Communications in **M**athematical **A**nalysis
Volume 7, Number 1, pp. 61–74 (2009) *www.commun-math-anal.org*
ISSN 1938-9787

SPHERICAL MEANS OF SUBHARMONIC FUNCTIONS

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(Communicated by Saburou Saitoh)

Abstract

This article is devoted to subharmonic functions u in the unit ball. It studies how growth conditions on u impact on the Riesz measure associated to u. More precisely, the growth hypotheses involve means of u over spheres. The results lead to applications about holomorphic functions in the unit disk and about the repartition of their zeros.

AMS Subject Classification: 26D15; 28A75; 30D50; 31B05; 40A10

Keywords: holomorphic functions in the unit disk, subharmonic functions in the unit ball, zeros of holomorphic functions, Riesz measures, Blaschke condition.

1 Introduction

For functions f holomorphic in the unit disk of $\mathbb{C}$, such that $\int_0^1 T(r)\,(1-r)^{\lambda}\,dr < +\infty$ with $\lambda > -1$ and $T(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(r\,e^{i\theta})|\,d\theta$, it is already known from [2] and [9] that the zeros $\{z_n\}_{n\in\mathbb{N}}$ of such functions satisfy:

$$\sum_{n\in\mathbb{N}} (1-|z_n|)^{\lambda+2} < +\infty$$

(see Example 2.5). In this case, $I(s) := \int_s^1 T(r)\,(1-r)^{\lambda}\,dr$ obviously tends towards 0 as $s \to 1$. When such an additional information as $I(s) = O((1-s)^{\beta})$ is available, the present paper obtains for $\beta \in]0, 1+\lambda[$ the convergence of

$$\sum_{n\in\mathbb{N}} (1-|z_n|)^{\lambda+2-\beta} \left(\log \frac{1}{1-|z_n|}\right)^{-\gamma}$$

$\forall\gamma > 1$ which can not be deduced from the convergence of the previous series (see Remark 2.9). Actually this result occurs as a special case (see Corollary 2.8) in a more general

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study devoted to subharmonic functions in the open Euclidean unit ball in $\mathbb{R}^N$. These functions are subject to growth conditions controlling their behavior near the border of the unit ball B_N. In Sections 2, 3 and 4, these conditions are formulated through means over spheres centered at the origin, whereas in Section 5 they involve means on spheres centered at the point $A = (\frac{1}{2}, 0, ..., 0) \in \mathbb{R}^N$. The aim of the paper is to study how these growth conditions on some subharmonic function u influence the Riesz measure μ associated to u. For instance, Theorem 2.2 provides a necessary and sufficient condition for the boundedness of the means on spheres centered at the origin. This condition

$$\int_{B_N} (1 - |\zeta|)\, d\mu(\zeta) < +\infty$$

(with $|\zeta|$ the Euclidean norm of ζ) is a generalization of the Blaschke condition known for holomorphic functions (see Example 2.5). In Theorems 2.6 and 2.7, the spherical means of u are controlled by integral conditions. These statements provide such results as:

$$\text{Theorem 2.6 :} \qquad \int_{B_N} (1 - |\zeta|)\, \varphi(|\zeta|^{\alpha})\, d\mu(\zeta) < +\infty \qquad \forall \alpha \in]0, 1[$$

$$\text{Theorem 2.7 :} \qquad \int_{B_N} (1 - |\zeta|)^{\beta} \left(\log \frac{1}{1 - |\zeta|} \right)^{-\gamma} d\mu(\zeta) < +\infty \qquad \forall \gamma > 1$$

where the function φ and the parameter $\beta \geq 0$ stem from the growth condition on u (see Section 2 for more details). In Sections 3 and 4, several counterexamples are built in order to investigate the sharpness of the hypotheses and of the conclusions in Theorem 2.6.

At the end of Section 2, Corollary 2.10 and Example 2.11 recover and generalize a previous result of [1] about holomorphic functions f in the unit disk: [1] obtained a substitute of the Blaschke condition when f has such a growth as: $|f(z)| = O(\log \frac{1}{1-|z|})$ as $|z| \to 1^-$ (for instance when f belongs to the Bloch space).

Section 5 is devoted to a Blaschke–type result on the ball P_N centered at the point $A' = (\frac{2}{3}, 0, \ldots, 0) \in \mathbb{R}^N$, with radius $\frac{1}{3}$. If some subharmonic function u has bounded means over spheres centered at A with radii $r \in [0, \frac{1}{2}[$, then Theorem 5.3 shows that its Riesz measure μ satisfies

$$\int_{P_N} (1 - |\zeta|)\, d\mu(\zeta) < +\infty.$$

For instance, this result holds for all subharmonic functions u with such a growth as: $u(x) = O((1 - |x|)^{-\alpha})$ as $|x| \to 1^-$ for some constant $\alpha \in [0, \frac{N-1}{2}[$ (Theorem 5.4). For $\alpha \geq \frac{N-1}{2}$, it does not compulsorily hold, as pointed out by Proposition 5.10. This is related to previous results due to [6] and [4] about the positive zeros of holomorphic functions in the unit disk (see Example 5.8).

2 Means over spheres centered at the origin

Definition 2.1. Given a function u subharmonic in $B_N = \{x \in \mathbb{R}^N : |x| < 1\}$ (with $N \in \mathbb{N}$, $N \geq 2$ and $|\,.\,|$ the Euclidean norm in $\mathbb{R}^N$), let $\mathcal{M}_u$ and $\mathcal{M}_u^+$ be defined on $[0, 1[$ by:

$$\mathcal{M}_u(r) = \frac{1}{\sigma_N} \int_{S_N} u(r\eta)\, d\sigma_\eta \qquad \text{and} \qquad \mathcal{M}_u^+ = \mathcal{M}_{u^+},$$

with $d\sigma$ the area element on the unit sphere $S_N \subset \mathbb{R}^N$, $\sigma_N = \int_{S_N} d\sigma$ and u^+ the subharmonic function defined by $u^+(x) = \max(u(x), 0)$ $\forall x \in B_N$. The function u is said to satisfy the $\mathcal{H}$ condition if u is moreover harmonic in some neighborhood of the origin, with $u(O) = 0$.

For the notion of Riesz measure, we refer to [3] (p.104).

Theorem 2.2. *Let μ denote the Riesz measure associated to a subharmonic function u in B_N, satisfying the $\mathcal{H}$ condition. If $\mathcal{M}_u$ is bounded by $K \geq 0$ on $[0,1[$, then*

$$\int_{B_N} h(|\zeta|)\, d\mu(\zeta) \leq K$$

where the function h is defined by: $h(s) = \log \frac{1}{s}$ if $N = 2$ or $h(s) = \frac{1}{s^{N-2}} - 1$ if $N \geq 3$. The converse also holds.

Remark 2.3. From the $\mathcal{H}$ condition, there exists $\varepsilon(u) > 0$ (written here ε for sake of brevity) such that $\mu(\overline{B}(O,\varepsilon)) = 0$ with $\overline{B}(O,\varepsilon) = \{x \in \mathbb{R}^N : |x| \leq \varepsilon\}$. The subharmonicity of u implies $\mathcal{M}_u(r) \geq u(O) = 0$.

Proof. The $\mathcal{H}$ condition allows to apply the Jensen–Privalov formula (see [5] p.44 and [3] p.29), whence:

$$\mathcal{M}_u(r) = \tau_N \int_0^r \frac{\rho(t)}{t^{N-1}}\, dt \qquad \forall r \in]0,1[,$$

where $\tau_N = \max(1, N-2)$ and ρ is defined by $\rho(t) = \mu(\overline{B}(O,t))$ $\forall t \in [0,1[$. The function $\mathcal{M}_u$ being non–decreasing on $[0,1[$, its boundedness by K is equivalent to: $\lim\limits_{r\to 1^-} \mathcal{M}_u(r) \leq K$. Now

$$\mathcal{M}_u(r) = \int_{|\zeta|\leq r} h_r(\zeta)\, d\mu(\zeta) = \int_{B_N} h_r(\zeta)\, \mathbb{1}_{\overline{B}(O,r)}(\zeta)\, d\mu(\zeta) \qquad \forall r \in]0,1[\qquad \forall s \in]0,1[$$

thanks to Lemma 2 of [8] (see also [7]), with $\mathbb{1}_{\overline{B}(O,r)}$ the indicator function of $\overline{B}(O,r)$ and h_r defined on $\mathbb{R}^N \setminus \{O\}$ by:

$$h_r(\zeta) = \log \frac{r}{|\zeta|} \qquad \text{if } N = 2$$

$$h_r(\zeta) = \frac{1}{|\zeta|^{N-2}} - \frac{1}{r^{N-2}} \qquad \text{if } N \geq 3.$$

Now $\lim\limits_{r\to 1^-} h_r(\zeta)\, \mathbb{1}_{\overline{B}(O,r)}(\zeta) = h(|\zeta|)$ increasingly $\forall \zeta \neq O$. The monotonic convergence theorem ([3] p.84) then applies:

$$\lim_{r\to 1^-} \int_{|\zeta|\leq r} h_r(\zeta)\, d\mu(\zeta) = \int_{B_N} h(|\zeta|)\, d\mu(\zeta).$$

□

Remark 2.4. Since $h(s) \sim \tau_N(1-s)$ as $s \to 1$, the result may also be formulated as:

$$\int_{B_N} (1 - |\zeta|)\, d\mu(\zeta) < +\infty.$$

Example 2.5. In the case $N=2$ and $u=\log|f|$ where f is holomorphic in the unit disk D of $\mathbb{C}$, with bounded $T(r)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(r\,e^{i\theta})|\,d\theta$, the previous theorem thus includes the known result:

$$\sum_{n\in\mathbb{N}}(1-|z_n|)<+\infty \qquad \text{(see [9] p.202)},$$

since the Riesz measure of such a function u is given by: $\mu=\sum_{n\in\mathbb{N}}\delta_{z_n}$ (Dirac masses at the points z_n: the zeros of f in D, taking multiplicities into account). The $\mathcal{H}$ condition is fulfilled as soon as f does not vanish at the origin. Given $\lambda>-1$, it was also known that

$$\sum_{n\in\mathbb{N}}(1-|z_n|)^{\lambda+2}<+\infty \qquad \text{(see [2] p.339 and [9] p.204)} \tag{2.1}$$

provided that $\int_0^1 T(r)\,(1-r)^\lambda\,dr<+\infty$. This result too is contained (as a special case when $N=2$) in Theorem 7 of [8] established for subharmonic functions u in B_N, satisfying the $\mathcal{H}$ condition, together with $\int_{B_N}u^+(x)\,[-\omega'(|x|^2)]\,dx<+\infty$ where ω is a $\mathcal{C}^1$ decreasing function on $[0,1[$ such that $\lim_{t\to1^-}\omega(t)=0$. This theorem then asserts that:

$$\int_{B_N}h(|\zeta|^{1-\alpha})\,\omega(|\zeta|^{2\alpha})\,d\mu(\zeta)<+\infty \qquad \forall\alpha\in]0,1[. \tag{2.2}$$

It is applied here with ω defined by $\omega(t^2)=(1-t)^{\lambda+1}$, thus $2t\,\omega'(t^2)=-(\lambda+1)(1-t)^\lambda$. Now

$$\int_{B_2}u^+(x)\,[-\omega'(|x|^2)]\,dx=\int_0^1\left(\int_0^{2\pi}u^+(r\,e^{i\theta})\,d\theta\right)[-\omega'(r^2)]\,r\,dr$$

hence $T(r)$ is recognized when $u=\log|f|$. Then (2.2) becomes

$$(1-\alpha)\sum_{n\in\mathbb{N}}\log\frac{1}{|z_n|}\,(1-|z_n|^\alpha)^{\lambda+1}<+\infty.$$

Since $\log\frac{1}{t}\sim(1-t)$ and $1-t^\alpha\sim\alpha(1-t)$ as $t\to1$, the result (2.1) follows.

More generally, we obtain:

Theorem 2.6. *Given μ the Riesz measure associated to a subharmonic function u in B_N, satisfying the $\mathcal{H}$ condition. Let φ denote a $\mathcal{C}^1$ decreasing function on $[0,1[$ such that $\lim_{t\to1^-}\varphi(t)=0$. If*

$$\int_0^1\mathcal{M}_u^+(r)\,[-\varphi'(r)]\,dr<+\infty$$

then

$$\int_{B_N}(1-|\zeta|)\,\varphi(|\zeta|^\alpha)\,d\mu(\zeta)<+\infty \qquad \forall\alpha\in]0,1[.$$

Proof. Let $\varepsilon:=\varepsilon(u)$ as in Remark 2.3. Then the above integral becomes

$$\int_{\varepsilon\le|\zeta|<1}(1-|\zeta|)\,\varphi(|\zeta|^\alpha)\,d\mu(\zeta).$$

Let ω be defined by $\omega(r)=\varphi(\sqrt{r})$ $\forall r\in[\varepsilon^2,1[$ and continuated on $[0,\varepsilon^2[$ in order to be $\mathcal{C}^1$ decreasing on $[0,1[$, for instance:

$$\omega(r)=\omega(\varepsilon^2)+(r-\varepsilon^2)\omega'(\varepsilon^2)=\omega(\varepsilon^2)+(r-\varepsilon^2)\,\tfrac{\varphi'(\varepsilon)}{2\varepsilon}\qquad\forall r\in[0,\varepsilon^2[.$$

Hence $\int_0^1\mathcal{M}_u^+(r)\,[-r\,\omega'(r^2)]\,dr<+\infty$. Since $r^{N-2}\le 1$, we obtain:

$$\int_0^1\mathcal{M}_u^+(r)\,[-\omega'(r^2)]\,r^{N-1}\,dr<+\infty$$

thus $\int_{B_N}u^+(x)\,[-\omega'(|x|^2)]\,dx<+\infty$ hence (2.2) from Theorem 7 of [8]. The result follows from $h(|\zeta|^{1-\alpha})\sim\tau_N(1-|\zeta|^{1-\alpha})\sim\tau_N(1-\alpha)(1-|\zeta|)$ as $|\zeta|\to 1$. □

Theorem 2.7. *Let μ denote the Riesz measure associated to a subharmonic function u in B_N, satisfying the $\mathcal{H}$ condition. If there exists constants $c>0$, $C>0$, $\alpha\ge\beta\ge 0$ and an integrable function $\omega:[0,1[\to[0,+\infty[$ satisfying $c(1-t)^\alpha\le\int_t^1\omega(r)\,dr<+\infty$ $\forall t\in[0,1[$ and such that: $\int_s^1\mathcal{M}_u^+(r)\,\omega(r)\,dr\le C\,(1-s)^\beta$ $\forall s\in[0,1[$ then*

$$\int_{B_N}(1-|\zeta|)^{\alpha-\beta+1}\left(\log\frac{1}{1-|\zeta|}\right)^{-\gamma}d\mu(\zeta)<+\infty\qquad\forall\gamma>1.$$

When moreover $N\ge 3$, the following also holds:

$$\int_{B_N}\left[\frac{1}{(1-|\zeta|)^{N-2}}-1\right]^{-\gamma}d\mu(\zeta)<+\infty\qquad\forall\gamma>\frac{\alpha-\beta+1}{N-2}.$$

Proof. The change of variables $dx=r^{N-1}\,dr\,d\sigma$ leads to:

$$\frac{1}{\sigma_N}\int_{s\le|x|<1}u^+(x)\,\omega(|x|)\,dx=\int_s^1\mathcal{M}_u^+(r)\,\omega(r)\,r^{N-1}\,dr\le C\,(1-s)^\beta\qquad\forall s\in[0,1[$$

since $r^{N-1}\le 1$. The result then follows from Theorem 8 of [8]. □

Corollary 2.8. *Let f be an holomorphic function in the unit disk $D\subset\mathbb{C}$, with $f(0)=1$, and such that $\int_s^1 T(r)\,(1-r)^\lambda\,dr=O((1-s)^\beta)$ as $s\to 1$, for some constants $\lambda>-1$ and $\beta\in]0,1+\lambda]$, with $T(r)$ defined as in Example 2.5. Then the zeros $\{z_n\}_{n\in\mathbb{N}}$ of f in D (repeated according to their multiplicities) satisfy:*

$$\sum_{n\in\mathbb{N}}(1-|z_n|)^{\lambda+2-\beta}\left(\log\frac{1}{1-|z_n|}\right)^{-\gamma}<+\infty\qquad\forall\gamma>1.\tag{2.3}$$

Proof. The subharmonic function $u=\log|f|$ fulfills the $\mathcal{H}$ condition. With ω defined on $[0,1[$ by $\omega(r)=(1-r)^\lambda$, we have $\int_t^1\omega(r)\,dr=\frac{(1-t)^{\lambda+1}}{\lambda+1}$ thus Theorem 2.7 applies with $\alpha=\lambda+1$ and $N=2$. The corollary follows since $\mu=\sum_{n\in\mathbb{N}}\delta_{z_n}$. □

Remark 2.9. The well–known convergence of $\sum_{n\in\mathbb{N}}(1-|z_n|)^{\lambda+2}$ (see Example 2.5) does not imply the convergence of this new series (2.3). Choosing $\varepsilon\in]0,\beta]$, we have

$$\left(\frac{1}{1-|z_n|}\right)^{\varepsilon}\left(\log\frac{1}{1-|z_n|}\right)^{-\gamma}\ge 1$$

for all sufficiently large integers n, thus $(1-|z_n|)^{\lambda+2}$ is smaller than the general term of the series (2.3), because $(1-|z_n|)^{\lambda+2} \leq (1-|z_n|)^{\lambda+2-\beta+\varepsilon}$, since $\lambda+2 \geq \lambda+2-\beta+\varepsilon$ and $1-|z_n| \in]0,1[$.

Corollary 2.10. *Let μ denote the Riesz measure associated to some subharmonic function u in B_N, fulfilling the $\mathcal{H}$ condition together with the following growth estimation: $u(x) \leq \log\log\frac{1}{1-|x|}$ for all $x \in B_N$ such that $|x| \geq 1-e^{-2}$. Then*

$$\int_{1-e^{-2}\leq|x|<1} (1-|\zeta|)\left(\log\log\frac{1}{1-|\zeta|}\right)^{-a} d\mu(\zeta) < +\infty \qquad \forall a > 1.$$

Proof. Given $a>1$, let φ denote a $\mathcal{C}^1$ decreasing function on $[0,1[$ such that $\varphi(t) = [\psi(t)]^{-a}$ for every $t \in J := [1-e^{-2},1[$, where $\psi(t) = \log\log\frac{1}{1-t}$. Since $\psi(t) > 0$ $\forall t \in J$, we obtain $u^+(x) \leq \psi(|x|)$ for all x such that $|x| \in J$, hence $\mathcal{M}_u^+(r) \leq \psi(r)$ $\forall r \in J$. Besides that, we have $\varphi'(t) = -a\,[\psi(t)]^{-a-1}\psi'(t)$ $\forall t \in J$, hence

$$\int_J \mathcal{M}_u^+(r)\,[-\varphi'(r)]\,dr \leq a\int_J [\psi(r)]^{-a}\psi'(r)\,dr = a\left[\frac{[\psi(r)]^{-a+1}}{-a+1}\right]_{1-\frac{1}{e^2}}^{\to 1} = \tfrac{a}{a-1}\,[\psi(1-\tfrac{1}{e^2})]^{-a+1}$$

since $-a+1<0$ and $\psi(r) \to +\infty$ as $r \to 1^-$. For the same reasons $\varphi(t) \to 0$ as $t \to 1^-$ thus Theorem 2.6 applies. With $\alpha = 1/2$, it provides: $\int_{B_N}(1-|\zeta|)\,\varphi(\sqrt{|\zeta|})\,d\mu(\zeta) < +\infty$. Now

$$\psi(\sqrt{t}) = \log\log\tfrac{1+\sqrt{t}}{1-t} = \log\left[\log(1+\sqrt{t}) + \log\tfrac{1}{1-t}\right] = \log\left[\left(\log\tfrac{1}{1-t}\right) j(t)\right],$$

with $j(t) = 1 - \frac{\log(1+\sqrt{t})}{\log(1-t)}$. Thus $\psi(\sqrt{t}) = \psi(t) + \log j(t)$ and

$$[\psi(\sqrt{t})]^{-a} = [\psi(t)]^{-a}\left(1+\tfrac{\log j(t)}{\psi(t)}\right)^{-a}.$$

As $t \to 1^-$, we observe that $\log(1+\sqrt{t}) \to \log 2$ and $\log(1-t) \to -\infty$, thus $j(t) \to 1$ and $\log j(t) \to 0$. Now $\psi(t) \to +\infty$ hence $1+\frac{\log j(t)}{\psi(t)} \to 1$. Finally $\varphi(\sqrt{t}) \sim [\psi(t)]^{-a}$ as $t \to 1^-$ and the result follows. □

Example 2.11. When $N=2$ and $u = \log|f|$, with some function f holomorphic in the unit disk of $\mathbb{C}$, growing as: $|f(z)| \leq \log\frac{1}{1-|z|}$ for all $z \in \mathbb{C}$ such that $1-e^{-2} \leq |z| < 1$, Corollary 2.10 thus contains the following result of [1] (p.120):

$$\sum_{n\in\mathbb{N}} (1-|z_n|)\left(\log\log\frac{1}{1-|z_n|}\right)^{-a} < +\infty \qquad \forall a > 1$$

with $\{z_n\}_{n\in\mathbb{N}}$ the zeros of f with moduli in $[1-e^{-2},1[$, multiplicities taken into account.

3 Sharpness of the condition in Theorem 2.6

In the situation where φ is moreover subject to the additional hypothesis:

$$\exists b \in \mathbb{N} \qquad \exists C > 0 \qquad \text{such that} \qquad \varphi(t) \geq C\,(1-t)^b \qquad \forall t \in [0,1[\tag{3.1}$$

we are going to build a subharmonic function u, fulfilling the $\mathcal{H}$ condition, such that both

$$\int_0^1 \mathcal{M}_u^+(r)\,[-\varphi'(r)]\,dr = +\infty \qquad \text{and} \qquad \int_{B_N} (1-|\zeta|)\,\varphi(|\zeta|^a)\,d\mu(\zeta) = +\infty$$

hold for some $a \in]0,1[$.

Proposition 3.1. *Given* $\alpha \in]0,1[$ *and* $\beta \in \mathbb{N}$, *let* $c_k = \frac{1}{k}\prod_{n=0}^{\beta+1}[n+(k+N-2)/\alpha]$ $\forall k \in \mathbb{N}^*$. *Then the power series* $g(r) = \sum_{k\geq 6} c_k\, r^k$ *has its radius of convergence equal to* 1. *Moreover the following numerical series diverges* $\forall \varepsilon \in]0,1[$*:*

$$\sum_{k\geq 6} c_k \int_\varepsilon^1 (1-r^\alpha)^{\beta+1}\, r^{k+N-3}\,dr.$$

Proof. The radius of convergence is equal to $\lim\limits_{k\to+\infty} \frac{c_k}{c_{k+1}} = 1$.

$$\int_\varepsilon^1 (1-r^\alpha)^{\beta+1}\, r^{k+N-3}\,dr = \int_0^1 (1-r^\alpha)^{\beta+1}\, r^{k+N-3}\,dr - \int_0^\varepsilon \underbrace{(1-r^\alpha)^{\beta+1}}_{\leq 1} r^{k+N-3}\,dr \geq$$

$$\geq \frac{1}{\alpha}\int_0^1 (1-t)^{\beta+1}\, t^{\frac{k+N-3}{\alpha}}\, t^{\frac{1}{\alpha}-1}\,dt - \frac{\varepsilon^{k+N-2}}{k+N-2}.$$

The power series $\sum_k \frac{c_k}{k+N-2} r^k$ also has its radius of convergence equal to 1, hence the convergence of the series $\sum_k c_k \frac{\varepsilon^{k+N-2}}{k+N-2}$ as $\varepsilon \in]0,1[$. Now it remains to establish the divergence of $\sum_k c_k \int_0^1 (1-t)^{\beta+1}\, t^{\frac{k+N-2}{\alpha}-1}\,dt$. The last integral is related to the Beta and Gamma functions:

$$B(\beta+2, \tfrac{k+N-2}{\alpha}) = \frac{\Gamma(\beta+2)\,\Gamma(\frac{k+N-2}{\alpha})}{\Gamma(\frac{k+N-2}{\alpha}+\beta+2)}.$$

According to formula $\Gamma(s+1) = s\,\Gamma(s)$ $\forall s > 0$, the denominator is equal to: $k\,c_k\,\Gamma(\frac{k+N-2}{\alpha})$. Finally the last series is merely $\sum_k \frac{\Gamma(\beta+2)}{k}$ which obviously diverges. □

Proposition 3.2. *Given* $\varepsilon \in]0,1[$, $\alpha \in]0,1[$, $\beta \in \mathbb{N}$ *and* g *as in Proposition 3.1, let* $G : [0,1[\to [0,+\infty[$ *be the* $\mathcal{C}^2$ *function defined by:* $G(r) = 0$ $\forall r \in [0,\varepsilon/2[$, $G''(r) = g''(r)$ $\forall r \in [\varepsilon,1[$ *and* $G''(r) = (\frac{2r}{\varepsilon}-1)\,g''(\varepsilon)$ $\forall r \in [\varepsilon/2,\varepsilon[$. *Thus* $G'(r) \geq g'(r)$ *and* $G(r) \geq g(r)$ $\forall r \in [\varepsilon,1[$. *Then the function* u *defined by* $u(x) = G(|x|)$ $\forall x \in B_N$ *is subharmonic and* ≥ 0 *in* B_N, *fulfills the* $\mathcal{H}$ *condition and its Riesz measure* μ *satisfies:*

$$\int_{B_N} (1-|x|)\,(1-|x|^\alpha)^\beta\,d\mu(x) = +\infty.$$

Remark 3.3. More precisely: $G'(r) = g'(r) + A$ and $G(r) = g(r) + Ar + A_0$ $\forall r \in [\varepsilon,1[$, with $A = \frac{\varepsilon}{4}g''(\varepsilon) - g'(\varepsilon)$ and $A_0 = \frac{\varepsilon^2}{24}g''(\varepsilon) - A\varepsilon - g(\varepsilon)$.

Proof. The $\mathcal{H}$ condition is fulfilled since $(\Delta u)(x) = 0$ whenever $|x| < \frac{\varepsilon}{2}$. The affine continuation of G'' on $[\varepsilon/2,\varepsilon[$ ensures $G'' \geq 0$ which implies both $G' \geq 0$ and $G \geq 0$ on $[0,1[$. With

$r = |x| \neq 0$, [3] (p.26) leads to: $(\Delta u)(x) = G''(r) + \frac{N-1}{r} G'(r) \geq 0$ and the subharmonicity of u follows. For $r \in [\varepsilon/2, \varepsilon]$, we have

$$G'(r) = \tfrac{\varepsilon}{4} g''(\varepsilon) \left(\tfrac{2r}{\varepsilon} - 1\right)^2 \qquad \text{and} \qquad G(r) = \tfrac{\varepsilon^2}{24} g''(\varepsilon) \left(\tfrac{2r}{\varepsilon} - 1\right)^3.$$

Hence

$$G'(\varepsilon) = \frac{\varepsilon}{4} g''(\varepsilon) = \sum_{k \geq 6} k \frac{k-1}{4} c_k \varepsilon^{k-1} \geq \sum_{k \geq 6} k c_k \varepsilon^{k-1} = g'(\varepsilon)$$

$$G(\varepsilon) = \frac{\varepsilon^2}{24} g''(\varepsilon) = \sum_{k \geq 6} \frac{k(k-1)}{24} c_k \varepsilon^k \geq \sum_{k \geq 6} c_k \varepsilon^k = g(\varepsilon) \qquad \text{since } k(k-1) \geq 30.$$

For all $r \in [\varepsilon, 1[$, we have $G'(r) = G'(\varepsilon) + \int_\varepsilon^r g''(t)\,dt = G'(\varepsilon) + g'(r) - g'(\varepsilon) \geq g'(r)$ and $G(r) = G(\varepsilon) + \int_\varepsilon^r G'(t)\,dt \geq G(\varepsilon) + \int_\varepsilon^r g'(t)\,dt = G(\varepsilon) + g(r) - g(\varepsilon) \geq g(r)$. Hence

$$(\Delta u)(x) \geq g''(r) + \frac{N-1}{r} g'(r) = \sum_{k \geq 6} k(k+N-2)\, c_k\, r^{k-2} \qquad \text{if } r = |x| \in [\varepsilon, 1[.$$

Now $d\mu = \frac{1}{\vartheta_N} \Delta u\, dx = \frac{1}{\vartheta_N} \Delta u\, r^{N-1}\, dr\, d\sigma$ with $\vartheta_N = \tau_N \sigma_N$ (see [5] p.43), thus:

$$\int_{B_N} (1-|x|)\,(1-|x|^\alpha)^\beta\, d\mu(x) \geq \int_{\varepsilon \leq |x| < 1} (1-|x|^\alpha)^{\beta+1}\, d\mu(x) \qquad (\text{since } |x|^\alpha \geq |x|\ )$$

$$\geq \frac{1}{\tau_N} \int_\varepsilon^1 (1-r^\alpha)^{\beta+1} \left(\sum_{k \geq 6} k(k+N-2)\, c_k\, r^{k-2} \right) r^{N-1}\, dr \geq$$

$$\geq \frac{1}{\tau_N} \sum_{k \geq 6} c_k \int_\varepsilon^1 (1-r^\alpha)^{\beta+1}\, r^{k+N-3}\, dr$$

since $k(k+N-2) \geq 1$. The result follows from Proposition 3.1. □

Proposition 3.4. *Let φ denote a $\mathcal{C}^1$ decreasing function on $[0,1[$ such that $\varphi(t) \to 0$ as $t \to 1^-$ and fulfilling moreover condition (3.1). Given $\varepsilon \in]0,1[$, $\alpha \in]0,1[$, let u be defined as in Proposition 3.2 with $\beta = b$ from (3.1). Then $\int_0^1 \mathcal{M}_u^+(r)\,[-\varphi'(r)]\,dr = +\infty$.*

Remark 3.5. From (3.1) and Proposition 3.2, we already know that the Riesz measure μ of this function u satisfies: $\int_{B_N} (1-|x|)\,\varphi(|x|^\alpha)\,d\mu(x) = +\infty$.

Proof. Here we have $u^+ = u$ hence $\mathcal{M}_u^+(r) = G(r)\ \forall r \in [0,1[$ thus

$$\int_0^1 \mathcal{M}_u^+(r)\,[-\varphi'(r)]\,dr \geq \int_\varepsilon^1 \mathcal{M}_u^+(r)\,[-\varphi'(r)]\,dr \geq \int_\varepsilon^1 g(r)\,[-\varphi'(r)]\,dr = \sum_{k \geq 6} c_k I_k$$

with g and c_k (resp. G) as in Proposition 3.1 (resp. 3.2) and

$$I_k = \int_\varepsilon^1 r^k\,[-\varphi'(r)]\,dr = \Big[-r^k \varphi(r)\Big]_\varepsilon^{\to 1} + k \int_\varepsilon^1 r^{k-1} \varphi(r)\,dr.$$

Now $\varphi(\varepsilon) \geq 0$ and $\lim\limits_{r \to 1^-} \varphi(r) = 0$, thus

$$I_k \geq kC \int_\varepsilon^1 r^{k-1}\,(1-r)^\beta\,dr \geq C \int_\varepsilon^1 r^{k+N-3}\,(1-r^\alpha)^{\beta+1}\,dr$$

because $k+N-3 \geq k-1$ and $r \in [0,1[$ hence $r^{k-1} \geq r^{k+N-3}$. Similarly $r^\alpha \geq r$ thus $1-r \geq 1-r^\alpha$. Finally $(1-r^\alpha)^\beta \geq (1-r^\alpha)^{\beta+1}$ because $1-r^\alpha \leq 1$, the conclusion follows through Proposition 3.1 as in the previous proof. □

4 Sharpness of the conclusion in Theorem 2.6

When there exist constants $b > 0$, $C > 0$ and $C' > 0$ such that the $\mathcal{C}^1$ decreasing function $\varphi : [0,1[\to [0,+\infty[$ satisfies:

$$C\,(1-t)^b \le \varphi(t) \le C'\,(1-t)^b \qquad \forall t \in [0,1[\tag{4.1}$$

we are going to build here (given $\varepsilon \in]0,\min(1,b)[$) a subharmonic function u such that

$$\int_0^1 \mathcal{M}_u^+(r)\,[-\varphi'(r)]\,dr < +\infty \tag{4.2}$$

but

$$\int_{B_N} (1-|\zeta|)^{1-\varepsilon}\,\varphi(|\zeta|^\alpha)\,d\mu(\zeta) = +\infty \qquad \forall \alpha \in]0,1[\tag{4.3}$$

where μ denotes the Riesz measure associated to u.

Lemma 4.1. *With φ and b as in (4.1), let $\varepsilon > 0$ and g be defined by $g(r) = (1-r)^{-b+\varepsilon} - 1$. Then $\int_0^1 g(r)\,[-\varphi'(r)]\,dr < +\infty$ and $\int_0^1 (1-r)^{1-\varepsilon}\,\varphi(r^\alpha)\,g''(r)\,dr = +\infty$ $\forall \alpha > 0$.*

Proof. Since $g(r)\,\varphi(r) \le C'(1-r)^\varepsilon$ then $g(r)\,\varphi(r) \to 0$ as $r \to 1^-$. Moreover $g(0) = 0$, thus the following holds:

$$\int_0^1 g(r)\,[-\varphi'(r)]\,dr = \underbrace{\Big[-g(r)\,\varphi(r)\Big]_0^{\to 1}}_{=0} + \int_0^1 g'(r)\,\varphi(r)\,dr \le C' \int_0^1 \frac{b-\varepsilon}{(1-r)^{b+1-\varepsilon}}\,(1-r)^b\,dr$$

and $\int_0^{\to 1} \frac{dr}{(1-r)^{1-\varepsilon}}$ converges. Besides that

$$\int_0^1 (1-r)^{1-\varepsilon}\,\varphi(r^\alpha)\,g''(r)\,dr \ge C\,(b-\varepsilon)\,(b+1-\varepsilon) \int_0^1 (1-r)^{1-\varepsilon}\,(1-r^\alpha)^b\,\frac{dr}{(1-r)^{b+2-\varepsilon}}\,.$$

Now $(1-r^\alpha)^b \sim [\alpha(1-r)]^b$ as $r \to 1^-$. Hence the divergence of the above integral since $\int_0^{\to 1} (1-r)^{1-\varepsilon+b}\,\frac{dr}{(1-r)^{b+2-\varepsilon}} = \int_0^{\to 1} \frac{dr}{(1-r)}$. □

Proposition 4.2. *With φ and b as in (4.1), let $\varepsilon \in]0,\min(1,b)[$ and g as in Lemma 4.1. Let u be the function given by $u(x) = G(|x|)$ $\forall x \in B_N$ with G defined from this new function g on the same pattern as in Proposition 3.2. Then u is subharmonic in B_N, fulfills the $\mathcal{H}$ condition as well as (4.2) and (4.3).*

Proof. Since $\varepsilon \le b$, we have $g \ge 0$, $g' \ge 0$ and $g'' \ge 0$ on $[0,1[$. That is why G can be built such that $G'' \ge 0$, $G' \ge 0$ and $G \ge 0$ on $[0,1[$. The subharmonicity of u, the $\mathcal{H}$ condition and the Riesz measure μ of u are obtained in the same way as in Proposition 3.2. Now $u^+ = u$ since $G \ge 0$. Together with Remark 3.3, it leads to:

$$\int_0^1 \mathcal{M}_u^+(r)\,[-\varphi'(r)]\,dr = \int_0^1 G(r)\,[-\varphi'(r)]\,dr =$$

$$= \int_0^\varepsilon G(r)\,[-\varphi'(r)]\,dr + \int_\varepsilon^1 (Ar + A_0)\,[-\varphi'(r)]\,dr + \int_\varepsilon^1 g(r)\,[-\varphi'(r)]\,dr.$$

The first and second above integrals are finite and the third integral converges according to Lemma 4.1. Besides that, given $\alpha > 0$ and noting that $(\Delta u)(x) \geq G''(r)$, we obtain

$$\int_{B_N} (1-|x|)^{1-\varepsilon} \varphi(|x|^\alpha)\, d\mu(x) \geq \frac{1}{\tau_N} \int_0^1 (1-r)^{1-\varepsilon} \varphi(r^\alpha)\, G''(r)\, r^{N-1}\, dr \geq$$

$$\geq \frac{\varepsilon^{N-1}}{\tau_N} \int_\varepsilon^1 (1-r)^{1-\varepsilon} \varphi(r^\alpha)\, g''(r)\, dr.$$

Lemma 4.1 provides the divergence of this integral. □

5 Means over spheres centered at the point $(\frac{1}{2}, 0, \ldots, 0)$

Definition 5.1. Let $B_N^* = B(A, \frac{1}{2}) = \{x \in \mathbb{R}^N : |x - A| < \frac{1}{2}\}$ with $A = (\frac{1}{2}, 0, \ldots, 0) \in \mathbb{R}^N$ and $P_N = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \frac{1}{4}(1 + 3|x|^2) < x_1\} = B(A', \frac{1}{3})$ with $A' = (\frac{2}{3}, 0, \ldots, 0)$. A subharmonic function u in B_N^* is said to fulfill the $\mathcal{H}^*$ condition if u is harmonic in some neighborhood of the point A, with $u(A) = 0$. Let $\mathcal{M}_u^*$ be defined on $[0, \frac{1}{2}[$ by:

$$\mathcal{M}_u^*(r) = \frac{1}{\sigma_N} \int_{S_N} u(A + r\eta)\, d\sigma_\eta.$$

Lemma 5.2. *The following statements hold:*

(i) $\frac{1}{3} < |x| < 1 \quad \forall x \in P_N$ *(ii)* $|x - A| < \frac{|x|}{2} \quad \forall x \in P_N$

(iii) $P_N \subset B_N^*$ *(iv)* $1 - |x| < 2(\frac{1}{2} - |x - A|) \quad \forall x \in P_N$

Proof. From $0 > 1 + 3|x|^2 - 4x_1 \geq 1 + 3|x|^2 - 4|x| = (3|x| - 1)(|x| - 1)$, it follows that $3|x| > 1$ and $|x| < 1$, since the converse inequalities $3|x| < 1$ and $|x| > 1$ are incompatible. Hence *(i)*. Now

$$|x - A|^2 = (x_1 - \tfrac{1}{2})^2 + \sum_{j=2}^N x_j^2 = |x|^2 - x_1 + \tfrac{1}{4} < \tfrac{1}{4}|x|^2 < \tfrac{1}{4}.$$

Hence *(ii)* and *(iii)*. The last inequality follows from $1 - |x| = 2(\frac{1}{2} - \frac{|x|}{2})$ and *(ii)*. □

Theorem 5.3. *Let μ denote the Riesz measure associated to a subharmonic function u in B_N^* which fulfills the $\mathcal{H}^*$ condition.*

(i) *If* $\int_{B_N^*} (1 - |\zeta|)\, d\mu(\zeta) < +\infty$, *then* $\sup_{0 \leq r < \frac{1}{2}} \mathcal{M}_u^*(r) < +\infty$.

(ii) *If* $\sup_{0 \leq r < \frac{1}{2}} \mathcal{M}_u^*(r) < +\infty$, *then* $\int_{P_N} (1 - |\zeta|)\, d\mu(\zeta) < +\infty$.

Proof. It works as in the proof of Theorem 2.2. Here Jensen–Privalov formula is applied on balls centered at A, with $\rho(t) = \mu(\overline{B}(A, t))\ \forall t \in [0, \frac{1}{2}[$. Thus

$$\mathcal{M}_u^*(r) = \int_{|\zeta - A| \leq r} h_r(\zeta - A)\, d\mu(\zeta) \quad \to \quad 2^{N-2} \int_{B_N^*} h(2|\zeta - A|)\, d\mu(\zeta) \qquad \text{as } r \to (\tfrac{1}{2})^-.$$

According to Remark 2.4, the boundedness of $\mathcal{M}_u^*$ on $[0,\frac{1}{2}[$ is equivalent to

$$\int_{B_N^*}(\tfrac{1}{2}-|\zeta-A|)\,d\mu(\zeta)<+\infty.$$

Now $|\zeta-A|\geq|\zeta|-|A|=|\zeta|-\frac{1}{2}$ thus $0<\frac{1}{2}-|\zeta-A|\leq\frac{1}{2}-(|\zeta|-\frac{1}{2})$ $\forall\zeta\in B_N^*$, hence *(i)*. According to Lemma 5.2, we have $0<\frac{1}{2}(1-|\zeta|)\leq\frac{1}{2}-|\zeta-A|$ $\forall\zeta\in P_N$, hence *(ii)*. □

Theorem 5.4. *Let u be a subharmonic function in B_N fulfilling both the $\mathcal{H}^*$ condition and*

$$\exists\alpha\in[0,\tfrac{N-1}{2}[\qquad\textit{such that}\qquad u(x)\leq\left(\frac{1}{1-|x|}\right)^{\alpha}\qquad\forall x\in B_N.$$

Then its Riesz measure μ satisfies $\displaystyle\int_{P_N}(1-|\zeta|)\,d\mu(\zeta)<+\infty.$

Remark 5.5. The previous statement remains valid if the growth condition is formulated as: $u(x)\leq C(1-|x|)^{-\alpha}$ $\forall x\in B_N$ for some constant $C>0$. Actually it is enough to apply Theorem 5.4 to the function $\frac{u}{C}$ whose Riesz measure is merely $\frac{\mu}{C}$.

The proof of Theorem 5.4 will require the following lemma:

Lemma 5.6. *There exists $C>0$ such that $\frac{t^2}{1-|z|}\leq C$ for all $z=\frac{1}{2}+re^{it}$ ($\forall t\in[-\pi,\pi]$, $\forall r\in[\frac{1}{4},\frac{1}{2}[$). For information: $C=\frac{16}{1-\frac{\pi^2}{12}}$.*

Proof. As $|z|^2=(\frac{1}{2}+r\cos t)^2+(r\sin t)^2=\frac{1}{4}+r^2+r\cos t$, there exists $\theta\in]0,1[$ such that

$$1-|z|^2=\tfrac{3}{4}-r^2-r[1-\tfrac{t^2}{2}+\tfrac{t^4}{4!}\cos(\theta t)]=\underbrace{1-(\tfrac{1}{2}+r)^2}_{\geq 0}+\tfrac{rt^2}{2}[1-\tfrac{t^2}{12}\cos(\theta t)].$$

Now $\frac{t^2}{12}\cos(\theta t)\leq\frac{t^2}{12}\leq\frac{\pi^2}{12}<1$, thus $2(1-|z|)\geq(1+|z|)(1-|z|)=1-|z|^2\geq\frac{t^2}{8}(1-\frac{\pi^2}{12})$. □

In Lemma 5.6, the exponent of t^2 can not be removed, as pointed out by the following:

Lemma 5.7. *There does NOT exist any constant $D>0$ such that $\frac{|t|}{1-|z|}\leq D$ for all $z=\frac{1}{2}+re^{it}$ ($\forall t\in[-\pi,\pi]$, $\forall r\in[\frac{1}{4},\frac{1}{2}[$).*

Proof. Let us assume on the contrary that such a constant D would exist. We would then obtain: $\frac{|t|}{1-|z|^2}\leq\frac{D}{1+|z|}\leq D$. With the notation $\lambda=\min(\frac{1}{D},\frac{1}{4})$, this would lead to:

$$\tfrac{1}{4}+r^2+r\cos t=|z|^2\leq 1-\lambda|t|\qquad\forall t\in[-\pi,\pi]\qquad\forall r\in[\tfrac{1}{4},\tfrac{1}{2}[.$$

Given $r\in[\frac{1}{4},\frac{1}{2}[$, let f_r be the function defined on $[0,\pi]$ by $f_r(t)=-\frac{3}{4}+r^2+r\cos t+\lambda t$. Now $f_r'(t)=-r\sin t+\lambda$ vanishes at the point $t=\arcsin\frac{\lambda}{r}\in[0,\frac{\pi}{2}]$, well-defined since $\frac{\lambda}{r}\in[0,1]$. The maximum attained by f_r is

$$M_r=f_r(\arcsin\tfrac{\lambda}{r})=-\tfrac{3}{4}+r^2+r\sqrt{1-\tfrac{\lambda^2}{r^2}}+\lambda\arcsin\tfrac{\lambda}{r}.$$

We should have $M_r\leq 0$ for all $r\in[\frac{1}{4},\frac{1}{2}[$. Letting $r\to(\frac{1}{2})^-$, it would provide the inequality: $-\frac{1}{2}+\frac{1}{2}\sqrt{1-4\lambda^2}+\lambda\arcsin(2\lambda)\leq 0$. But $\arcsin x\geq x+\frac{x^3}{6}$ and $\sqrt{x}\geq x$ $\forall x\in[0,1]$. Hence $2\lambda\arcsin(2\lambda)+\sqrt{1-4\lambda^2}\geq(2\lambda)^2+\frac{(2\lambda)^4}{6}+1-4\lambda^2=1+\frac{8\lambda^4}{3}>1$, the contradiction follows. □

Proof. We are now ready to prove Theorem 5.4. Since $\mathcal{M}_u^*$ is non–decreasing on $[0,\frac{1}{2}[$, it is enough, according to Theorem 5.3 *(ii)*, to prove that

$$\sup_{\frac{1}{4}\leq r<\frac{1}{2}} \mathcal{M}_u^*(r) < +\infty.$$

Let $x = A + r\eta$ with $\eta = (\eta_1,\eta_2,...,\eta_N) \in S_N$. Polar coordinates in $\mathbb{R}^N$ provide: $\eta_1 = \cos\theta_1$ and $d\sigma = (\sin\theta_1)^{N-2}(\sin\theta_2)^{N-3}\dots(\sin\theta_{N-2})\,d\theta_1\,d\theta_2\dots d\theta_{N-1}$ where $\theta_{N-1} \in [0,2\pi[$ and $\theta_1,\ \theta_2,\ \dots,\ \theta_{N-2} \in [0,\pi[$ (see [10] p.15). Thus $|x|^2 = (\frac{1}{2}+r\eta_1)^2 + r^2(\eta_2^2+...+\eta_N^2) = \frac{1}{4}+r\eta_1+r^2|\eta|^2 = \frac{1}{4}+r\cos\theta_1+r^2 = |z|^2$ with $z = \frac{1}{2}+re^{i\theta_1}$. If $N>2$ then $\theta_1 \in [0,\pi[$ thus

$$u(A+r\eta) \leq \left(\frac{1}{1-|z|}\right)^{\alpha} \leq \left(\frac{C}{\theta_1^2}\right)^{\alpha}$$

according to Lemma 5.6. Now $(\sin\theta_2)^{N-3}(\sin\theta_3)^{N-4}\dots(\sin\theta_{N-2})\,d\theta_2\,d\theta_3\dots d\theta_{N-1}$ is the area element on S_{N-1}. Hence

$$\sigma_N\,\mathcal{M}_u^*(r) \leq C^{\alpha}\int_{S_N}\frac{d\sigma}{(\theta_1)^{2\alpha}} = C^{\alpha}\,\sigma_{N-1}\int_{\to 0}^{\pi}\frac{(\sin\theta_1)^{N-2}}{(\theta_1)^{2\alpha}}\,d\theta_1.$$

This integral behaves as:

$$\int_{\to 0}^{\pi}\frac{(\theta_1)^{N-2}}{(\theta_1)^{2\alpha}}\,d\theta_1 = \int_{\to 0}^{\pi}\frac{d\theta_1}{(\theta_1)^{2\alpha-N+2}}$$

which converges since $2\alpha-N+2<1$. In the case $N=2$, we have $\theta_1 \in [0,2\pi[$ and

$$\sigma_N\,\mathcal{M}_u^*(r) \leq \int_0^{2\pi}\frac{d\theta_1}{(1-|z|)^{\alpha}}.$$

As in the previous case

$$\int_0^{\pi}\frac{d\theta_1}{(1-|z|)^{\alpha}} \leq C^{\alpha}\int_{\to 0}^{\pi}\frac{d\theta_1}{(\theta_1)^{2\alpha}} < +\infty \qquad \text{since } 2\alpha<1.$$

When $\theta_1 \in [\pi,2\pi[$, then $z = \frac{1}{2}+re^{it}$ with $t = \theta_1-2\pi \in [-\pi,0[$ and Lemma 5.6 leads to

$$\int_{\pi}^{2\pi}\frac{d\theta_1}{(1-|z|)^{\alpha}} = \int_{-\pi}^{0}\frac{dt}{(1-|z|)^{\alpha}} \leq C^{\alpha}\int_{-\pi}^{\to 0}\frac{dt}{t^{2\alpha}} < +\infty.$$

□

Example 5.8. With $N=2$, $\alpha \in [0,\frac{1}{2}[$, $u = \log|f|$, f an holomorphic function in D and $\{z_n\}_{n\in\mathbb{N}}$ the zeros of f in D repeated according to their multiplicities, Theorem 5.4 thus leads to $\sum_{n:z_n\in P_N}(1-|z_n|) < +\infty$ when f has such a growth as:

$$\exists C>0 \qquad \log|f(z)| \leq C\left(\frac{1}{1-|z|}\right)^{\alpha} \qquad \forall z\in D.$$

In particular

$$\sum_{n:z_n\in[0,1[}(1-|z_n|) < +\infty,$$

previously proved for $\alpha \in [0,1[$ by [6] (p.225) and [4] where it is also shown that this Blaschke–type result on $[0,1[$ does not remain valid for $\alpha \geq 1$.

Remark 5.9. Theorem 5.4 does not hold any longer with $\alpha \geq \frac{N-1}{2}$, as shown by the following counterexample:

Proposition 5.10. *Given $\alpha \geq \frac{N-1}{2}$, let $k_\alpha = \frac{\alpha(\alpha+1)}{2^{\alpha+2}}$ and $G : [0,1[\to [0,+\infty[$ be the $\mathcal{C}^2$ function defined by: $G(r) = 0\ \forall r \in [0,\frac{2}{3}]$, $G''(r) = k_\alpha (1-r)^{-(\alpha+2)}\ \forall r \in [\frac{5}{6},1[$ and $G''(r) = 4\,G''(\frac{5}{6})\,(\frac{3r}{2}-1)\ \forall r \in [\frac{2}{3},\frac{5}{6}]$. Then the function u defined by $u(x) = G(|x|)\ \forall x \in B_N$ is subharmonic in B_N, fulfills the $\mathcal{H}^*$ condition and grows as: $u(x) \leq (1-|x|)^{-\alpha}\ \forall x \in B_N$. Moreover the Riesz measure μ associated to u satisfies*

$$\int_{P_N} (1-|\zeta|)\, d\mu(\zeta) = +\infty.$$

Proof. We first notice that $G''(r) \leq \alpha(\alpha+1)\,(1-r)^{-\alpha-2}\ \forall r \in [0,1[$. On $[\frac{5}{6},1[$, it follows from $2^{\alpha+2} \geq 1$ hence $k_\alpha \leq \alpha(\alpha+1)$. On $[\frac{2}{3},\frac{5}{6}]$, we have $G''(r) \leq G''(\frac{5}{6}) = \frac{\alpha(\alpha+1)}{2^{\alpha+2}}\,6^{\alpha+2} = \alpha(\alpha+1)\,3^{\alpha+2} = \alpha(\alpha+1)\,(1-\frac{2}{3})^{-\alpha-2} \leq \alpha(\alpha+1)\,(1-r)^{-\alpha-2}$, this function being increasing, as well as G''. We next obtain: $G'(r) = G'(r) - G'(0) \leq \int_0^r \alpha(\alpha+1)\,(1-t)^{-(\alpha+2)} dt = \alpha\left[(1-t)^{-(\alpha+1)}\right]_0^r = \alpha\,(1-r)^{-(\alpha+1)} - \alpha \leq \alpha\,(1-r)^{-(\alpha+1)}$ for every $r \in [0,1[$. Similarly $G(r) = \int_0^r G'(t)\,dt \leq \int_0^r \alpha\,(1-t)^{-(\alpha+1)} dt = (1-r)^{-\alpha} - 1 \leq (1-r)^{-\alpha}\ \forall r \in [0,1[$.

The affine continuation of G'' on $[\frac{2}{3},\frac{5}{6}]$ implies $G'' \geq 0$, $G' \geq 0$ and $G \geq 0$ on $[0,1[$. The $\mathcal{H}^*$ condition and the subharmonicity of u are obvious, moreover $(\Delta u)(x) \geq G''(r)$ with $r = |x|$ from [3] (p.26) and $d\mu$ is given by [5] (p.43) as in the proof of Proposition 3.2. Hence

$$\int_{P_N} (1-|x|)\, d\mu(x) \geq \frac{1}{\vartheta_N} \int_{P_N} (1-|x|)\, G''(|x|)\, dx.$$

Given $r \in]\frac{1}{3},1[$, let $A_r = \arccos \frac{1+3r^2}{4r}$. For $(x_1,x_2,\ldots,x_N) \in P_N$ with $|x| = r$ and $x_1 = r\cos\theta_1$ (polar coordinates as in [10] p.15), we have $\frac{1}{4}(1+3r^2) < r\cos\theta_1$. Hence $\theta_1 \in [0,A_r[$ if $N > 2$ or $\theta_1 \in I_r := [0,A_r[\cup]2\pi - A_r, 2\pi[$ if $N = 2$, with $(x_2,\ldots,x_N) \in (r\sin\theta_1)S_{N-1}$. Now

$$(\cos A_r) - 1 = \tfrac{(3r-1)(r-1)}{4r} \sim \tfrac{r-1}{2} \qquad \text{as } r \to 1.$$

Simultaneously $A_r \to 0$, thus $(\cos A_r) - 1 \sim -\frac{A_r^2}{2}$, hence $A_r \sim \sqrt{1-r}$ and $\sin A_r \sim \sqrt{1-r}$. In the case $N = 2$, we obtain

$$\int_{P_2} (1-|x|)\, d\mu(x) \geq \frac{1}{\vartheta_2} \int_{1/3}^{1} \int_{\theta \in I_r} (1-r)\, G''(r)\, d\theta\, r\, dr \geq$$

$$\geq \frac{1}{2\pi} \int_{5/6}^{1} (1-r)\, k_\alpha\, (1-r)^{-(\alpha+2)}\, 2A_r\, r\, dr \geq \frac{5\,k_\alpha}{6\pi} \int_{5/6}^{\to 1} (1-r)^{-(\alpha+1)}\, A_r\, dr.$$

This integral behaves as

$$\int_{5/6}^{\to 1} (1-r)^{-\alpha-1} \sqrt{1-r}\, dr = \int_{5/6}^{\to 1} \frac{dr}{(1-r)^{\alpha+\frac{1}{2}}}$$

which diverges since $\alpha + \frac{1}{2} \geq 1$. When $N > 2$, we use the same notations for polar coordinates as in the previous proof. Thus

$$\int_{P_N} (1-|x|)\, d\mu(x) \geq \frac{1}{\vartheta_N} \int_{1/3}^{1} \int_0^{A_r} (1-r)\, G''(r)\, \sigma_{N-1}\, (\sin\theta_1)^{N-2}\, d\theta_1\, r^{N-1}\, dr \geq$$

$$\geq \frac{\sigma_{N-1}}{3^{N-1}\vartheta_N}\int_{5/6}^{1}(1-r)\,G''(r)\left(\int_0^{A_r}\cos\theta_1\,(\sin\theta_1)^{N-2}\,d\theta_1\right)dr =$$

$$= \frac{\sigma_{N-1}}{3^{N-1}\vartheta_N}\int_{5/6}^{1}(1-r)\,k_\alpha\,(1-r)^{-(\alpha+2)}\left[\frac{(\sin\theta_1)^{N-1}}{N-1}\right]_0^{A_r}dr =$$

$$= \frac{k_\alpha\,\sigma_{N-1}}{(N-1)\,3^{N-1}\vartheta_N}\int_{5/6}^{\to 1}(1-r)^{-\alpha-1}\,(\sin A_r)^{N-1}\,dr.$$

This integral behaves as

$$\int_{5/6}^{\to 1}(1-r)^{-\alpha-1}\,(1-r)^{\frac{N-1}{2}}\,dr = \int_{5/6}^{\to 1}\frac{dr}{(1-r)^{1+\alpha-\frac{N-1}{2}}}$$

which diverges since $1+\alpha-\frac{N-1}{2}\geq 1$. □

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