Communications in **M**athematical **A**nalysis
Volume 7, Number 1, pp. 75–93 (2009)
ISSN 1938-9787
www.commun-math-anal.org

APPROXIMATION OF ENTIRE FUNCTIONS OVER JORDAN DOMAINS

RAMESH GANTI *
Department of Mathematics
National Institute of Technology Silchar
SILCHAR, 788010, INDIA

G.S.SRIVASTAVA †
Department of Mathematics
Indian Institute of Technology Roorkee
ROORKEE, 247667, INDIA

(Communicated by Toka Diagana)

Abstract

In the present paper, we study the polynomial approximation of entire functions over Jordan domains by using Faber polynomials. The coefficient characterizations of generalized order and generalized type of entire functions have been obtained in terms of the approximation errors.

AMS Subject Classification: 30B10, 30D15.

Keywords: Entire function, generalized order, generalized type, Faber polynomial approximation, approximation error.

1 Introduction

Let C be an analytic Jordan curve, D its interior and E be its exterior. Let φ map E conformally onto $\{w : |w| > 1\}$ such that $\varphi(\infty) = \infty$ and $\varphi^{'}(\infty) > 0$. Then for sufficiently large $|z|$, $\varphi(z)$ can be expressed as

$$w \;=\; \varphi(z) = \frac{z}{d} + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots. \tag{1.1}$$

An arbitrary Jordan curve can be approximated from the inside as well as from the outside by analytic Jordan curves. Since C is analytic, φ is holomorphic on C as well. The n th

*E-mail address: ramesh.ganti@gmail.com
†E-mail address: girssfma@iitr.ernet.in

Faber polynomial $F_n(z)$ of C is the principal part of $(\varphi(z))^n$ at ∞, so that

$$F_n(z) = \frac{z^n}{d^n} + \dots.$$

Faber [2] proved that as $n \to \infty$,

$$F_n(z) \sim (\varphi(z))^n \tag{1.2}$$

uniformly for $z \in E$ and

$$\lim_{n\to\infty} \left(\max_{z\in C} |F_n(z)| \right)^{1/n} = 1. \tag{1.3}$$

A function f holomorphic in D can be represented by its Faber series

$$f(z) = \sum_{n=0}^{\infty} a_n F_n(z) \tag{1.4}$$

where

$$a_n = \frac{1}{2\pi i} \int_{|w|=r} f(\varphi^{-1}(w)) w^{-(n+1)} dw$$

and $r < 1$ is sufficiently close to 1 so that φ^{-1} is holomorphic and univalent in $|w| \geq r$, the series converging uniformly on compact subsets of D.

Let, for an entire function f, $M(r, f) = \max_{|z|=r} |f(z)|$ be the maximum modulus of $f(z)$. The growth of $f(z)$ is measured in terms of its order ρ and type τ defined as under

$$\limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r} = \rho, \tag{1.5}$$

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{r^{\rho}} = \tau, \tag{1.6}$$

for $0 < \rho < \infty$. Various workers have given different characterizations for entire functions of fast growth $(\rho = \infty)$.

M. N. Seremeta [4] defined the generalized order and generalized type with the help of general functions as follows. Let L^o denote the class of functions h satisfying the following conditions

(i) $h(x)$ is defined on $[a, \infty)$ and is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$,

(ii)

$$\lim_{x\to\infty} \frac{h\{(1 + 1/\psi(x))x\}}{h(x)} = 1,$$

for every function $\psi(x)$ such that $\psi(x) \to \infty$ as $x \to \infty$.

Let Λ denote the class of functions h satisfying condition (i) and

$$\lim_{x\to\infty} \frac{h(cx)}{h(x)} = 1$$

for every $c > 0$, that is, $h(x)$ is slowly increasing.
For the entire function $f(z) = \sum_{n=0}^{\infty} c_n z^n$, and functions $\alpha(x) \in \Lambda, \beta(x) \in L^o$, Seremeta [4, Th .1] proved that

$$\rho(\alpha,\ \beta,\ f) \ = \ \limsup_{r\to\infty} \frac{\alpha[\log M(r,f)]}{\beta(\log r)} \ = \ \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|c_n|\right)}. \tag{1.7}$$

Further, for $\alpha(x) \ \in L^o, \beta^{-1}(x) \ \in L^o, \gamma(x) \ \in L^o$,

$$\tau(\alpha,\ \beta,\ f) \ = \ \limsup_{r\to\infty} \frac{\alpha[\log M(r,f)]}{\beta[(\gamma(r))^{\rho}]} \ = \ \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{[\gamma(e^{1/\rho}|c_n|^{-1/n})]^{\rho}\}}. \tag{1.8}$$

where $\rho,\ 0 < \rho < \infty$, is a fixed number.

Let $L^p(D)$ denote the set of functions f holomorphic in D and such that

$$\|f\|_{L^p(D)} = \left(\frac{1}{A}\int\int_D |f(z)|^p dxdy\right)^{1/p} < \infty$$

where A is the area of D. For $f \in L^p(D)$, set

$$E_n^p = E_n^p(f;D) = \min_{\pi_n} \|f - \pi_n\|_{L^p(D)}$$

where π_n is an arbitrary polynomial of degree at most n.

Giroux [3] obtained the characterizations of order and type of entire functions in terms of the coefficients of their Faber expansions over Jordan domains. He also obtained necessary and sufficient conditions in terms of polynomial approximation errors.To the best of our knowledge, coefficient characterization for generalized order and generalized type of entire functions over Jordan domain have not been obtained so far.

In this paper, we have made an attempt to bridge this gap. First we obtain coefficient characterization for generalized order and generalized type of entire functions over Jordan domains. Next we obtain necessary and sufficient conditions of generalized order and generalized type of entire functions in terms of approximation errors.

2 Generalized Order and Generalized Type

In this section we obtain the growth characterizations in terms of the coefficients $\{a_n\}$ of the Faber series (1.4). We first prove

Theorem 2.1. *Let $\alpha(x) \in \Lambda,\ \beta(x) \in L^0$. Set $H(x;c) = \beta^{-1}[c\ \alpha(x)]$, then f is restriction to the domain D of an entire function of finite generalized order ρ if and only if*

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|a_n|\right)} \ = \ \rho, \tag{2.1}$$

provided $dH(x;c)/d\ln x = O(1)$ as $x \to \infty$ for all c, $0 < c < \infty$.

Proof. Let $f(z)=\sum_{n=0}^{\infty} a_n F_n(z)$ be an entire function of finite generalized order ρ, where

$$a_n = \frac{1}{2\pi i}\int_{|w|=R} f(\varphi^{-1}(w))\, w^{-(n+1)} dw$$

with arbitrarily large R. From (1.1) , we have

$$\lim_{|w|\to\infty} \frac{\varphi^{-1}(w)}{w} = d.$$

Hence for sufficiently large $|w|$,

$$(d-\varepsilon)|w| \;\le\; |\varphi^{-1}(w)| \le (d+\varepsilon)|w|.$$

Therefore

$$|f(\varphi^{-1}(w))| \;\le\; \exp\left\{\alpha^{-1}\left[\overline{\rho}\,\beta\left(\ln\left(d+\varepsilon\right)|w|\right)\right]\right\},\ \overline{\rho}=\rho+\varepsilon,$$

and from Cauchy's inequality, we have

$$|a_n| \;\le\; R^{-n}\exp\left\{\alpha^{-1}\left[\overline{\rho}\,\beta\left(\ln\left(d+\varepsilon\right)|w|R\right)\right]\right\}.$$

for all R sufficiently large. To minimize the right member of this inequality, choose $R = R(n) = \frac{1}{d+\varepsilon}\exp\left\{H(n;\frac{1}{\overline{\rho}})\right\}$. Substituting this value of R in the above inequality, we have

$$-\ln|a_n| \;\ge\; nH(n;\frac{1}{\overline{\rho}}) - n\ln(d+\varepsilon) - \alpha^{-1}\left[\overline{\rho}\,\beta\left(H(n;\frac{1}{\overline{\rho}})\right)\right]$$

$$\Rightarrow \qquad -\frac{1}{n}\ln|a_n| \;\ge\; H\left(n;\frac{1}{\overline{\rho}}\right) \;=\; \beta^{-1}\left[\frac{1}{\overline{\rho}}\,\alpha(n)\right]$$

$$\Rightarrow \qquad \beta\left(-\frac{1}{n}\ln|a_n|\right) \;\ge\; \frac{1}{\overline{\rho}}\alpha(n)$$

$$\Rightarrow \qquad \rho+\varepsilon \;\ge\; \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|a_n|\right)}.$$

Now proceeding to limits and since ε is arbitrary, we have

$$\rho \;\ge\; \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|a_n|\right)}. \tag{2.2}$$

Conversely, let

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|a_n|\right)} = \sigma.$$

Suppose $\sigma<\infty$. Then for every $\varepsilon>0$, there exists $N(\varepsilon)$ such that $\forall\, n \;\ge\; N(\varepsilon)$, we have

$$\frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|a_n|\right)} \;\le\; \sigma+\varepsilon \;=\; \overline{\sigma},$$

$$\Rightarrow \qquad |a_n| \;\le\; \exp\left\{-n\,H\left(n;\frac{1}{\overline{\sigma}}\right)\right\}.$$

Since $f(z) = \sum_{n=0}^{\infty} a_n F_n(z)$, therefore

$$|f(z)| \le \sum_{n=0}^{\infty} \exp\left\{-nH\left(n;\frac{1}{\bar{\sigma}}\right)\right\} |F_n(z)|.$$

But from (1.2), we have for some $K>0$, $|F_n(z)| \le K|\varphi(z)|^n \quad \forall z \in E$ and from (1.1), for all sufficiently large $|z|$, we have

$$|\varphi(z)| \le \frac{|z|}{d-\varepsilon}. \tag{2.3}$$

Therefore the above inequality reduces to

$$|f(z)| \le K\sum_{n=0}^{\infty} \exp\left\{-nH\left(n;\frac{1}{\bar{\sigma}}\right)\right\} \left(\frac{|z|}{d-\varepsilon}\right)^n. \tag{2.4}$$

By considering the function $\psi(x) = \left(\frac{R}{d-\varepsilon}\right)^x \exp\left\{-xH\left(x;\frac{1}{\bar{\rho}}\right)\right\}$ and proceeding on the lines of proof of Theorem 1 of Seremeta [4, p 294], we obtain

$$M(R;f)(1+o(1)) \le \exp\{(G+o(1))\alpha^{-1}[\bar{\sigma}\,\beta(\ln R+G)]\}$$
$$\Rightarrow \quad \frac{\alpha[(G+o(1))^{-1}\ln M(R;f)]}{\beta(\ln R+G)} \le \bar{\sigma} = \sigma+\varepsilon.$$

Since $\alpha(x) \in \Lambda$ and $\beta(x) \in L^0$, on letting $R \to \infty$ and since ε is arbitrary, we get

$$\limsup_{R\to\infty} \frac{\alpha(\ln M(R;f))}{\beta(\ln R)} \le \sigma = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|a_n|\right)}. \tag{2.5}$$

The above inequality holds obviously of $\sigma = \infty$. From (2.2) and (2.5), we obtain the required result (2.1). This completes the proof of Theorem 1. □

Next we prove

Theorem 2.2. *Let* $\alpha(x)$, $\beta^{-1}(x)$, $\gamma(x) \in L^0$; *let* ρ *be a fixed number,* $0<\rho<\infty$. *Set* $H(x;\sigma,\rho) = \gamma^{-1}\left\{\left[\beta^{-1}(\sigma\,\alpha(x))\right]^{1/\rho}\right\}$, *then f is the restriction to the domain D of an entire function of generalized order* ρ *and finite generalized type* τ *if and only if*

$$\limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|a_n|^{-1/n}\right)\right]^{\rho}\right\}} = \tau \tag{2.6}$$

provided if $\gamma(x) \in \Lambda$ *and* $\alpha(x) \in \Lambda$, $dH(x;\sigma,\rho)/d\ln x = O(1)$ *as* $x \to \infty$.

Proof. Proceeding as in the proof of Theorem 1, we have

$$(d-\varepsilon)|w| \le |\varphi^{-1}(w)| \le (d+\varepsilon)|w|.$$

Let f be an entire function of generalized type τ having finite generalized order ρ. Then we have

$$|f(\varphi^{-1}(w))| \le \exp\left\{\alpha^{-1}\left\{\bar{\tau}\,\beta\left[(\gamma((d+\varepsilon)|w|))^{\rho}\right]\right\}\right\},$$

and from Cauchy's inequality, we have

$$|a_n| \leq R^{-n} \exp\left\{\alpha^{-1}\left\{\overline{\tau}\,\beta\left[(\gamma((d+\varepsilon)|w|))^{\rho}\right]\right\}\right\},$$

for all R sufficiently large. To minimize the right hand side of this inequality, choose $R = R(n) = \frac{1}{(d+\varepsilon)}H\left(\frac{n}{\rho};\frac{1}{\overline{\tau}},\rho\right)$. Substituting value of R in the above inequality, we have

$$|a_n| \leq \frac{\exp\left(\frac{n}{\rho}\right)}{\left[(d+\varepsilon)\,H\left(\frac{n}{\rho};\frac{1}{\overline{\tau}},\rho\right)\right]^n}$$

$$\Rightarrow \quad (d+\varepsilon)e^{1/\rho}|a_n|^{-1/n} \geq H\left(\frac{n}{\rho};\frac{1}{\overline{\tau}},\rho\right).$$

Now proceeding to limits and since ε is arbitrary, we have

$$\limsup_{n\to\infty}\frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|a_n|^{-1/n}\right)\right]^{\rho}\right\}} \leq \tau. \tag{2.7}$$

Conversely let

$$\limsup_{n\to\infty}\frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|a_n|^{-1/n}\right)\right]^{\rho}\right\}} = \eta.$$

Suppose $\eta < \infty$. Then for every $\varepsilon > 0$, there exists $Y(\varepsilon)$ such that for all $n \geq Y(\varepsilon)$, we have

$$\frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|a_n|^{-1/n}\right)\right]^{\rho}\right\}} \leq \eta+\varepsilon = \overline{\eta}.$$

$$\Longrightarrow \quad |a_n| \leq \frac{d^n\exp\left(\frac{n}{\rho}\right)}{\left[H\left(\frac{n}{\rho};\frac{1}{\overline{\eta}},\rho\right)\right]^n}.$$

Since $f(z) = \sum_{n=0}^{\infty} a_n\,F_n(z)$, therefore

$$|f(z)| \leq \sum_{n=0}^{\infty}\frac{d^n\exp\left(\frac{n}{\rho}\right)}{\left[H\left(\frac{n}{\rho};\frac{1}{\overline{\eta}},\rho\right)\right]^n}\,|F_n(z)|.$$

As in (2.4), we have on using the estimate of $F_n(z)$,

$$\begin{aligned}|f(z)| &\leq \sum_{n=0}^{\infty}\frac{d^n\exp\left(\frac{n}{\rho}\right)}{\left[H\left(\frac{n}{\rho};\frac{1}{\overline{\eta}},\rho\right)\right]^n}\left(\frac{d\,|z|}{d-\varepsilon}\right)^n\\ &\leq \sum_{n=0}^{\infty}\frac{d^n\exp\left(\frac{n}{\rho}\right)}{\left[H\left(\frac{n}{\rho};\frac{1}{\overline{\eta}},\rho\right)\right]^n}R^n.\end{aligned}$$

To estimate the summation of the left hand side of above inequality, we consider the function $\psi(x) = \left(R\, e^{1/\rho}\right)^x \left[H\left(\frac{n}{\rho};\frac{1}{\overline{\eta}},\rho\right)\right]^{-x}$. Then following the proof of Seremeta [4, Th .2, Page 296], we obtain

$$M(R;f) \leq \exp\left\{(A\,\rho + o(1))\ \alpha^{-1}\left\{\overline{\eta}\,\beta\left[\left(\gamma\left(Re^{\frac{1}{\rho}}\right)\right)^{\rho}\right]\right\}\right\}.$$

By using the definition of the class L^o and Λ, and proceeding to limits, we obtain

$$\tau = \limsup_{R\to\infty} \frac{\alpha\left[\ln M(R;f)\right]}{\beta\left[(\gamma(R))^{\rho}\right]} \leq \eta. \tag{2.8}$$

From (2.7) and (2.8), we get the required result. This completes the proof of Theorem 2. □

3 L^p - Approximation

In this section we consider the approximations of an entire function over the domain D.

Consider the polynomials

$$p_n(z) = \lambda_n z^n + \ldots (\lambda_n > 0)$$

defined through

$$\frac{1}{A}\int\int_D p_n(z)\,\overline{p_m(z)}dx\,dy = \delta_{n,m}.$$

These polynomials were first considered by T. Carleman [1] who proved that

$$p_n(z) \sim \left(\frac{(n+1)A}{\pi}\right)^{1/2} \varphi^{'}(z)\,(\varphi(z))^n \qquad \text{as } n\to\infty \tag{3.1}$$

uniformly for $z \in E$ where A and $\varphi(z)$ are as defined earlier. Any function $f \in L^2(D)$ can be expanded in terms of these polynomials in a series

$$f(z) = \sum_{n=0}^{\infty} b_n\, p_n(z) \tag{3.2}$$

where

$$b_n = \frac{1}{A}\int\int_D f(z)\,\overline{p_n(z)}dx\,dy$$

and the series converges uniformly on compact subsets of D.

Parseval's relation yields

$$E_n^2 = \left(\sum_{k=n+1}^{\infty} |b_k|^2\right)^{1/2}. \tag{3.3}$$

We now prove

Lemma 3.1. *Let* $\alpha(x) \in \Lambda,\ \beta(x) \in L^o$, *then*

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)} = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln E_n^2\right)} \tag{3.4}$$

Proof. From (3.3), we have

$$|b_{n+1}| \leq E_n^2,$$

$$\Longrightarrow \frac{1}{n} \ln\frac{1}{|b_{n+1}|} \geq \frac{1}{n} \ln\frac{1}{E_n^2}$$

Since $\beta \in L^o$, we have

$$\beta\left(-\frac{1}{n} \ln|b_{n+1}|\right) \geq \beta\left(-\frac{1}{n} \ln E_n^2\right).$$

Since $\alpha \in \Lambda$, proceeding to limits, we have

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)} \leq \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln E_n^2\right)}. \tag{3.5}$$

Conversely, let

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)} = \rho.$$

Suppose $\rho < \infty$. Then for every $\varepsilon > 0$, there exists $G(\varepsilon)$, such that for all $n \geq G(\varepsilon)$, we have

$$\frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)} \leq \rho+\varepsilon = \bar{\rho}$$

$$\begin{aligned}\Longrightarrow |b_n| &\leq G \exp\left\{-n\,\beta^{-1}\left[\frac{1}{\bar{\rho}}\,\alpha(n)\right]\right\}\\ &\leq G \exp\left\{-n\,H(n;\frac{1}{\bar{\rho}})\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}\left(E_n^2\right)^2 &\leq G \sum_{k=n+1}^{\infty} \exp\left\{-2kH(k;\frac{1}{\bar{\rho}})\right\}\\ &\leq G \exp\left\{-2(n+1)H(n+1;\frac{1}{\bar{\rho}})\right\}\left(1-\frac{1}{e^{2H(n+1;\frac{1}{\bar{\rho}})}}\right)\\ &\leq G\exp\left\{-2(n+1)H(n+1;\frac{1}{\bar{\rho}})\right\}\end{aligned}$$

$$\Longrightarrow \quad \ln\frac{1}{E_n^2} \geq (n+1)\,H(n+1;\frac{1}{\bar{\rho}})$$

$$\text{or } \beta\left(-\frac{1}{n+1}\ \ln E_n^2\right) \ \ge\ \frac{1}{\rho}\,\alpha(n+1).$$

Now proceeding to limits, we obtain

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln E_n^2\right)} \ \le\ \rho = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)}. \tag{3.6}$$

From (3.5) and (3.6), we get the required result. This completes the proof of Lemma 1. □

Now we prove

Lemma 3.2. *Let* $\alpha(x)\,,\beta^{-1}(x)\,,\gamma(x) \in L^o$*; let* ρ *be a fixed number,* $0<\rho<\infty$*. Set* $H(x;\sigma,\rho) = \gamma^{-1}\left\{\left[\beta^{-1}\left(\sigma\,\alpha(x)\right)\right]^{1/\rho}\right\}$*, then*

$$\limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|b_n|^{-1/n}\right)\right]^{\rho}\right\}} = \limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}(E_n^2)^{-1/n}\right)\right]^{\rho}\right\}} \tag{3.7}$$

Proof. From (3.3), we have

$$|b_{n+1}| \ \le\ E_n^2,$$

$$e^{1/\rho}\ |b_{n+1}|^{-1/n} \ \ge\ e^{1/\rho}\ (E_n^2)^{-1/n}.$$

Since $\gamma \in L^o$, we have

$$\gamma\Big[d\ e^{1/\rho}\ |b_{n+1}|^{-1/n}\Big] \ \ge\ \gamma\Big[d\ e^{1/\rho}\ (E_n^2)^{-1/n}\Big]$$

$$\Longrightarrow\ \beta\Big\{\Big(\gamma\Big[d\ e^{1/\rho}\ |b_{n+1}|^{-1/n}\Big]\Big)^{\rho}\Big\} \ \ge\ \beta\Big\{\Big(\gamma\Big[d\ e^{1/\rho}\ (E_n^2)^{-1/n}\Big]\Big)^{\rho}\Big\}.$$

Hence

$$\frac{\alpha\left(\frac{n+1}{\rho}\right)}{\beta\Big\{\Big(\gamma\Big[d\ e^{1/\rho}\ |b_{n+1}|^{-1/n}\Big]\Big)^{\rho}\Big\}} \ \le\ \frac{\alpha\left(\frac{n+1}{\rho}\right)}{\beta\Big\{\Big(\gamma\Big[d\ e^{1/\rho}\ (E_n^2)^{-1/n}\Big]\Big)^{\rho}\Big\}}.$$

By applying limits, since $\alpha \in L^o$, we obtain

$$\limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|b_n|^{-1/n}\right)\right]^{\rho}\right\}} \ \le\ \limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}(E_n^2)^{-1/n}\right)\right]^{\rho}\right\}}. \tag{3.8}$$

Conversely, let

$$\limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|b_n|^{-1/n}\right)\right]^{\rho}\right\}} \ =\ \tau.$$

Suppose $\tau < \infty$. Then for every $\varepsilon > 0$, there exists $V(\varepsilon)$, such that for all $n \geq V(\varepsilon)$, we have

$$\gamma^{-1}\left[\left\{\beta^{-1}\left[\frac{1}{\tau+\varepsilon}\,\alpha\left(\frac{n}{\rho}\right)\right]\right\}^{1/\rho}\right] \leq d\, e^{1/\rho}\, |b_n|^{-1/n}$$

implying

$$|b_n| \leq d^n\, e^{n/\rho}\left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\tau}\,,\,\rho\right)\right]^{-n}.$$

Therefore

$$\begin{aligned}\left(E_n^2\right)^2 &\leq \sum_{k=n+1}^{\infty} d^{2k}\, e^{2k/\rho}\left[H\left(\frac{k}{\rho}\,;\,\frac{1}{\tau}\,,\,\rho\right)\right]^{-2k} \\ &\leq \left[\frac{d\, e^{1/\rho}}{H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\tau}\,,\,\rho\right)}\right]^{2(n+1)}\left[1-\left(\frac{d\, e^{1/\rho}}{H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\tau}\,,\,\rho\right)}\right)^2\right]^{-1} \\ &\leq O(1)\left[\frac{d\, e^{1/\rho}}{H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\tau}\,,\,\rho\right)}\right]^{2(n+1)}\end{aligned}$$

for $n > 2de^{1/\rho}$. Thus

$$\begin{aligned} d\, e^{1/\rho}\,\left(E_n^2\right)^{-1/(n+1)} &\geq H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\tau}\,,\,\rho\right) \\ &\geq \gamma^{-1}\left\{\left[\beta^{-1}\left(\frac{1}{\tau}\,\alpha\left(\frac{n+1}{\rho}\right)\right)\right]^{1/\rho}\right\}\end{aligned}$$

$$\beta\left[\left\{\gamma\left[d\, e^{1/\rho}\,\left(E_n^2\right)^{-1/(n+1)}\right]\right\}^{1/\rho}\right] \geq \frac{1}{\tau}\alpha\left(\frac{n+1}{\rho}\right)$$

and

$$\tau+\varepsilon \geq \frac{\alpha\left(\frac{n+1}{\rho}\right)}{\beta\left[\left\{\gamma\left[d\, e^{1/\rho}\,\left(E_n^2\right)^{-1/(n+1)}\right]\right\}^{1/\rho}\right]}.$$

Since $\alpha(x)$, $\beta^{-1}(x)$ and $\gamma(x) \in L^o$, proceeding to limits, we have

$$\limsup_{n\to\infty}\frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|b_n|^{-1/n}\right)\right]^{\rho}\right\}} \geq \limsup_{n\to\infty}\frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}(E_n^2)^{-1/n}\right)\right]^{\rho}\right\}}. \tag{3.9}$$

From (3.8) and (3.9), we get the required result (3.7). This completes the proof of Lemma 2. □

Now we prove

Theorem 3.3. *Let* $2 \le p \le \infty$. $\alpha(x) \in \Lambda$, $\beta(x) \in L^o$. *Set* $H(x;c) = \beta^{-1}[c\,\alpha(x)]$, *then f is restriction to the domain D of an entire function of finite generalized order* ρ *if and only if*

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln(E_n^p)\right)} = \rho, \tag{3.10}$$

provided $dH(x;c)/d\ln x = O(1)$ *as* $x \to \infty$ *for all* c, $0 < c < \infty$.

Proof. We prove the theorem in two steps. First we consider the case for $p = 2$. Assume f is of finite generalized order ρ. Then from (2.1), we have

$$|a_n| \le e^{-n\,H(n;\frac{1}{\rho})}.$$

Now by considering the orthonormality of the polynomials $p_n(z)$, we have

$$b_n = \frac{1}{A}\sum_{k=n+1}^{\infty} a_k \int\int_D F_k(z)\,\overline{p_n(z)}dx\,dy.$$

Hence

$$|b_n| \le \sum_{k=n+1}^{\infty} |a_k| \max_{z\in C}|F_k(z)|.$$

Since from (1.3), we have

$$\max_{z\in C}|F_k(z)| \le L\,(1+\varepsilon)^k, \tag{3.11}$$

therefore, we have

$$\begin{aligned} |b_n| &\le L\sum_{k=n+1}^{\infty} e^{-k\,H(k;\frac{1}{\rho})}\,(1+\varepsilon)^k \\ &\le L\,e^{-(n+1)\,H((n+1);\frac{1}{\rho})}\,(1+\varepsilon)^{(n+1)}\left[1-\frac{1+\varepsilon}{e}\right] \\ &\le O(1)\,L\,e^{-(n+1)\,H((n+1);\frac{1}{\rho})} \end{aligned}$$

since $H(x;\frac{1}{\rho})$ is an increasing function $\to\infty$ as $x\to\infty$. Hence

$$\ln\frac{1}{|b_n|} \ge (n+1)\,H((n+1);\frac{1}{\rho})$$

so that

$$-\frac{1}{n+1}\ln|b_n| \ge \beta^{-1}\left[\frac{1}{\rho}\,\alpha(n+1)\right]$$

$$\text{or } \beta\left(-\frac{1}{n+1}\ln|b_n|\right) \ge \frac{1}{\rho}\,\alpha(n+1)$$

and

$$\rho+\varepsilon \;\geq\; \frac{\alpha(n+1)}{\beta\left(-\frac{1}{n+1}\,\ln|b_n|\right)}.$$

Since $\beta \in L^o$, proceeding to limits, we get

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\,\ln|b_n|\right)} \;\leq\; \rho. \tag{3.12}$$

Conversely, let

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)} \;=\; \sigma.$$

Suppose $\sigma < \infty$. Then for each $\varepsilon > 0$, there exists $Z(\varepsilon)$ such that for all $n \geq Z(\varepsilon)$, we have

$$|b_n| \;\leq\; e^{-n\,H\left(n;\frac{1}{\bar{\sigma}}\right)}.$$

By Carleman's result, as $n \to \infty$, we have

$$p_n(z) \;\sim\; \left(\frac{(n+1)A}{\pi}\right)^{1/2} \varphi^{'}(z)\;(\varphi(z))^n$$

uniformly for $z \in E$. Therefore for all $z \in E$, we have

$$|p_n(z)| \;\leq\; L^{'}\;(n+1)^{1/2}\;|\varphi^{'}(z)|\;|\varphi(z)|^n,$$

$$|\varphi^{'}(z)| \;\leq\; T \quad \forall\, z \in E,$$

$$\text{and} \qquad |\varphi(z)| \;\leq\; \frac{|z|}{d-\varepsilon}$$

for all z with sufficiently large modulus. Therefore

$$|f(z)| \;\leq\; L\,\sum_{n=0}^{\infty} e^{-n\,H\left(n;\frac{1}{\bar{\sigma}}\right)}\;(n+1)^{1/2}\;\left(\frac{|z|}{d-\varepsilon}\right)^n.$$

Now consider $n\,\beta^{-1}\left[\frac{1}{\bar{\sigma}}\,\alpha(n)\right] - \frac{1}{2}\ln(n+1) \;=\; g(n)$.
Hence

$$\beta^{-1}\left[\frac{1}{\bar{\sigma}}\,\alpha(n)\right] \;=\; \frac{g(n)}{n}\left[1+\frac{1}{2}\;\frac{\ln(n+1)}{g(n)}\right]$$

$$\text{or } \frac{1}{\bar{\sigma}}\,\alpha(n) \;=\; \beta\left[\frac{1}{n}\,g(n)\left\{1+\frac{1}{2}\;\frac{\ln(n+1)}{g(n)}\right\}\right].$$

Since $\beta \in L^o$, we have

$$\frac{1}{\overline{\sigma}}\,\alpha(n) \simeq \beta\left[\frac{1}{n}\,g(n)\right]$$

$$\text{or} \quad \alpha(n) = (\overline{\sigma}+\varepsilon)\,\beta\left[\frac{1}{n}\,g(n)\right] + o(\beta(n))$$

implying

$$g(n) = n\beta^{-1}\left[\frac{1}{\sigma+2\varepsilon}\,\alpha(n)\right].$$

Therefore

$$|f(z)| \le L \sum_{n=0}^{\infty} e^{-n\,H\left(n;\frac{1}{\sigma}\right)} \left(\frac{|z|}{d-\varepsilon}\right)^n.$$

Consider the function $\chi(x) = \left(\frac{R}{d-\varepsilon}\right)^x exp\left[-x\,H\left(x;\frac{1}{\sigma+2\varepsilon}\right)\right]$. Take its logarithmic derivative and set it equal to zero. Then we have

$$\frac{\chi'(x)}{\chi(x)} = \ln\left(\frac{R}{d-\varepsilon}\right) - H\left(x;\frac{1}{\sigma+2\varepsilon}\right) - \frac{d\,H\left(x;\frac{1}{\sigma+2\varepsilon}\right)}{d\,\ln x} = 0.$$

By assumption of the theorem there exists $K' > 0$ such that for $x \ge x_1$

$$\left|\frac{d\,H\left(x;\frac{1}{\sigma+2\varepsilon}\right)}{d\,\ln x}\right| \le K'.$$

Let $K_1(R) = E\left[\alpha^{-1}\left\{(\sigma+2\varepsilon)\,\beta(\ln R+K')\right\}\right]+1$ and $k_0 = \max\left(K'(\varepsilon), E[x_1]+1\right)$. For $R > R_1(k_0)$, $\psi'(k_0)/\psi(k_0) > 0$, and $\psi'(K_1(R))/\psi(K_1(R)) < 0$.

Let $x^*(R)$ be the point where the function ψ attains its maximum such that $\psi(x^*(R)) = \max_{k_0 \le x \le K_1(R)} \psi(x)$, then $k_0 < x^*(R) < K_1(R)$ and $x^*(R) = \alpha^{-1}((\sigma+2\varepsilon)\,\beta(\ln R - a(R)))$, where

$$-K' < a(R) = \frac{d\,H\left(x;\frac{1}{\sigma+2\varepsilon}\right)}{d\,\ln x}\Big|_{x=x^*(R)} < K'.$$

Further

$$\max_{k_0 \le k \le K_1(R)} (|b_k|\,R^k) \le \max_{k_0 \le x \le K_1(R)} \psi(x) = \frac{R^{\alpha^{-1}\{(\sigma+2\varepsilon)\,\beta(\ln R-\ln(d-\varepsilon)-a(R))\}}}{e^{\alpha^{-1}\{(\sigma+2\varepsilon)\,\beta(\ln R-\ln(d-\varepsilon)-a(R))\}\,[\ln R-\ln(d-\varepsilon)-a(R)]}}$$

$$= \exp\left\{K'\,\alpha^{-1}\left[(\sigma+2\varepsilon)\,\beta(\ln R-\ln(d-\varepsilon)-a(R))\right]\right\}$$

$$\le \exp\left\{K'\,\alpha^{-1}\left[(\sigma+2\varepsilon)\,\beta(Y)\right]\right\}$$

where $\beta(Y) \ = \ \beta(\ln R - \ln(d-\varepsilon) + K^{'})$. Therefore for $R > R_1(k_0)$, we have

$$M(R;f) \ \leq \ \sum_{k=0}^{\infty} |b_k| R^k \ = \ \sum_{k=0}^{k_0} |b_k| R^k \ + \sum_{k=k_0+1}^{k_1(R)} |b_k| R^k \ + \sum_{k=k_1(R)+1}^{\infty} |b_k| R^k$$

$$\leq \ O(R^{k_0}) + 1 + k_1(R) \max_{k_0 \leq k \leq K_1(R)} (|b_k| \, R^k).$$

Hence

$$\begin{aligned} M(R;f)\,(1+o(1)) \quad &\leq \quad \left(\alpha^{-1}\left[(\sigma+2\varepsilon)\,\beta(Y)\right]+1\right)\ \exp\left\{K^{'}\,\alpha^{-1}\left[(\sigma+2\varepsilon)\,\beta(Y)\right]\right\} \\ &\leq \quad \exp\left\{(K^{'}+o(1))\,\alpha^{-1}\left[(\sigma+2\varepsilon)\,\beta(Y)\right]\right\}. \end{aligned}$$

Then, we have

$$\frac{\alpha\left[(K^{'}+o(1))^{-1}\ \ln M(R;f)\right]}{\beta(Y)} \ \leq \ \sigma + 2\varepsilon.$$

Since $\alpha(x) \in \Lambda$ and $\beta(x) \in L^o$, proceeding to limits as $R \to \infty$, we obtain

$$\rho \ = \ \limsup_{R\to\infty} \frac{\alpha(\ln M(R;f))}{\beta(\ln R)} \ \leq \ \sigma. \tag{3.13}$$

Combining (3.12) and (3.13),we obtain

$$\limsup_{n\to\infty} \ \frac{\alpha(n)}{\beta\left(-\frac{1}{n}\ln|b_n|\right)} \ = \ \rho.$$

The result now follows on using Lemma 1 for the case of $p = 2$.

Now we consider the case $p > 2$. Since, we have

$$E_n^2 \ \leq \ E_n^p \ \leq \ E_n^{\infty} \tag{3.14}$$

for $2 \leq p \leq \infty$, it is sufficient to consider the case $p = \infty$. Suppose f is an entire function of generalized type having finite generalized order ρ. Then

$$\begin{aligned} E_n^{\infty} \quad &\leq \quad \max_{z\in C} \left| f(z) - \sum_{k=0}^{n} a_k\, F_k(z) \right| \\ &\leq \quad \sum_{k=n+1}^{\infty} |a_k| \ \max_{z\in C} |F_k(z)|. \end{aligned}$$

Since by Theorem 1, we have

$$|a_n| \ \leq \ e^{-n\,H(n;\frac{1}{\rho})}.$$

and since we have

$$\max_{z\in C}|F_k(z)| \;\leq\; K(1+\varepsilon)^k,$$

therefore the above inequality becomes

$$\begin{aligned} E_n^\infty &\leq K \sum_{k=n+1}^{\infty} e^{-n\,H(n;\frac{1}{\bar\rho})}\,(1+\varepsilon)^k \\ &\leq K\,e^{-(n+1)\,H(n+1;\frac{1}{\bar\rho})}\,(1+\varepsilon)^{(n+1)}\left[1-\frac{(1+\varepsilon)}{e}\right]^{-1} \\ &\leq O(1)\,K\,e^{-(n+1)\,H(n+1;\frac{1}{\bar\rho})}\,(1+\varepsilon)^{(n+1)} \end{aligned}$$

$$\Longrightarrow \ln\frac{1}{E_n^\infty} \;\geq\; (n+1)\,H((n+1);\frac{1}{\bar\rho})$$

$$\text{or}\quad \beta\!\left(-\frac{1}{n+1}\,\ln E_n^\infty\right) \;\geq\; \frac{1}{\bar\rho}\,\alpha(n+1).$$

Since $\alpha\in L^o$, proceeding to limits, we get

$$\limsup_{n\to\infty}\frac{\alpha(n)}{\beta\left(-\frac{1}{n}\,\ln E_n^\infty\right)} \;\leq\; \rho.$$

In view of inequalities (3.19) and the fact that (3.10) holds for $p=2$, this last inequality is an equality. This completes the proof of Theorem 3. □

Next we prove

Theorem 3.4. *Let* $2\leq p\leq\infty$. $\alpha(x)$, $\beta^{-1}(x)$, $\gamma(x)\in L^o$*; and* ρ *be a fixed number,* $0<\rho<\infty$. *Set* $H(x;\sigma,\rho) \;=\; \gamma^{-1}\left\{\left[\beta^{-1}\left(\sigma\,\alpha(x)\right)\right]^{1/\rho}\right\}$*, then f is restriction to the domain D of an entire function of generalized order* ρ *and finite generalized type* τ *if and only if*

$$\limsup_{n\to\infty}\frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}(E_n^p)^{-1/n}\right)\right]^{\rho}\right\}} \;=\; \tau \tag{3.15}$$

provided if $\gamma(x)\in\Lambda$ *and* $\alpha(x)\in\Lambda$, $dH(x;\sigma,\rho)/d\ln x = O(1)$ *as* $x\to\infty$.

Proof. We prove the result in two steps. First we consider the case when $p=2$. Assume f is an entire function of generalized type τ having finite generalized order ρ. From equation (2.6), we have

$$|a_n| \;\leq\; d^n\,e^{n/\rho}\,\left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\bar\tau}\,,\,\rho\right)\right]^{-n},$$

where $\bar\tau=\tau+\varepsilon$. As before we mentioned in the proof of Theorem 3, we obtain

$$|b_n| \;\leq\; \sum_{k=n+1}^{\infty}|a_k|\;\max_{z\,\in\,C}|F_k(z)|.$$

Since from (1.3), we have

$$\max_{z\in C}|F_k(z)| \le L^{'}\,(1+\varepsilon)^k.$$

Therefore, we have

$$\begin{aligned}|b_n| &\le L^{'}\sum_{k=n+1}^{\infty} d^k\, e^{k/\rho}\left[H\left(\frac{k}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]^{-k}(1+\varepsilon)^k\\ &\le L^{'}\,d^{(n+1)}\,e^{\frac{n+1}{\rho}}\left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]^{-(n+1)}(1+\varepsilon)^{(n+1)}\left[1-\frac{(1+\varepsilon)\,e^{1/\rho}}{d\left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]}\right]\\ &\le O(1)\,L^{'}\,d^{(n+1)}\,e^{\frac{n+1}{\rho}}\left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]^{-(n+1)}(1+\varepsilon)^{(n+1)}\end{aligned}$$

$$\Longrightarrow |b_n|^{1/(n+1)} \le \frac{d\,e^{1/\rho}\,(1+\varepsilon)}{\left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]}$$

$$\text{or}\quad \frac{d\,e^{1/\rho}\,(1+\varepsilon)}{|b_n|^{1/(n+1)}} \ge \left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]$$

$$\text{or}\quad \beta\left\{\left[\gamma\left(d\,e^{1/\rho}\,(1+\varepsilon)\,|b_n|^{-1/(n+1)}\right)\right]^{\rho}\right\} \ge \frac{1}{\overline{\tau}}\,\alpha\left(\frac{n+1}{\rho}\right).$$

Since $\alpha(x),\ \beta^{-1}(x),\ \gamma(x)\in L^{o}$, proceeding to limits, since ε is arbitrary, we obtain

$$\tau \ge \limsup_{n\to\infty}\frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(d\,e^{1/\rho}\,|b_n|^{-1/n}\right)\right]^{\rho}\right\}}. \tag{3.16}$$

Conversely, let

$$\limsup_{n\to\infty}\frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(d\,e^{1/\rho}\,|b_n|^{-1/n}\right)\right]^{\rho}\right\}} = \sigma.$$

Suppose $\sigma<\infty$. Then for each $\varepsilon>0$, there exists $L(\varepsilon)$ such that for all $n \ge L(\varepsilon)$, we have

$$|b_n| \le d^n\,e^{n/\rho}\left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\overline{\sigma}}\,,\,\rho\right)\right]^{-n},$$

where $\overline{\sigma}=\sigma+\varepsilon$. As in the proof of Theorem 1, we obtain

$$|f(z)| \le L\sum_{n=0}^{\infty} d^n\,e^{n/\rho}\left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\overline{\sigma}}\,,\,\rho\right)\right]^{-n}(n+1)^{1/2}\left(\frac{|z|}{d-\varepsilon}\right)^n.$$

Consider

$$\frac{(n+1)^{\rho/2n}}{\left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\overline{\overline{\sigma}}}\,,\,\rho\right)\right]^{\rho}} = g(n/\rho)$$

Then

$$\left(\frac{g(n/\rho)}{e}\right)^{-1/\rho} = \frac{\left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\overline{\overline{\sigma}}}\,,\,\rho\right)\right]}{(n+1)^{1/2n}}$$

and

$$n^{1/2n}\,(1+1/n)^{1/2n}\,\left(\frac{g(n/\rho)}{e}\right)^{-1/\rho} = \gamma^{-1}\left\{\left[\beta^{-1}\left(\frac{1}{\overline{\overline{\sigma}}}\,\alpha(\frac{n}{\rho})\right)\right]^{1/\rho}\right\}.$$

As $n\to\infty$, we have

$$\gamma\left[(1+o(1))\,\left(\frac{g(n/\rho)}{e}\right)^{-1/\rho}\right] = \left[\beta^{-1}\left(\frac{1}{\overline{\overline{\sigma}}}\,\alpha(\frac{n}{\rho})\right)\right]^{1/\rho}.$$

Since $\gamma(x)\in L^{o}$, by using the property of L^{o} class, we have

$$\simeq\,\gamma\left[\left(\frac{g(n/\rho)}{e}\right)^{-1/\rho}\right] = \left[\beta^{-1}\left(\frac{1}{\overline{\overline{\sigma}}}\,\alpha(\frac{n}{\rho})\right)\right]^{1/\rho}$$

implying

$$\alpha(\frac{n}{\rho}) = (\overline{\sigma}+\varepsilon)\,\left\{\left(\beta\left[\gamma\left(\frac{g(n/\rho)}{e}\right)^{-1/\rho}\right]^{\rho}\right)\right\}+o(\gamma(\upsilon))$$

$$g(n/\rho) = \left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\overline{\overline{\sigma}}+\varepsilon}\,,\,\rho\right)\right]^{-\rho}.$$

Therefore, using above approximation of $g(n/\rho)$, we get

$$|f(z)| \leq L\,\sum_{n=0}^{\infty}e^{n/\rho}\,\left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\overline{\overline{\sigma}}+\varepsilon}\,,\,\rho\right)\right]^{-n}\,\left(\frac{d\,|z|}{d-\varepsilon}\right)^{n}.$$

Consider the function $\zeta(x) = (Re^{1/\rho})^{x}\,\left[H\left(\frac{x}{\rho}\,;\,\frac{1}{\overline{\overline{\sigma}}+\varepsilon}\,,\,\rho\right)\right]^{-x}$. Set its logarithmic derivative equal to zero. Then

$$\frac{\zeta'(x)}{\zeta(x)} = \ln R+\frac{1}{\rho}-\ln\left(H\left(\frac{x}{\rho}\,;\,\frac{1}{\overline{\overline{\sigma}}+\varepsilon}\,,\,\rho\right)\right)-\frac{d\,\ln\left(H\left(\frac{x}{\rho}\,;\,\frac{1}{\overline{\overline{\sigma}}+\varepsilon}\,,\,\rho\right)\right)}{d\,\ln x} = 0.$$

If $\alpha(x),\gamma(x)\in\Lambda$, then by hypothesis of theorem, there exists $A>0$, such that for $x>x_1$, we have

$$\left|\frac{d\ \ln\left(H\left(\frac{x}{\rho}\,;\,\frac{1}{\overline{\sigma}+\varepsilon}\,,\,\rho\right)\right)}{d\ \ln x}\right| < A.$$

By replacing $\overline{\sigma}$ by $\overline{\sigma}+\varepsilon$, the rest of the minimization process follows from the proof of converse part of Seremeta [4, Th .2, Page 296]. Then we get,

$$M(R;f)\ \le\ \exp\left\{(A\rho+o(1))\ \alpha^{-1}\ \left\{(\overline{\sigma}+\varepsilon)\,\beta\left[\left(\gamma\left(R\,e^{\frac{1}{\rho}+A}\right)\right)^{\rho}\right]\right\}\right\}.$$

Since $\alpha(x),\ \gamma(x)\in L^{o}$, proceeding to limits, we obtain

$$\tau\ =\ \limsup_{R\to\infty}\frac{\alpha\,(\ln M(R;f))}{\beta[(\gamma(R))^{\rho}]}\ \le\ \sigma. \tag{3.17}$$

Combining (3.16) and (3.17), we obtain

$$\limsup_{n\to\infty}\frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(de^{1/\rho}|b_n|^{-1/n}\right)\right]^{\rho}\right\}}\ =\ \tau. \tag{3.18}$$

The result now follows on using Lemma 2 for the case of $p=2$.

Now we consider the other case $p>2$. Since, we have

$$E_n^2\ \le\ E_n^p\ \le\ E_n^{\infty} \tag{3.19}$$

for $2\le p\le\infty$, it is sufficient to consider the case $p=\infty$. Suppose f is an entire function of generalized type having finite generalized order ρ. Then

$$\begin{aligned} E_n^{\infty} &\le \max_{z\in C}\left|f(z)-\sum_{k=0}^{n}a_k\,F_k(z)\right| \\ &\le \sum_{k=n+1}^{\infty}|a_k|\ \max_{z\in C}|F_k(z)|. \end{aligned}$$

Since by Theorem 2, we have

$$|a_n|\ \le\ K\,d^n\,e^{n/\rho}\ \left[H\left(\frac{n}{\rho}\,;\,\frac{1}{\overline{\tau}}\,,\,\rho\right)\right]^{-n},$$

and since we have

$$\max_{z\in C}|F_k(z)|\ \le\ K(1+\varepsilon)^k,$$

therefore the above inequality becomes

$$\begin{aligned}
E_n^\infty &\le K \sum_{k=n+1}^{\infty} d^k e^{k/\rho} \left[H\left(\frac{k}{\rho}\,;\,\frac{1}{\tau}\,,\rho\right)\right]^{-k} (1+\varepsilon)^k \\
&\le K\, d^{n+1} e^{(n+1)/\rho} \left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\tau}\,,\rho\right)\right]^{-(n+1)} (1+\varepsilon)^{(n+1)} \left[1-\frac{d\,(1+\varepsilon)\,e^{1/\rho}}{\left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\tau}\,,\rho\right)\right]}\right]^{-1} \\
&\le O(1)\, K\, d^{n+1} e^{(n+1)/\rho} \left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\tau}\,,\rho\right)\right]^{-(n+1)} (1+\varepsilon)^{(n+1)}
\end{aligned}$$

$$\Longrightarrow (E_n^\infty)^{1/(n+1)} \le \frac{d\,e^{1/\rho}\,(1+\varepsilon)}{\left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\tau}\,,\rho\right)\right]}$$

$$\frac{d\,e^{1/\rho}\,(1+\varepsilon)}{(E_n^\infty)^{1/(n+1)}} \ge \left[H\left(\frac{n+1}{\rho}\,;\,\frac{1}{\tau}\,,\rho\right)\right]$$

$$\text{or} \quad \beta\left\{\left[\gamma\left(d\,e^{1/\rho}\,(1+\varepsilon)\,(E_n^\infty)^{-1/(n+1)}\right)\right]^{\rho}\right\} \ge \frac{1}{\tau}\,\alpha\left(\frac{n+1}{\rho}\right).$$

Since $\alpha(x),\ \beta^{-1}(x),\ \gamma(x) \in L^o$, proceeding to limits we obtain

$$\tau \ge \limsup_{n\to\infty} \frac{\alpha\left(\frac{n}{\rho}\right)}{\beta\left\{\left[\gamma\left(d\,e^{1/\rho}\,(E_n^\infty)^{-1/n}\right)\right]^{\rho}\right\}}.$$

In view of inequalities (3.19) and the fact that (3.15) holds for $p=2$, this last inequality is an equality. This completes the proof of Theorem 4. □

Acknowledgments

The author thanks the referees for their careful reading of the manuscript and insightful comments.

References

[1] T. Carleman, U̇ber die Approximation analytischer Funnkti onen durch lineare Aggregatevon vorgegebenen Potenzen. *Ark. Mat. Astron. Fys.* **17** (1923), pp 30.

[2] G. Faber, U̇ber Tschebyscheffsche Polynome. *J. Reine Angew. Math.* **150** (1919), pp 79-106.

[3] A. Giroux, Approximation of Entire Functions over Bounded Domains. *J. Approx. Theory* **28** (1980), pp 45-53.

[4] M. N. Seremeta, On the connection between the growth of the maximum modulus of an entire function and the moduli of the coefficients of its power series expansion. *Amer. Math. Soc. Transl. Ser. 2* **88** (1970), pp 291-301.