Communications in Mathematical Analysis
Volume 7, Number 1, pp. 50–54 (2009)
ISSN 1938-9787
www.commun-math-anal.org

ASYMPTOTICS OF THE q-THETA FUNCTION

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(Communicated by Toka Diagana)

Abstract

An asymptotic formula is given to the q-Theta function

$$\Theta_q(x) := \sum_{k=-\infty}^{\infty} q^{k^2} x^k$$

as $q \to 1^-$, where $x > 0$ is fixed.

AMS Subject Classification: 41A60; 33D99.

Keywords: asymptotics; q-Theta function; Jacobi Theta functions; asymptotic formula; asymptotically equal.

1 Introduction

For $0 < q < 1$ and $x \in \mathbb{C}$, the q-Theta function is defined by [2]

$$\Theta_q(x) := \sum_{k=-\infty}^{\infty} q^{k^2} x^k. \tag{1.1}$$

It satisfies the Jacobi triple product identity [1, Theorem 12.3.2]

$$\Theta_q(x) = \prod_{k=0}^{\infty} (1 - q^{2k+2})(1 + q^{2k+1} x)(1 + q^{2k+1}/x). \tag{1.2}$$

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The q-Theta function can be written in terms of the Jacobi Theta functions [5, Chapter 21]. For example, by choosing any $z \in \mathbb{C}$ such that $x = e^{2iz}$, we have

$$\Theta_q(x) = \vartheta_3(z,q). \tag{1.3}$$

For more properties of the q-Theta function and the Jacobi Theta functions, please refer to [5, Chapter 21] and references therein.

Note that the definition (1.1) and the Jacobi triple product (1.2) of the q-Theta function are also valid for $q \in \mathbb{C}$ with $|q| < 1$. However, as in the theory of q-orthogonal polynomials (or basic hypergeometric orthogonal polynomials), we will always assume $0 < q < 1$; see [1] and [3]. With the aid of the q-Theta function, Ismail and Zhang [2] derive several asymptotic formulas for three classes of q-orthogonal polynomials. Their results have been improved by Wang and Wong in [4], where again, the q-Theta function plays a significant role. Therefore, it will be useful to investigate asymptotic behavior of the q-Theta function in a stand-alone manner. This paper is dedicated to give an asymptotic formula for the q-Theta function as $q \to 1^-$ with fixed $x > 0$. As far as we are aware, this result has not been obtained previously.

2 Main Results

Our main theorem is stated below.

Theorem 2.1. *As $q \to 1^-$, we have*

$$\Theta_q(x) \sim \sqrt{\frac{\pi}{-\ln q}}\exp\{\frac{(\ln x)^2}{-4\ln q}\} \tag{2.1}$$

for $x > 0$. Here the symbol "$\sim$" means asymptotically equal, that is, we write $A_q \sim B_q$ if $\lim\limits_{q\to 1^-} A_q/B_q = 1$.

For preparation, we study the sum

$$I(\lambda,a) := \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak} \tag{2.2}$$

for $a \in \mathbb{R}$ and $\lambda > 0$. It is easily seen from (1.1) and (2.2) that

$$\Theta_q(x) = I(-1/\ln q, \ln x) + I(-1/\ln q, -\ln x) - 1. \tag{2.3}$$

Lemma 2.2. *As $\lambda \to +\infty$,*

$$I(\lambda,0) := \sum_{k=0}^{\infty} e^{-k^2/\lambda} \sim \sqrt{\pi\lambda}/2. \tag{2.4}$$

If the integer-valued function $N = N(\lambda) \in \mathbb{N}$ satisfies

$$\lim_{\lambda\to+\infty} N/\lambda = c$$

for some positive constant $c > 0$, *we have*

$$\sum_{k=0}^{N} e^{-k^2/\lambda} \sim \sqrt{\pi\lambda}/2 \tag{2.5}$$

as $\lambda \to +\infty$.

Proof. Consider the auxiliary integral

$$\widetilde{I}(\lambda) := \int_0^\infty e^{-t^2/\lambda} dt = \sum_{k=0}^{\infty} \int_k^{k+1} e^{-t^2/\lambda} dt.$$

Since $e^{-(k+1)^2/\lambda} \le e^{-t^2/\lambda} \le e^{-k^2/\lambda}$ for $k \le t \le k+1$, it follows that

$$\sum_{k=1}^{\infty} e^{-k^2/\lambda} \le \widetilde{I}(\lambda) \le \sum_{k=0}^{\infty} e^{-k^2/\lambda}.$$

On account of (2.2), we have $\widetilde{I}(\lambda) \le I(\lambda, 0) \le \widetilde{I}(\lambda) + 1$. Multiply this by $\lambda^{-1/2}$ and then let $\lambda \to +\infty$. Formula (2.4) follows from the fact $\widetilde{I}(\lambda) = \sqrt{\pi\lambda}/2$.

To prove (2.5), we shall estimate the sum

$$\sum_{k=N+1}^{\infty} e^{-k^2/\lambda} = \sum_{k=1}^{\infty} e^{-(k+N)^2/\lambda} \le \sum_{k=0}^{\infty} e^{-N^2/\lambda - 2Nk/\lambda} = \frac{e^{-N^2/\lambda}}{1 - e^{-2N/\lambda}}.$$

As $\lambda \to +\infty$, the right-hand side of the last inequality vanishes since $N/\lambda \to c > 0$ by assumption. This implies

$$\lim_{\lambda \to +\infty} \sum_{k=N+1}^{\infty} e^{-k^2/\lambda} = 0.$$

Therefore, formula (2.5) follows from (2.4). □

Lemma 2.3. *For* $a < 0$, *we have*

$$I(\lambda, a) := \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak} \sim (1 - e^a)^{-1} \tag{2.6}$$

as $\lambda \to +\infty$.

Proof. Since $k^2/\lambda \le 1/\sqrt{\lambda}$ for $0 \le k \le \lfloor \lambda^{1/4} \rfloor$, we have

$$I(\lambda, a) = \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak} \ge \sum_{k=0}^{\lfloor \lambda^{1/4} \rfloor} e^{-1/\sqrt{\lambda} + ak} = \frac{e^{-1/\sqrt{\lambda}}(1 - e^{a(\lfloor \lambda^{1/4} \rfloor + 1)})}{1 - e^a}.$$

By letting $\lambda \to +\infty$, we obtain from the assumption $a < 0$ that

$$\liminf_{\lambda \to +\infty} I(\lambda, a) \ge (1 - e^a)^{-1}.$$

Moreover, it is easily seen that

$$I(\lambda, a) = \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak} \le \sum_{k=0}^{\infty} e^{ak} = (1 - e^a)^{-1}.$$

Coupling the last two inequalities yields our desired result. □

Lemma 2.4. *For $a > 0$, we have*

$$I(\lambda, a) := \sum_{k=0}^{\infty} e^{-k^2/\lambda + ak} \sim \sqrt{\pi\lambda} e^{\lambda a^2/4} \tag{2.7}$$

as $\lambda \to +\infty$.

Proof. Consider the sum

$$e^{-\lambda a^2/4} I(\lambda, a) = \sum_{k=0}^{\infty} e^{-(k-\lambda a/2)^2/\lambda} = \sum_{k=0}^{m} + \sum_{k=m+1}^{\infty} =: I_1 + I_2, \tag{2.8}$$

where $m := \lfloor \lambda a/2 \rfloor$. We intend to show $I_1 \sim \sqrt{\pi\lambda}/2$ and $I_2 \sim \sqrt{\pi\lambda}/2$ as $\lambda \to +\infty$.

Firstly, it follows from $m \leq \lambda a/2 \leq m+1$ that

$$I_1 := \sum_{k=0}^{m} e^{-(k-\lambda a/2)^2/\lambda} \leq \sum_{k=0}^{m} e^{-(m-k)^2/\lambda} = \sum_{k=0}^{m} e^{-k^2/\lambda},$$

and

$$I_1 \geq \sum_{k=0}^{m} e^{-(m+1-k)^2/\lambda} = \sum_{k=0}^{m+1} e^{-k^2/\lambda} - 1.$$

Since $m/\lambda \to a/2 > 0$ as $\lambda \to +\infty$, we obtain from (2.5) and the last two inequalities that

$$\lim_{\lambda \to +\infty} I_1/\sqrt{\lambda} = \sqrt{\pi}/2. \tag{2.9}$$

Secondly, since $m \leq \lambda a/2 \leq m+1$, we have

$$I_2 := \sum_{k=m+1}^{\infty} e^{-(k-\lambda a/2)^2/\lambda} \leq \sum_{k=m+1}^{\infty} e^{-(k-m-1)^2/\lambda} = \sum_{k=0}^{\infty} e^{-k^2/\lambda},$$

and

$$I_2 \geq \sum_{k=m+1}^{\infty} e^{-(k-m)^2/\lambda} = \sum_{k=0}^{\infty} e^{-k^2/\lambda} - 1.$$

applying (2.4) to the last two inequalities gives

$$\lim_{\lambda \to +\infty} I_2/\sqrt{\lambda} = \sqrt{\pi}/2. \tag{2.10}$$

Finally, a combination of (2.8)-(2.10) yields (2.7) immediately. □

Proof of Theorem 2.1. For $x = 1$, we obtain from (2.3) that

$$\Theta_q(1) = 2I(-1/\ln q, 0) - 1.$$

Coupling this and (2.4) gives

$$\lim_{q \to 1^-} \Theta_q(1) \sqrt{-\ln q} = \sqrt{\pi}. \tag{2.11}$$

For $x > 1$, it follows from (2.6) that

$$\lim_{q\to 1^-}[I(-1/\ln q, -\ln x) - 1]\sqrt{-\ln q}\exp\{\frac{(\ln x)^2}{4\ln q}\} = 0.$$

On the other hand, from (2.7) we have

$$\lim_{q\to 1^-} I(-1/\ln q, \ln x)\sqrt{-\ln q}\exp\{\frac{(\ln x)^2}{4\ln q}\} = \sqrt{\pi}.$$

Therefore, applying the last two equations to (2.3) yields

$$\lim_{q\to 1^-} \Theta_q(x)\sqrt{-\ln q}\exp\{\frac{(\ln x)^2}{4\ln q}\} = \sqrt{\pi}. \tag{2.12}$$

Similarly, for $0 < x < 1$, a combination of (2.3), (2.6) and (2.7) implies

$$\lim_{q\to 1^-} \Theta_q(x)\sqrt{-\ln q}\exp\{\frac{(\ln x)^2}{4\ln q}\} = \sqrt{\pi}. \tag{2.13}$$

Thus, formula (2.1) follows from (2.11)-(2.13). This ends the proof of Theorem 2.1. □

Acknowledgments

The author thanks the referees for their careful reading of the manuscript and helpful comments.

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