)$, we have

$$f(x) = c_\kappa^2 \int_{\mathbb{R}^d} \mathcal{F}_\kappa f(y) E_\kappa(iy,x) w_\kappa(y) dy, \qquad \text{a.e } x \in \mathbb{R}^d. \tag{2.2}$$

(iii) (Plancherel's theorem) The normalized Dunkl transform $c_\kappa \mathcal{F}_\kappa$ can be uniquely extended to an isometric isomorphism on $L^2_\kappa(\mathbb{R}^d)$. In particular, we have the following Plancherel's formula

$$\int_{\mathbb{R}^d} |f(x)|^2 \, w_\kappa(x)dx = c_\kappa^2 \int_{\mathbb{R}^d} |\mathcal{F}_\kappa f(y)|^2 \, w_\kappa(y)dy\,, \qquad f \in L^2_\kappa(\mathbb{R}^d)\,. \tag{2.3}$$

K. Trimèche has introduced in [18] the Dunkl translation operators by defining them, on $\mathcal{E}(\mathbb{R}^d)$, by

$$\tau_x f(y) = V_\kappa^x \otimes V_\kappa^y [(V_\kappa)^{-1} f(x+y)]\,, \qquad f \in \mathcal{E}(\mathbb{R}^d)\,,\ \ x,y \in \mathbb{R}^d\,,$$

here the superscript denotes the relevant variable. These operators satisfy for all $f \in \mathcal{E}(\mathbb{R}^d)$ and $x,y \in \mathbb{R}^d$, the following properties

$$\tau_0 f = f\,, \quad T_j(\tau_x f) = \tau_x(T_j f) \quad \text{and} \quad \tau_x f(y) = \tau_y f(x)\,, \quad 1 \le j \le d\,.$$

It is obvious from the definition that we have for all $x,y \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$

$$\tau_x(E_\kappa(\,.\,,z))(y) = E_\kappa(x,z)E_\kappa(y,z)\,.$$

Further properties of the Dunkl translation are in the below proposition which is shown in [18]

Proposition 2.4. *Assume that f belongs to $\mathcal{S}(\mathbb{R}^d)$. Then we have*

$$\mathcal{F}_\kappa(\tau_x f)(y) = E_\kappa(ix,y)\mathcal{F}_\kappa f(y)\,, \qquad x,y \in \mathbb{R}^d\,. \tag{2.4}$$

Moreover,

$$\tau_x f(y) = c_\kappa^2 \int_{\mathbb{R}^d} \mathcal{F}_\kappa f(\xi)E_\kappa(ix,\xi)E_\kappa(iy,\xi)w_\kappa(\xi)d\xi\,. \tag{2.5}$$

As the Dunkl transform is an isometry of $L^2_\kappa(\mathbb{R}^d)$ onto itself and the function $E_\kappa(ix,y)$ is bounded, we can define the Dunkl translation operators on $L^2_\kappa(\mathbb{R}^d)$ by the relation (2.4) of the above proposition (see S. Thangavelu and Y. Xu [16].) On the other hand, by combining the relation (2.4) with Plancherel formula we deduce that τ_x, $x \in \mathbb{R}^d$, is bounded as an operator on $L^2_\kappa(\mathbb{R}^d)$ into itself. For p different from 2 the L^p boundedness of τ_x is still an open problem. At the moment an explicit formula for $\tau_x f$ is unknown in general. However, a such formula is known when the reflection group is $W = \mathbb{Z}_2^d$ (see [11] and [16]) and when f is radial (see [13].)

In the case when $d = 1$, M. Rösler has shown in [11] that the Dunkl translation operators τ_x, $x \in \mathbb{R}$ possess an integral representation

$$\tau_x f(y) = \int_{\mathbb{R}} f(t)d\nu_{x,y}(t)\,, \qquad f \in \mathcal{E}(\mathbb{R})\,, \tag{2.6}$$

where $\nu_{x,y}$ is a finite signed measure on $\mathbb{R}$, with uniformly bounded total variation norm and supported in $[-|x|-|y|\,, -||x|-|y||] \cup [||x|-|y||\,, |x|+|y|]$.
An immediate consequence of the explicit formula (2.6) is the L^p boundedness of the Dunkl translation operator on $\mathbb{R}$. S. Thangavelu and Y. Xu have shown in [16] that the explicit

formula (2.6) extends to the case where the reflection group is $W=\mathbb{Z}_2^d$. They have proved, in this case, that the operator τ_x , $x\in\mathbb{R}^d$, is bounded on $L^p_\kappa(\mathbb{R}^d)$, $1\le p\le+\infty$, and we have

$$\|\tau_x f\|_{p,\kappa}\le 3\|f\|_{p,\kappa}, \qquad x\in\mathbb{R}^d, \quad f\in L^p_\kappa(\mathbb{R}^d). \tag{2.7}$$

The Dunkl convolution product $\star_\kappa$ of two functions f and g in $L^2_\kappa(\mathbb{R}^d)$, (see [16]), is given by

$$f\star_\kappa g(x)=\int_{\mathbb{R}^d}\tau_x f(-y)g(y)w_\kappa(y)dy, \qquad x\in\mathbb{R}^d.$$

For f and g belonging in $\mathcal{D}(\mathbb{R}^d)$ the following equalities of the Dunkl convolution product $\star_\kappa$ are shown in [18]

$$f\star_\kappa g=g\star_\kappa f \quad\text{and}\quad \mathcal{F}_\kappa(f\star_\kappa g)=(\mathcal{F}_\kappa f)(\mathcal{F}_\kappa g). \tag{2.8}$$

Lemma 2.5. *Let $f\in L^2_\kappa(\mathbb{R}^d)$ and $g\in L^1_\kappa(\mathbb{R}^d)$. Then*

$$\mathcal{F}_\kappa\left(f\star_\kappa g\right)(x)=\mathcal{F}_\kappa f(x)\mathcal{F}_\kappa g(x) \qquad\qquad \text{a.e } x\in\mathbb{R}^d. \tag{2.9}$$

Proof. For every $f\in\mathcal{D}(\mathbb{R}^d)$ and $g\in L^1_\kappa(\mathbb{R}^d)$, the identity (2.9) holds.
Fix $g\in L^1_\kappa(\mathbb{R}^d)$. For every $f\in L^2_\kappa(\mathbb{R}^d)$, $f\star_\kappa g\in L^2_\kappa(\mathbb{R}^d)$ and we have

$$\|\mathcal{F}_\kappa\left(f\star_\kappa g\right)\|_{2,\kappa}=\|f\star_\kappa g\|_{2,\kappa}\le\|g\|_{1,\kappa}\|f\|_{2,\kappa}, \quad f\in L^2_\kappa(\mathbb{R}^d).$$

On the other hand, $\mathcal{F}_\kappa g\in L^\infty_\kappa(\mathbb{R}^d)$ then we have also

$$\|\mathcal{F}_\kappa f\mathcal{F}_\kappa g\|_{2,\kappa}\le\|g\|_{\infty,\kappa}\|f\|_{2,\kappa}, \qquad f\in L^2_\kappa(\mathbb{R}^d).$$

Therefore, for every $g\in L^1_\kappa(\mathbb{R}^d)$ each term of the identity (2.9) define a bounded operator from $L^2_\kappa(\mathbb{R}^d)$ into itself. Since $\mathcal{D}(\mathbb{R}^d)$ is a dense subset of $L^2_\kappa(\mathbb{R}^d)$ we get the identity (2.9) for every $f\in L^2_\kappa(\mathbb{R}^d)$ and $g\in L^1_\kappa(\mathbb{R}^d)$. □

Lemma 2.6. *Let $f\in L^1_\kappa(\mathbb{R}^d)$ and $g\in L^p_\kappa(\mathbb{R}^d)$, $1\le p<+\infty$. Then we have*

$$\tau_v(f\star_\kappa g)=\tau_v f\star_\kappa g=f\star_\kappa\tau_v g, \quad v\in\mathbb{R}^d. \tag{2.10}$$

Proof. Since $\mathcal{D}(\mathbb{R}^d)$ is a dense subset of $L^q_\kappa(\mathbb{R}^d)$ $(1\le q<+\infty)$ it is sufficient to prove (2.10) when f and g are in $\mathcal{D}(\mathbb{R}^d)$. By using the relations (2.5), (2.4) and the second equality of (2.8) , we can write

$$\tau_v(f\star_\kappa g)(x)=c_\kappa^2\int_{\mathbb{R}^d}\mathcal{F}_\kappa f(\xi)\mathcal{F}_\kappa(\tau_v g)(\xi)E_\kappa(ix,\xi)w_\kappa(\xi)d\xi, \quad x,v\in\mathbb{R}^d.$$

Next invoking the Fubini's theorem and again the relation (2.5), we deduce that we have

$$\tau_v(f\star_\kappa g)(x)=\int_{\mathbb{R}^d}\tau_v g(z)\,\tau_x f(-z)\,w_\kappa(z)dz=f\star_\kappa\tau_v g(x), \quad x,v\in\mathbb{R}^d.$$

With the help of the commutativity of the convolution operator $\star_\kappa$ we complete the proof. □

The Dunkl translation operator, $\tau_x \ (x \in \mathbb{R}^d)$, can be extended to all radial functions in $L^p_\kappa(\mathbb{R}^d)$ provided that $1 \le p \le 2$. In this sense, the following proposition collects some results demonstrated by Thangavelu and Xu in [16]

Proposition 2.7. *(i) Let f be a radial function in $L^p_\kappa(\mathbb{R}^d)$, $1 \le p \le 2$. For each $x \in \mathbb{R}$, the following inequality holds*

$$\|\tau_x f\|_{p,\kappa} \le \|f\|_{p,\kappa}. \tag{2.11}$$

(ii) If f is a bounded radial function belonging to $L^1_\kappa(\mathbb{R}^d)$ then the convolution product $f \star_\kappa g$ can be defined for all $g \in L^p_\kappa(\mathbb{R}^d)$, $1 \le p \le +\infty$, and we have

$$\|f \star_\kappa g\|_{p,\kappa} \le \|f\|_{1,\kappa} \|g\|_{p,\kappa}. \tag{2.12}$$

At the end of this section we recall a result which was shown by the first author (see [10, Lemma 2.3])

Proposition 2.8. *Let f be a radial function in $L^1_\kappa(\mathbb{R}^d)$ such that the function F, given by $F(\|x\|) = f(x)$, $x \in \mathbb{R}^d$, is differentiable on $\mathbb{R}^+$. Assume that the function g defined by $g(x) = F'(\|x\|)$, $x \in \mathbb{R}^d$, belongs to $L^1_\kappa(\mathbb{R}^d)$. Then, for all $u \in S^{d-1}$,*

$$\|\tau_{tu} f - f\|_{1,\kappa} \le t\, \|g\|_{1,\kappa}, \qquad t > 0.$$

3 Lipschitz-Dunkl spaces associated with the reflection group $\mathbb{Z}_2^d$

Throughout this section, we assume that the reflection group is $W = \mathbb{Z}_2^d$. Furthermore, we represent by C a positive constant whose value can vary from one line to another. We introduce new function spaces that we call Lipschitz-Dunkl spaces as follows. For $0 < \beta \le 1$, the Lipschitz-Dunkl space ΛD_β is the set of functions $f \in L^\infty_\kappa(\mathbb{R}^d)$ verifying

$$\sup_{z \in \mathbb{R}^d \setminus \{0\}} \left(\|z\|^{-\beta} \|\tau_z f - f\|_{\infty,\kappa}\right) < +\infty.$$

Remark 3.1. If $f \in \Lambda D_\beta$ then we have

$$\sup_{t>0} \left[t^{-\beta} \int_{S^{d-1}} \|\tau_{tu} f - f\|_{\infty,\kappa}\, d\sigma(u) \right] < +\infty.$$

S^{d-1} being the unit sphere on $\mathbb{R}^d$ with the normalized surface measure $d\sigma$.

Let $T > 0$ and $\delta \ge 0$. We define the Bochner-Riesz mean $\sigma^\delta_T f$ of a function $f \in L^1_\kappa(\mathbb{R}^d)$ by

$$\sigma^\delta_T f(x) = c_\kappa^2 \int_{\|y\| \le T} \left(1 - \frac{\|y\|^2}{T^2}\right)^\delta \mathcal{F}_\kappa f(y)\, E_\kappa(ix, y)\, w_\kappa(y) dy, \quad x \in \mathbb{R}^d.$$

The results of the two below lemmas were proved in [10, proposition 3.1, Lemma 3.1 and Lemma 3.2]

Lemma 3.2. *Let $f \in L^1_\kappa(\mathbb{R}^d)$. If $\delta > \gamma + \frac{d-1}{2}$ then for $T > 0$, the Bochner-Riesz mean of f is given by the following convolution relation*

$$\sigma_T^\delta f = \Phi_{T,\delta} \star_\kappa f, \tag{3.1}$$

where

$$\Phi_{T,\delta}(x) = \frac{c_\kappa \Gamma(\delta+1)\, T^{2\gamma+d}}{2^{\gamma+\frac{d}{2}}\, \Gamma\left(\gamma+\frac{d}{2}+\delta+1\right)}\, j_{\gamma+\frac{d}{2}+\delta}(T\,\|x\|), \quad x \in \mathbb{R}^d. \tag{3.2}$$

and

$$\int_{\mathbb{R}^d} \Phi_{T,\delta}(x)\, w_\kappa(x)\, dx = 1, \qquad T > 0.$$

Lemma 3.3. *Let $\delta \geq \frac{d+1}{2}$, $1 \leq p < +\infty$ and $f \in L^p_\kappa(\mathbb{R}^d)$. We have,*

$$f(x)\mathrm{Log}2 = \int_0^{+\infty} \left[\sigma_{2T}^\delta f(x) - \sigma_T^\delta f(x)\right] \frac{dT}{T}, \quad a.e\ x \in \mathbb{R}^d.$$

Theorem 3.4. *Let $f \in L^2_\kappa(\mathbb{R}^d)$, $\delta \geq \frac{d+1}{2}$ and $0 < \beta < 1$ such that $\gamma + \frac{d-2}{2} < \delta - \beta - \frac{1}{2}$. Then $f \in \Lambda D_\beta$ if and only if $T^\beta \|\sigma_T^\delta f - f\|_{\infty,\kappa}$ is bounded on $(0,+\infty)$.*

Proof. Assume that $f \in \Lambda D_\beta$. According to Lemma 3.2, we can write

$$\sigma_T^\delta f(x) - f(x) = \int_{\mathbb{R}^d} \Phi_{T,\delta}(y)\, [\tau_y f(x) - f(x)]\, w_\kappa(y) dy, \quad x \in \mathbb{R}^d,$$

where $\Phi_{T,\delta}$ is given by the relation (3.2).
It follows that, for all $x \in \mathbb{R}^d$,

$$\sigma_T^\delta f(x) - f(x) =$$

$$CT^{2\gamma+d} \int_0^{+\infty} \left(\int_{S^{d-1}} [\tau_{tu} f(x) - f(x)] w_\kappa(u) d\sigma(u)\right) j_{\gamma+\frac{d}{2}+\delta}(tT) t^{2\gamma+d-1} dt,$$

By taking into account that w_κ is bounded on S^{d-1} we deduce that

$$\left|\sigma_T^\delta f(x) - f(x)\right| \leq CT^{2\gamma+d} \int_0^{+\infty} t^\beta \left|j_{\gamma+\frac{d}{2}+\delta}(tT)\right| t^{2\gamma+d-1}\, dt, \;\; x \in \mathbb{R}^d.$$

Next a change of variable leads

$$\left|\sigma_T^\delta f(x) - f(x)\right| \leq CT^{-\beta} \int_0^{+\infty} \left|j_{\gamma+\frac{d}{2}+\delta}(t)\right| t^{\beta+2\gamma+d-1}\, dt, \;\; x \in \mathbb{R}^d.$$

On the other hand $j_{\gamma+\frac{d}{2}+\delta}$ and $t^{\gamma+\frac{d}{2}+\delta+\frac{1}{2}}\, j_{\gamma+\frac{d}{2}+\delta}$ are bounded on $(0,+\infty)$. Therefore,

$$T^\beta \|\sigma_T^\delta f - f\|_{\infty,\kappa} \leq C\left[\int_0^1 t^{\beta+2\gamma+d-1}\, dt + \int_1^{+\infty} t^{\beta+\frac{d}{2}-\frac{3}{2}+\gamma-\delta}\, dt\right],$$

Hence $T^\beta \|\sigma_T^\delta f - f\|_{\infty,\kappa}$ is bounded on $(0,+\infty)$.

Conversely suppose that $T^{\beta}\|\sigma_T^{\delta} f - f\|_{\infty,\kappa}$ is bounded on $(0,+\infty)$. According to (3.1), (2.11) and the Hölder inequality , we get for all $T>0$,

$$\|\sigma_T^{\delta} f\|_{\infty,\kappa} \le \|\Phi_{T,\delta}\|_{2,\kappa}\|f\|_{2,\kappa}.$$

Hence,

$$\|f\|_{\infty,\kappa} \le \|\sigma_T^{\delta} f - f\|_{\infty,\kappa} + \|\sigma_T^{\delta} f\|_{\infty,\kappa} \le CT^{-\beta} + \|\Phi_{T,\delta}\|_{2,\kappa}\|f\|_{2,\kappa}.$$

This means that $f \in L^{\infty}_{\kappa}(\mathbb{R}^d)$. We define the operator Δ on $L^2_{\kappa}(\mathbb{R}^d)$ as follows

$$\Delta(f,t,u)(x) = \tau_{tu} f(x) - f(x), \qquad t>0,\ u \in S^{d-1},\ x \in \mathbb{R}^d.$$

Under the condition $\delta \ge \frac{d+1}{2}$ we obtain from Lemma 3.3 and the relation (2.10) , for almost everywhere $x \in \mathbb{R}^d$, $t>0$ and $u \in S^{d-1}$,

$$\Delta(f,t,u)(x)\,\mathrm{Log}2 = \int_0^{+\infty} \Delta(\sigma_{2T}^{\delta} f - \sigma_T^{\delta} f, t, u)(x)\,\frac{dT}{T}. \tag{3.3}$$

It follows from (2.7) that

$$\|\Delta(\sigma_{2T}^{\delta} f - \sigma_T^{\delta} f, t, u)\|_{\infty,\kappa} \le 4\,\|\sigma_{2T}^{\delta} f - \sigma_T^{\delta} f\|_{\infty,\kappa}, \quad t>0 \text{ and } T>0. \tag{3.4}$$

Choose an even smooth function g on $\mathbb{R}$ such that $g(t)=1$ if $|t|\le 1$ and $g(t)=0$ if $|t|\ge 2$. Designate by G and H the functions given by $G(x)=g(\|x\|)$ and $H(x)=c_{\kappa}^{-2}\mathcal{F}_{\kappa}G(x)$, $x \in \mathbb{R}^d$. For every $\varepsilon>0$ and $x \in \mathbb{R}^d$, we put $H_{\varepsilon}(x)=\varepsilon^{2\gamma+d}H(\varepsilon x)$. It is clear that $\mathcal{F}_{\kappa}(H_{\varepsilon})(x) = G\left(\frac{x}{\varepsilon}\right)$, $\varepsilon>0$ and $x \in \mathbb{R}^d$. In particular, if $\|x\|\le\varepsilon$ then $\mathcal{F}_{\kappa}(H_{\varepsilon})(x)=1$. By combining Lemma 2.5 with relations (2.4), (2.10) and (3.1) we can write, for all $T>0$, $t>0$ and $u \in S^{d-1}$,

$$\Delta(\sigma_{2T}^{\delta} f - \sigma_T^{\delta} f, t, u) = (\tau_{tu}H_{2T} - H_{2T}) \star_{\kappa} (\sigma_{2T}^{\delta} f - \sigma_T^{\delta} f).$$

Therefore, we get from relation (2.12)

$$\|\Delta(\sigma_{2T}^{\delta} f - \sigma_T^{\delta} f, t, u)\|_{\infty,\kappa} \le \|\tau_{tu}H_{2T} - H_{2T})\|_{1,\kappa}\|\sigma_{2T}^{\delta} f - \sigma_T^{\delta} f\|_{\infty,\kappa}, \tag{3.5}$$

where $T>0$, $t>0$ and $u \in S^{d-1}$. The function H_{2T} verifies the hypotheses of Proposition 2.8 . Then, a same manner as in [8, Corollary 2.2] and the relation (3.5) yield

$$\|\Delta(\sigma_{2T}^{\delta} f - \sigma_T^{\delta} f, t, u)\|_{\infty,\kappa} \le CtT\,\|\sigma_{2T}^{\delta} f - \sigma_T^{\delta} f\|_{\infty,\kappa}, \tag{3.6}$$

where $t>0$, $u \in S^{d-1}$ and $T>0$.
By using the relations (3.3) , (3.4) and (3.6), we obtain, for $t>0$ and $u \in S^{d-1}$,

$$\|\tau_{tu} f - f\|_{\infty,\kappa} \le C\left\{t\int_0^{1/t}\|\sigma_{2T}^{\delta} f - \sigma_T^{\delta} f\|_{\infty,\kappa}dT + \int_{1/t}^{+\infty}\|\sigma_{2T}^{\delta} f - \sigma_T^{\delta} f\|_{\infty,\kappa}\frac{dT}{T}\right\}.$$

Thus, by taking into account of the boundedness of $T^{\beta}\|\sigma_T^{\delta} f - f\|_{\infty,\kappa}$ on $(0,+\infty)$ we get, for all $t>0$ and $u \in S^{d-1}$,

$$\|\tau_{tu} f - f\|_{\infty,\kappa} \le C\left\{t\int_0^{1/t} T^{-\beta}dT + \int_{1/t}^{+\infty} T^{-\beta-1}dT\right\} \le Ct^{\beta}.$$

This means that $f \in \Lambda D_{\beta}$. □

Theorem 3.5. *Let f be a function in $\Lambda D_1 \bigcap L^2_\kappa(\mathbb{R}^d)$ and $\delta > \gamma + \frac{d-1}{2}$. Then as $T \longrightarrow +\infty$*

$$\|\sigma_T^\delta f - f\|_{\infty,\kappa} = \begin{cases} O(T^{-1}) & \text{if} \quad \delta > \gamma + \frac{d+1}{2} \\ O(T^{2\gamma+d-2\delta}) & \text{if} \quad \gamma + \frac{d-1}{2} < \delta \leq \gamma + \frac{d+1}{2} \end{cases}$$

Proof. Let $T \in (1,+\infty)$. From the well known asymptotic behavior of the first kind Bessel function we can assert that $\Phi_{T,\delta} \in L^2_\kappa(\mathbb{R}^d)$. Then we get, for $x \in \mathbb{R}^d$,

$\sigma_T^\delta f(x) - f(x) =$

$$CT^{2\gamma+d} \int_0^{+\infty} \left(\int_{S^{d-1}} [\tau_{tu} f(x) - f(x)] \, w_\kappa(u) d\sigma(u) \right) j_{\gamma+\frac{d}{2}+\delta}(Tt)\, t^{2\gamma+d-1}\, dt\,,$$

where C does not depend on T. We put

$$I_1 = T^{2\gamma+d} \int_0^{1/T} \left(\int_{S^{d-1}} [\tau_{tu} f(x) - f(x)] \, w_\kappa(u) d\sigma(u) \right) j_{\gamma+\frac{d}{2}+\delta}(Tt)\, t^{2\gamma+d-1}\, dt\,,$$

$$I_2 = T^{2\gamma+d} \int_{1/T}^{T} \left(\int_{S^{d-1}} [\tau_{tu} f(x) - f(x)] \, w_\kappa(u) d\sigma(u) \right) j_{\gamma+\frac{d}{2}+\delta}(Tt)\, t^{2\gamma+d-1}\, dt$$

and

$$I_3 = T^{2\gamma+d} \int_{T}^{+\infty} \left(\int_{S^{d-1}} [\tau_{tu} f(x) - f(x)] \, w_\kappa(u) d\sigma(u) \right) j_{\gamma+\frac{d}{2}+\delta}(Tt)\, t^{2\gamma+d-1}\, dt\,.$$

Since $\left| j_{\gamma+\frac{d}{2}+\delta}(z) \right| \leq 1$, $z \geq 0$ and w_κ is bounded on S^{d-1}, we can write

$$|I_1| \leq CT^{2\gamma+d} \int_0^{1/T} t^{2\gamma+d}\, dt = \frac{C}{T}\,. \tag{3.7}$$

Moreover, since $z^{\frac{1}{2}+\gamma+\frac{d}{2}+\delta} j_{\gamma+\frac{d}{2}+\delta}(z)$ is a bounded function on $(0,+\infty)$ we get

$$|I_2| \leq CT^{\gamma+\frac{d}{2}-\delta-\frac{1}{2}} \int_{1/T}^{T} t^{\gamma+\frac{d}{2}-\delta-\frac{1}{2}}\, dt = C\left(T^{2\gamma+d-2\delta} - \frac{1}{T} \right). \tag{3.8}$$

By virtue of the relation (2.7) and again according to the boundedness of the function $z^{\frac{1}{2}+\gamma+\frac{d}{2}+\delta} j_{\gamma+\frac{d}{2}+\delta}(z)$, we obtain whenever $\delta > \gamma + \frac{d-1}{2}$,

$$|I_3| \leq CT^{\gamma+\frac{d}{2}-\delta-\frac{1}{2}} \int_{T}^{+\infty} t^{\gamma+\frac{d}{2}-\delta-\frac{3}{2}}\, dt = CT^{2\gamma+d-2\delta-1}. \tag{3.9}$$

From relations (3.7), (3.8) and (3.9) we obtain the announced result. □

For $T > 0$, we introduce the partial Dunkl integral s_T as an operator on $L^2_\kappa(\mathbb{R}^d)$ by

$$s_T f(x) = c_\kappa^2 \int_{\|y\| \leq T} \mathcal{F}_\kappa f(y) E_\kappa(ix,y) w_\kappa(y) dy\,, \quad f \in L^2_\kappa(\mathbb{R}^d)\,,\ x \in \mathbb{R}^d\,.$$

Proposition 3.6. *Let $f \in L^2_\kappa(\mathbb{R}^d)$. We have*

$$s_T f(x) = \int_{\mathbb{R}^d} \tau_x f(-y)\Phi_{T,0}(y) w_\kappa(y)\, dy, \qquad x \in \mathbb{R}^d \tag{3.10}$$

Proof. Let $T > 0$ and $f \in L^2_\kappa(\mathbb{R}^d)$, we have

$$s_T f(x) = \int_{\mathbb{R}^d} c_\kappa^2 \chi_T(y) \mathcal{F}_\kappa f(y) E_\kappa(ix,y) w_\kappa(y) dy, \qquad x \in \mathbb{R}^d$$

where χ_T is the indicator function of the ball $\{y \in \mathbb{R}^d \quad | \quad \|y\| \leq T\}$.
On the other hand χ_T is radial. So, according to [14, Proposition 2.4] and the Sonine's formula [19, §12.11 (1)], we can assert that

$$\mathcal{F}_\kappa \left(c_\kappa^2 \chi_T \right) = \Phi_{T,0}.$$

But $\Phi_{T,0} \in L^2_\kappa(\mathbb{R}^d)$ and $c_\kappa \mathcal{F}_\kappa$ is an isometric isomorphism on $L^2_\kappa(\mathbb{R}^d)$ then by taking into account of the relation (2.4) we get

$$s_T f(x) = c_\kappa^2 \int_{\mathbb{R}^d} \mathcal{F}_\kappa (\tau_x f)(y) \mathcal{F}_\kappa (\Phi_{T,0})(y) w_\kappa(y) dy, \quad x \in \mathbb{R}^d.$$

Next, by using Plancherel's theorem we obtain the relation (3.10). □

Lemma 3.7. *Let $\delta > \gamma + \frac{d-1}{2}$ and f be a function in $L^2_\kappa(\mathbb{R}^d)$. Then we have, for every $0 < T \leq \nu$, the equality*

$$s_\nu(\sigma_T^\delta f) = \sigma_T^\delta f. \tag{3.11}$$

Proof. we can deduce the equality (3.11) from equalities (3.10), (3.1) and the Sonine's formula [19, §12.11 (1)]. □

Lemma 3.8. *Let $T > 0$ and $f \in L^2_\kappa(\mathbb{R}^d)$. For $1 < p < +\infty$ such that $\gamma < 1 - \frac{d}{2} - \left| \frac{1}{2} - \frac{1}{p} \right|$, we have*

$$\left(\frac{1}{T} \int_0^T |s_\nu f(x)|^p \, d\nu \right)^{1/p} \leq C \|f\|_{\infty,\kappa}, \quad x \in \mathbb{R}, \tag{3.12}$$

where C does not depend on T nor x.

Proof. For $\nu > 0$ and $x \in \mathbb{R}^d$, the partial Dunkl integral $s_\nu f(x)$ can be written,

$$s_\nu f(x) = \frac{c_\kappa \nu^{2\gamma+d}}{2^{\gamma+\frac{d}{2}} \Gamma(\gamma + \frac{d}{2} + 1)} \int_0^{+\infty} \int_{S^{d-1}} \tau_{-x} f(tu) w_\kappa(u) j_{\gamma+\frac{d}{2}}(t\nu) t^{2\gamma+d-1} dt\, d\sigma(u).$$

By taking into account that $j_{\gamma+\frac{d}{2}}(z)$ is a bounded function on $(0, +\infty)$ and using relation (2.7) we get

$$\frac{1}{T} \int_0^T \left| \int_0^{1/T} \left(\int_{S^{d-1}} \tau_{-x} f(tu) w_\kappa(u)\, d\sigma(u) \right) j_{\gamma+\frac{d}{2}}(t\nu) t^{2\gamma+d-1}\, dt \right|^p \nu^{2\gamma p + dp} d\nu$$

$$\leq \frac{C\|f\|_{\infty,\kappa}^p}{T}\int_0^T\left(\frac{\nu}{T}\right)^{p(2\gamma+d)}d\nu = C\|f\|_{\infty,\kappa}^p.$$

Assume now that $p \geq 2$ and take q such that $\frac{1}{p}+\frac{1}{q}=1$. Using the behavior asymptotic of Bessel function (see [19] or [2]), we get

$$(t\nu)^{\frac{1}{2}}J_{\gamma+\frac{d}{2}}(t\nu) = \sqrt{\frac{2}{\pi}}\cos\left(t\nu-\frac{\pi}{4}-\left(\gamma+\frac{d}{2}\right)\frac{\pi}{2}\right)+R(t\nu)$$

where

$$\begin{cases} R(t\nu)\leq A \quad if \quad t\leq \frac{1}{\nu} \\ R(t\nu)\leq \frac{B}{t\nu} \quad if \quad t>\frac{1}{\nu} \end{cases},$$

A and B being two positive constants. So, we obtain

$$\frac{1}{T}\int_0^T\left|\int_{1/T}^{+\infty}\left(\int_{S^{d-1}}\tau_{-x}f(tu)w_\kappa(u)\,d\sigma(u)\right)\nu^{2\gamma+d}j_{\gamma+\frac{d}{2}}(t\nu)t^{2\gamma+d-1}\,dt\right|^p d\nu$$

$$\leq \frac{1}{T}\int_0^T\left(|\phi_1(\nu)|^p+|\phi_2(\nu)|^p+|\phi_3(\nu)|^p\right)d\nu$$

where

$$\phi_1(\nu)=\int_{1/T}^{+\infty}\left(\int_{S^{d-1}}\tau_{-x}f(tu)w_\kappa(u)\,d\sigma(u)\right)\nu^{\gamma+\frac{d-1}{2}}t^{\gamma+\frac{d-3}{2}}\cos\left(t\nu-\frac{\pi}{4}-\left(\gamma+\frac{d}{2}\right)\frac{\pi}{2}\right)dt,$$

$$\phi_2(\nu)=A\int_{1/T}^{1/\nu}\left(\int_{S^{d-1}}\tau_{-x}f(tu)w_\kappa(u)\,d\sigma(u)\right)\nu^{\gamma+\frac{d-1}{2}}t^{\gamma+\frac{d-3}{2}}\,dt$$

and

$$\phi_3(\nu)=B\int_{1/\nu}^{+\infty}\left(\int_{S^{d-1}}\tau_{-x}f(tu)w_\kappa(u)\,d\sigma(u)\right)\nu^{\gamma+\frac{d-3}{2}}t^{\gamma+\frac{d-5}{2}}\,dt$$

Since $\cos(t\nu-\frac{\pi}{4}-(\gamma+\frac{d}{2})\frac{\pi}{2}) = \cos(t\nu)\cos(\frac{\pi}{4}+(\gamma+\frac{d}{2})\frac{\pi}{2})+\sin(t\nu)\sin(\frac{\pi}{4}(\gamma+\frac{d}{2})\frac{\pi}{2})$, by virtue of Hausdorff-Young inequality, we obtain

$$\frac{1}{T}\int_0^T|\phi_1(\nu)|^p\,d\nu \leq CT^{(\gamma+\frac{d-1}{2})p-1}\left(\int_{1/T}^{+\infty}\left|\int_{S^{d-1}}\tau_{-x}f(tu)w_\kappa(u)t^{(\gamma+\frac{d-3}{2})}\,d\sigma(u)\right|^q dt\right)^{\frac{p}{q}}$$

$$\leq C\,\|f\|_{\infty,\kappa}^p T^{(\gamma+\frac{d-1}{2})p-1}\left(\int_{1/T}^{+\infty}t^{(\gamma+\frac{d-3}{2})q}\,dt\right)^{\frac{p}{q}} = C\|f\|_{\infty,\kappa}^p$$

Furthermore, we have

$$\frac{1}{T}\int_0^T|\phi_2(\nu)|^p\,d\nu \leq \frac{C}{T}\,\|f\|_{\infty,\kappa}^p\int_0^T(1-\frac{\nu}{T})^{(\gamma+\frac{d-1}{2})p}\,d\nu \leq C\|f\|_\infty^p$$

and

$$\frac{1}{T}\int_0^T|\phi_3(\nu)|^p\,d\nu \leq \frac{C}{T}\,\|f\|_\infty^p\int_0^T d\nu = C\|f\|_\infty^p.$$

Hence the relationship (3.12) holds.

Assume now that $1 < p \leq 2$. Let q the real number verifying $\frac{1}{p} + \frac{1}{q} = 1$. Then, $q \geq 2$ and the Hölder inequality yields

$$\left(\frac{1}{T}\int_0^T |s_\nu f(x)|^p \, d\nu\right)^{1/p} \leq \left(\frac{1}{T}\int_0^T |s_\nu f(x)|^q \, d\nu\right)^{1/q} .$$

It follows that the relationship (3.12) holds. □

Theorem 3.9. *Let* $0 < \beta < 1$ *and* $f \in \Lambda D_\beta \bigcap L^2_\kappa(\mathbb{R}^d)$. *If* $1 < p < \frac{1}{\beta}$ *and* $\gamma < 1 - \frac{d}{2} - \left|\frac{1}{2} - \frac{1}{p}\right|$, *then*

$$\left\| \left(\frac{1}{T}\int_0^T |s_\nu f - f|^p \, d\nu\right)^{1/p} \right\|_{\infty,\kappa} \leq CT^{-\beta}, \quad T \in (0, +\infty).$$

Proof. Choose $\delta > \max(\gamma + \frac{d-1}{2} + \beta, \frac{d+1}{2})$. According to Lemma 3.7 , we are able to write for every $T > 0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} &\left(\frac{1}{T}\int_T^{2T} |s_\nu f(x) - f(x)|^p \, d\nu\right)^{1/p} \\ &= \left(\frac{1}{T}\int_T^{2T} \left|s_\nu(f - \sigma_T^\delta f)(x) + \sigma_T^\delta f(x) - f(x)\right|^p d\nu\right)^{1/p} \\ &\leq \left(\frac{1}{T}\int_T^{2T} \left|s_\nu(f - \sigma_T^\delta f)(x)\right|^p d\nu\right)^{1/p} + \|\sigma_T^\delta f - f\|_{\infty,\kappa} \end{aligned}$$

By virtue of Lemma 3.8, it follows

$$\begin{aligned} \left(\frac{1}{T}\int_T^{2T} \left|s_\nu(f - \sigma_T^\delta f)(x)\right|^p d\nu\right)^{1/p} &\leq \left(\frac{1}{T}\int_0^{2T} \left|s_\nu(f - \sigma_T^\delta f)(x)\right|^p d\nu\right)^{1/p} \\ &\leq C\|\sigma_T^\delta f - f\|_{\infty,\kappa} . \end{aligned}$$

Therefore, Theorem 3.4 yields

$$\left(\frac{1}{T}\int_T^{2T} |s_\nu f(x) - f(x)|^p \, d\nu\right)^{1/p} \leq CT^{-\beta}, \quad T > 0, \; x \in \mathbb{R}^d .$$

It follows that we have, for every $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ and $T > 0$,

$$\frac{1}{T}\int_{T/2^{n+1}}^{T/2^n} |s_\nu f(x) - f(x)|^p \, d\nu \leq CT^{-p\beta} 2^{(p\beta-1)(n+1)} .$$

Consequently, we obtain for every $x \in \mathbb{R}^d$ and $T > 0$,

$$\frac{1}{T}\int_0^T |s_\nu f(x) - f(x)|^p \, d\nu \leq C \frac{2^{(p\beta-1)}}{1 - 2^{(p\beta-1)}} T^{-p\beta} .$$

This completes the proof.

□

Acknowledgments

The authors thank the referees for their careful reading of the manuscript.

References

[1] J.J. Betancor and L. Rodríguez-Mesa, Lipschitz-Hankel spaces, partial Hankel integrals and Bochner-Riesz means, *Arch. Math.*, **71**, (1998), 115-122.

[2] S. Bochner, Vorlechungen ber Fouriersche Integrale, *Chelsea Pub. Company*, New York 1948.

[3] C.F. Dunkl, Differential-difference operators associated to reflection groups, *Trans. Amer. Math. Soc.* **311**, n^{o}1, (1989), 167-183.

[4] C.F. Dunkl, Integral kernels with reflection group invariance, *Can. J. Math.* **43**, n^{o}6, (1991), 1213-1227.

[5] C.F. Dunkl, Hankel transforms associated to finite reflection groups, in: Proc. of special session on hypergeometric functions on domains of positivity, Jack polynomials and applications. Proceedings, Tampa 1991, Contemporary Mathematics **138**, (1992), pp 123-138.

[6] C.F. Dunkl and Y. Xu, Orthogonal polynomials of several variables, *Cambridge Univ. Press*, Cambridge, 2001.

[7] D.V. Giang, Approximation on real line by Fourier transform, *Acta. Sci. Math.* (Szeged) **58**, (1993), 197-209.

[8] J. Gosselin and K. Stempak, A weak-type estimate for Fourier-Bessel multipliers, *Proc. Amer. Math.* Soc. **106**, n^{o}3 (1989), 655-662.

[9] M.F.E. de Jeu, The Dunkl transform, *Inv. Math.* **113**, (1993), 147-162.

[10] L. Kamoun and M. Sifi, On radial Besov-Dunkl spaces, *Math. Balkanica* **21**, n^{o}1-2,(2007), 201-220.

[11] M. Rösler, Bessel-type signed hypergroup on $\mathbb{R}$, in: Probability measure on groups and related structures (Proc. Conf. Oberwolfach, 1994), H.Heyer and A.Mukherjea (Eds.). World scientific Publ., 1995, 292-304.

[12] M. Rösler, Positivity of Dunkl's intertwining operator, *Duke Math. J.* **98**, n^{o}3, (1999), 445-463.

[13] M. Rösler, A positive radial product formula for the Dunkl kernel, *Trans. Amer. Math. Soc.* **355**, n^{o}6, (2003), 2413-2438.

[14] M. Rösler and M. Voit, Markov processes with Dunkl operators, *Adv. Appl. Math.* **21**, (1998), 575-643.

[15] E.M. Stein and G. Weiss, Introduction to Fourier analysis on euclidean spaces, *Prin. Univ. Press*, Princeton - New Jersey, 1990.

[16] S. Thangavelu and Y. Xu, Convolution operator and maximal function for Dunkl transform, *J. Anal. Math.* **97**, (2006), 25-55.

[17] K. Trimèche, The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual, *Integral Transforms Spec. Funct.* **12**, (2001), 349-374.

[18] K. Trimèche, Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators, *Integral Transforms Spec. Funct.* **13**, (2002), 17-38.

[19] G.N. Watson, A treatise on the theory of Bessel functions, *Cambridge Univ. Press*, Cambridge, 1966.

[20] Y. Xu, Orthogonal polynomials for a family of product weight functions on the spheres, *Can. J. Math*. **49**, n^{o}1, (1997), 175-192.