Communications in Mathematical Analysis
Volume 7, Number 1, pp. 55–60 (2009)
ISSN 1938-9787
www.commun-math-anal.org

FACTORS FOR ABSOLUTE NÖRLUND SUMMABILITY

HÜSEYIN BOR *
Department of Mathematics
Erciyes University
Kayseri, 38039, TURKEY

(Communicated by John M. Rassias)

Abstract

In this paper, a general theorem dealing with $\mid N,p_n\mid_k$ summability factors has been proved. This theorem also includes some known results.

AMS Subject Classification: 40D15, 40F05, 40G05 , 40G99.

Keywords: Nörlund summability, infinite series , sequences, summability factors.

1 Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n) and $w_n = na_n$. By u_n^α and t_n^α we denote the n-th Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (w_n), respectively. i.e,

$$u_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1.1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1.2}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad \mathit{and} \quad A_{-n}^\alpha = 0. \tag{1.3}$$

The series $\sum a_n$ is said to be summable $\mid C,\alpha\mid_k$, $k \geq 1$, if (see [5],[7])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k = \sum_{n=1}^{\infty} \frac{1}{n} \mid t_n^\alpha \mid^k < \infty. \tag{1.4}$$

If we take $\alpha = 1$, then $\mid C,\alpha\mid_k$ summability reduces to $\mid C,1\mid_k$ summability.
Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \ldots + p_n \neq 0, \ (n \geq 0). \tag{1.5}$$

*E-mail address: bor@erciyes.edu.tr ; hbor@gmail.com

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \tag{1.6}$$

defines the sequence (σ_n) of the Nörlund mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $\mid N, p_n \mid$, if (see [8])

$$\sum_{n=1}^{\infty} \mid \sigma_n - \sigma_{n-1} \mid < \infty, \tag{1.7}$$

and it is said to be summable $\mid N, p_n \mid_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} n^{k-1} \mid \sigma_n - \sigma_{n-1} \mid^k < \infty. \tag{1.8}$$

In the special case when

$$p_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \ \alpha \geq 0 \tag{1.9}$$

the Nörlund mean reduces to the (C, α) mean and $\mid N, p_n \mid_k$ summability becomes $\mid C, \alpha \mid_k$ summability. For $p_n = 1$, we get the $(C, 1)$ mean and then $\mid N, p_n \mid_k$ summability becomes $\mid C, 1 \mid_k$ summability. For any sequence (λ_n), we write $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

2 The known results

Concerning the $\mid C, 1 \mid$ and $\mid N, p_n \mid$ summabilities Kishore [6] has proved the following theorem.

Theorem 2.1. *Let $p_0 > 0$, $p_n \geq 0$ and (p_n) be a non-increasing sequence. If $\sum a_n$ is summable $\mid C, 1 \mid$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $\mid N, p_n \mid$.*

Varma [11] has also generalized Theorem 2.1 for $\mid N, p_n \mid_k$ summability.

Theorem 2.2. *Let $p_0 > 0$, $p_n \geq 0$ and (p_n) be a non-increasing sequence. If $\sum a_n$ is summable $\mid C, 1 \mid_k$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $\mid N, p_n \mid_k$, $k \geq 1$.*

Recently Bor [2] has proved the following theorem on this subject.

Theorem 2.3. *Let (p_n) be as in Theorem 2.1. If*

$$\sum_{v=1}^{n} \frac{1}{v} \mid t_v \mid^k = O(X_n) \ as \ n \to \infty, \tag{2.1}$$

where (t_n) is the n-th (C,1) mean of the sequence (na_n) , (X_n) is a positive non-decreasing sequence and (λ_n) is a sequence such that

$$\sum_{n=1}^{\infty} n \mid \Delta^2 \lambda_n \mid X_n < \infty, \tag{2.2}$$

$$\mid \lambda_n \mid X_n = O(1) \ as \ n \to \infty, \tag{2.3}$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $\mid N, p_n \mid_k$, $k \geq 1$.

3 Main Result

The aim of this paper is to prove Theorem 2.3 in a more general form for $\mid N,p_n\mid_k$ summability. Now we shall prove the following theorem.

Theorem 3.1. *Let (p_n) be as in Theorem 2.1 and (X_n) be a positive non - decreasing sequence. If the conditions (2.2)-(2.3) of Theorem 2.3 are satisfied and the sequence (w_n^α), defined by (see [10])*

$$w_n^\alpha = \mid t_n^\alpha \mid, \quad \alpha = 1 \tag{3.1}$$

$$w_n^\alpha = \max_{1\leq v\leq n} \mid t_v^\alpha \mid, \quad 0<\alpha<1 \tag{3.2}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-1}(w_n^\alpha)^k = O(X_m) \quad as \quad m\to\infty, \tag{3.3}$$

then the series $\sum a_nP_n\lambda_n(n+1)^{-1}$ is summable $\mid N,p_n\mid_k$, $k\geq 1$ and $0<\alpha\leq 1$.

It should be remarked that if we take $\alpha = 1$, then we get Theorem 2.3.
We need the following lemmas for the proof of our theorem.

Lemma 3.2. *([4]) If $0<\alpha\leq 1$ and $1\leq v\leq n$, then*

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1}a_p \mid \leq \max_{1\leq m\leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1}a_p \mid . \tag{3.4}$$

Lemma 3.3. *([2]) Under the conditions on (X_n) and (λ_n), as taken in the statement of Theorem 3.1, the following conditions hold :*

$$nX_n\Delta\lambda_n = O(1) \text{ as } n\to\infty, \tag{3.5}$$

$$\sum_{n=1}^{\infty} \Delta\lambda_n X_n < \infty. \tag{3.6}$$

Lemma 3.4. *([9]) If $-1<\alpha\leq\beta, k>1$ and the series $\sum a_n$ is summable $\mid C,\alpha\mid_k$, then it is also summable $\mid C,\beta\mid_k$.*

The case k =1 of this Lemma is due to Kogbetliantz [7]. The case $k>1$ is a special case of a theorem of Flett ([5], Theorem 1).

4 Proof of Theorem 3.1.

In order to prove the theorem, we need to consider only the special case in which (N,p_n) is (C,α), that is, we shall prove that $\sum a_n\lambda_n$ is summable $\mid C,\alpha\mid_k$. Our theorem will then follow by mean of Lemma 3.4 (for $\beta=1$) and Theorem 2.2. Let (T_n^α) be the n-th (C,α), with $0<\alpha\leq 1$, mean of the sequence $(na_n\lambda_n)$. Then, by (1.2), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{4.1}$$

By applying Abel's transformation, we find from (4.1) that

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,$$

which, in view of Lemma 3.2, yields

$$\begin{aligned}\mid T_n^\alpha \mid &\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\mid \Delta\lambda_v \mid\mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^\alpha}\mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \mid \\ &\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha \mid \Delta\lambda_v \mid + \mid \lambda_n \mid w_n^\alpha \\ &= T_{n,1}^\alpha + T_{n,2}^\alpha \quad .\end{aligned}$$

Since,

$$\mid T_{n,1}^\alpha + T_{n,2}^\alpha \mid^k \leq 2^k(\mid T_{n,1}^\alpha \mid^k + \mid T_{n,2}^\alpha \mid^k),$$

in order to complete the proof of the Theorem, by (1.4) it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-1} \mid T_{n,r}^\alpha \mid^k < \infty \quad for \quad r=1,2.$$

Whenever $k>1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{-1} \mid T_{n,1}^{\alpha} \mid^{k} &\leq \sum_{n=2}^{m+1} n^{-1}(A_n^{\alpha})^{-k}\{\sum_{v=1}^{n-1} A_v^{\alpha} w_v^{\alpha} \mid \Delta\lambda_v \mid\}^k \\
&= O(1)\sum_{n=2}^{m+1} n^{-1} n^{-\alpha k}\{\sum_{v=1}^{n-1} v^{\alpha k}(w_v^{\alpha})^k \mid \Delta\lambda_v \mid\} \times \{\sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid\}^{k-1} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^{\alpha})^k \mid \Delta\lambda_v \mid \sum_{n=v+1}^{m+1} \frac{1}{n^{\alpha k+1}} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^{\alpha})^k \mid \Delta\lambda_v \mid \int_v^{\infty} \frac{dx}{x^{\alpha k+1}} \\
&= O(1)\sum_{v=1}^{m} v \mid \Delta\lambda_v \mid v^{-1}(w_v^{\alpha})^k \\
&= O(1)\sum_{v=1}^{m-1} \Delta(v \mid \Delta\lambda_v \mid)\sum_{r=1}^{v} r^{-1}(w_r^{\alpha})^k \\
&+ O(1) m \mid \Delta\lambda_m \mid \sum_{v=1}^{m} v^{-1}(w_v^{\alpha})^k \\
&= O(1)\sum_{v=1}^{m-1} \mid \Delta(v \mid \Delta\lambda_v \mid) \mid X_v + O(1) m \mid \Delta\lambda_m \mid X_m \\
&= O(1)\sum_{v=1}^{m-1} \mid (v+1) \mid \Delta^2\lambda_v \mid - \mid \Delta\lambda_v \mid\mid X_v + O(1) m \mid \Delta\lambda_m \mid X_m \\
&= O(1)\sum_{v=1}^{m-1} v \mid \Delta^2\lambda_v \mid X_v + O(1)\sum_{v=1}^{m-1} \mid \Delta\lambda_v \mid X_v + O(1) m \mid \Delta\lambda_m \mid X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem and Lemma 3.3. Again, we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{-1} \mid T_{n,2}^{\alpha} \mid^{k} &= O(1)\sum_{n=1}^{m} \mid \lambda_n \mid n^{-1}(w_n^{\alpha})^k \\
&= O(1)\sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{-1}(w_v^{\alpha})^k + O(1) \mid \lambda_m \mid \sum_{n=1}^{m} n^{-1}(w_n^{\alpha})^k \\
&= O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of the Theorem and Lemma 3.3. Therefore, we get that

$$\sum_{n=1}^{m} n^{-1} \mid T_{n,r}^{\alpha} \mid^{k} = O(1) \quad as \quad m \to \infty, \quad for \quad r = 1,2.$$

This completes the proof of the Theorem.

References

[1] H. Bor, Absolute Nörlund summability factors. *Utilitas Math.* **40** (1991), pp 231-236.

[2] H. Bor On a summability factor theorem. *Commun. Math. Anal.* **1** (2006), pp 46-51.

[3] D. Borwein and F. P. Cass, Strong Nörlund summability. *Math. Zeith.***103** (1968), pp 94-111.

[4] L. S. Bosanquet, A mean value theorem.*J. London Math. Soc.* **16** (1941), pp 146-148.

[5] T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley.*Proc. London Math. Soc.* **7** (1957), pp 113-141.

[6] N. Kishore, On the absolute Nörlund summability factors.*Riv. Mat. Univ. Parma* **6** (1965), pp 129-134.

[7] E. Kogbentliantz, Sur lés series absolument sommables par la méthode des moyennes arithmétiques. *Bull. Sci. Math.* **49** (1925), pp 234-256.

[8] F. M. Mears, Some multiplication theorems for the Nörlund mean. *Bull. Amer. Math. Soc.* **41** (1935), pp 875-880.

[9] M. R. Mehdi, Linear transformations between the Banach spaces L^p and l^p with applications to absolute summability. *PhD.Thesis*, University College and Birkbeck College , London (1959).

[10] T. Pati, The summability factors of infinite series. *Duke Math. J.* **21** (1954), pp 271-284.

[11] R. S. Varma, On the absolute Nörlund summability factors. *Riv. Mat.Univ. Parma* **4** 3(1977), pp 27-33.