539· 374

ON THE BUCKLING OF A CENTRALLY LOADED STRAIGHT BAR IN THE VISCO-ELASTIC RANGE

by

I. SALA

University of Oulu

Abstract

As a sequel to a former investigation [1] having reference to some buckling tests with visco-elastic ice bars in this publication a brief theoretic account of the buckling phenomenon in general is given at first, after that there is a report upon some tests with centrally loaded prismatic wooden bars, and at last an attempt is made with the aid of the test results to elucidate the buckling problem in the visco-elastic range.

1. *Introduction*

With the artificial materials, *e.g.* plastics, it has become more and more necessary to have command of the visco-elastic phenomenon. In nature there exist some materials, too, *e.g.* wood and ice, which behave visco-elastically in some conditions.

As an addition to the buckling investigation with ice bars (SALA and OLKKONEN [1]) in this publication at first a brief theoretic presentation of the buckling of a straight bar in several conditions is given, after which there is a report upon some tests performed with several prismatic bars of Finnish pinewood, and at last an attempt, based on the test results, is made to explain the buckling problem in the visco-elastic range.

2. *Buckling of a centrally loaded straight bar when the material is perfectly elastic*

Let us consider a slender prismatic bar with hinged ends loaded centrally with a compressive force F (Fig. 1 a). If the force F is small enough the bar remains straight and undergoes only axial compression. This straight form of elastic equilibrium is stable, *i.e.*, if a lateral force is applied and a small deflection produced, this deflection disappears when the lateral force is removed and the bar becomes straight again. It is assumed that the material is perfectly elastic. By gradually in-

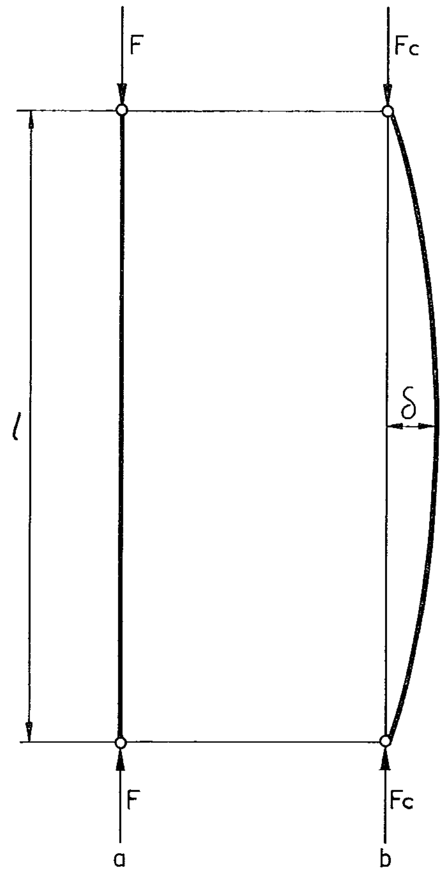

Fig. 1 a. A slender prismatic bar with hinged ends centrally loaded with an axial force F. b. F_c is called the critical load when it is sufficient to keep the bar in a slightly bent form. According to the linear theory the deflection δ remains indeterminate at the critical load, *i.e.*, it can be, for instance, infinitesimal.

creasing the compressive force F a condition is obtained in which the straight form of equilibrium becomes unstable, and a slight lateral force may produce a deflection which does not disappear when the lateral force is removed. This change of the straight form of elastic equilibrium from stable to instable is called buckling, as is well known, and the critical load is then defined as the axial load F_c which is sufficient to keep the bar in a slightly bent form (Fig. 1 b).

Let us further assume that the material follows Hooke's law for all values of the compressive stress σ, or

$$\sigma = E\varepsilon \, , \tag{1}$$

where $\sigma = \frac{F}{A}$ when A is the cross-sectional area of the bar, ε is the compressive strain, $\varepsilon = \frac{\Delta l}{l}$, where l is the length of the bar and Δl the total contraction of the bar in the straight form, and E is the elastic modulus of the material.

In the fundamental case of buckling of a prismatic bar, *i.e.*, in the case of a bar with hinged ends, the well known Euler's formula

$$F_c = \frac{\pi^2 E I}{l^2} \tag{2}$$

gives the critical value of the compressive force. In this formula I is the smallest moment of inertia of the cross-section with respect to the neutral axis, which because of symmetry in this case unites with the central axis of the cross-section. For other end conditions one must use a modified length $\frac{l}{\sqrt{\mu}}$, also called the free buckling length, instead of the length l of the bar, where μ is the so-called coefficient of restraint.

Dividing formula (2) by the cross-sectional area A of the bar and letting $i = \sqrt{\frac{I}{A}}$ be the smallest radius of gyration, the critical value of the compressive stress will be

$$\sigma_c = \frac{\pi^2 E}{\lambda^2} \tag{3}$$

where $\lambda = \frac{l}{i}$ is called the slenderness ratio.

3. *Buckling of a bar made of an actual structural material*

It was assumed in the previous discussion that the material follows Hooke's law without limits. By the structural materials the proportionality between stress and strain holds only up to a certain limiting value of the stress, called the proportional limit (σ_p in Fig. 2), which depends upon the properties of the material. Euler's buckling formula (3) is then applicable only when $\sigma_c < \sigma_p$, *i.e.*, when $\lambda > \lambda_p$, where

$$\lambda_p = \pi \sqrt{\frac{E}{\sigma_p}} \tag{4}$$

is called the limit slenderness.

Let us assume now that the structural material is perfectly elastic beyond the proportional limit, too. If a bar of such material is compressed up to a certain stress value beyond the proportional limit (σ_1 in Fig. 2), and then a small change of load is produced, the relation between the change in stress and the change in strain is given by the slope of the compression test curve at the point corresponding to the stress σ_1. The magnitude of the derivative $\dfrac{d\sigma}{d\varepsilon}$ can then be considered as a

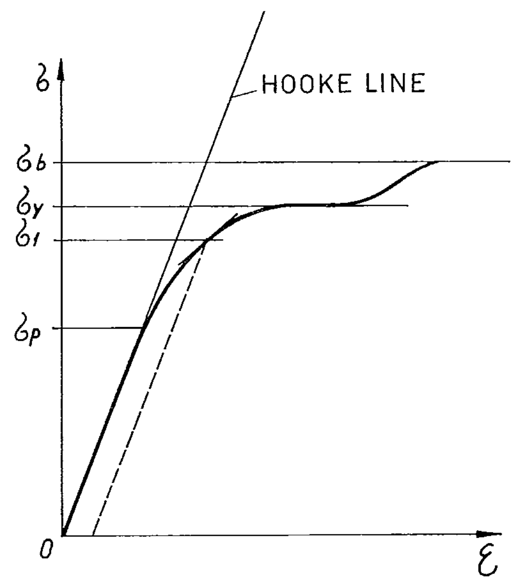

Fig. 2. The compressive stress-strain diagram schematically presented.

variable elastic modulus of the material which is a function of σ and called the tangent modulus E_t.

When the buckling of a prismatic bar takes place beyond the proportional limit, *i.e.*, when $\lambda < \lambda_p$, the critical stress can thus be calculated by the aid of the Engesser's formula

$$\sigma_c = \frac{\pi^2 E_t}{\lambda^2} . \tag{5}$$

In fact the problem is more complicated since the deformation is not a reversible process. By the diminishing load the stress-strain relation will follow a linear law as indicated by the dashed line in Fig. 2. In the case of the buckling of a centrally loaded prismatic bar dealt with in this paper the influence of the irreversibility upon the critical load, however, is negligible, as is well known. In Fig. 3 the diagram of the critical stress is schematically presented both within the proportional limit and beyond it (the thick continual curve).

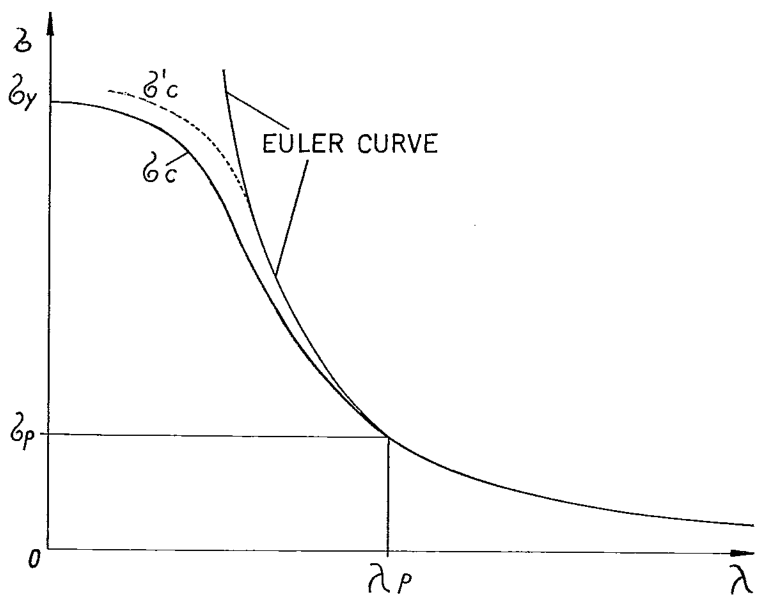

Fig. 3. The critical compressive stress of a centrally loaded prismatic bar as a function of slenderness ratio. The dotted curve after a visco-elastic creep period, and the thick continual curve without it.

Above it is assumed that the bar is loaded exactly centrally. In the loading of columns unavoidable, eventhough very small, eccentricities are, however, always present.

When there is an eccentricity in the compression there exists a definite deflection for any compressive load however small it may be. There is then no more a problem of stability in question, but of an ultimate bending strength. In this eccentric compression of a straight bar the critical load is defined as the load which makes the maximum compressive stress on the concave side of the deflected bar as great as the yielding stress (σ_y in Fig. 2), *i.e.*, the stress corresponding to the point on the compression test diagram at which the tangent of the curve is parallel to the ε-axis. If the eccentricity, however, is small enough its effect on the critical load can be disregarded, as is well known. In the laboratory tests this is possible to realize.

4. *Buckling in connection with a visco-elastic creep*

It is assumed above that the deformation does not depend on time. When loading a bar made of a visco-elastic material the effect of time upon the compressive strain can not be disregarded.

The visco-elastic strain can be supposed to be produced very slowly. Let us assume now that a prismatic bar of visco-elastic material has been compressed centrally with a constant load at a known time interval and then a small increase of the load is suddenly produced, the relation between the change in stress and the change in strain is again given by the slope of the compressive stress-strain diagram which would have begun if the increase of the load had continued. The magnitude of the derivative $\frac{d\sigma}{d\varepsilon}$ can still be considered as a variable elastic modulus, *i.e.*, as the tangent modulus of the material in the range of deformation in question, and according to Eq. (5) it determines the critical stress.

5. *The tests with visco-elastic wooden bars*

For the present publication the author had the following tests to use: First the tests made by Mr. Eero Olkkonen, Mech.Eng. in the Metals Laboratory of the Technical Research Centre of Finland in Otaniemi, and secondly the work required for a diploma by Mr. Jouko Kangas, Mech.Eng. in the Department of Mechanical Engineering of the University of Oulu. The author's thanks are due to both investigators for their assistance.

The test bars in Otaniemi were made of Finnish pinewood whose specific weight was 0.56, the percentage of moisture ca. 8, and the summerwood content ca. 30%. The Otaniemi tests, at the author's disposal, were in fact not purposed to investigate the buckling of a bar of Finnish pinewood in the visco-elastic range. In spite of that the author, however, could make use of them.

The compressive stress-strain diagram is obtained by allowing σ to increase at a set constant rate, as is well known, (in Fig. 4 marked with $t = 0$). In the first group of the tests both in Otaniemi and in Oulu a procedure which aimed at such a compressive stress-strain diagram was interrupted at σ_0 and the stress was held unchanged for a certain period of time after which the growth of σ occurred again at the original speed. Then it was observed that the slope of the newly beginning compressive stress-strain diagram was greater than the slope of the curve corresponding to the time value $t = 0$ at the point whose ordinate equalled the σ_0-value in question, and in the tests with smaller σ_0-values it was roughly the same as the elastic modulus E of the material in question, as it is schematically seen in Fig. 4. The corresponding test

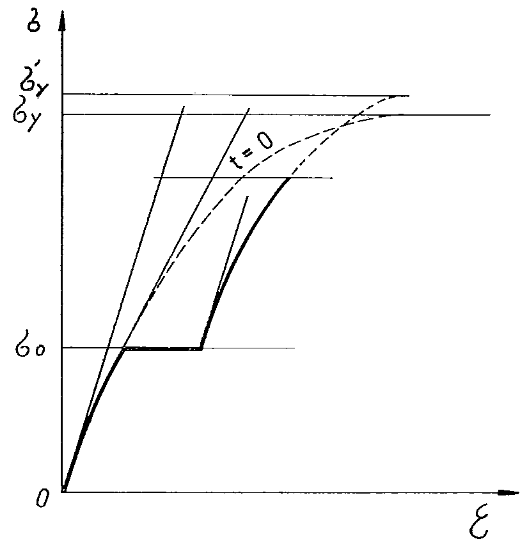

Fig. 4. A schematic compressive stress-strain diagram of a visco-elastic prismatic bar with an interruption in the growth of the compression stress for a certain time period.

was performed by decreasing σ after the visco-elastic creep, and as a result it was obtained that the slope was then not smaller than that given by the increasing stress at the same ordinate value.

In the former group of the Otaniemi tests the slenderness ratio of the bars was $\lambda = 17$ (Table 1), the constant compressive stress σ_0 varied from 450 to 600 kp/cm², and the time interval of the visco-elastic creep ranged from 1 to 5 days. The critical compressive stress σ_c after the visco-elastic creep period varied from 695 to 722 kp/cm² when it in the tests without any creep period was on an average $\sigma_y = 624$ kp/cm². It may be pointed out that the yielding stress σ_y and the ultimate compressive stress σ_b are the same by wood, and that with so low a slenderness ratio as $\lambda = 17$ the critical stress is about ca. 98 per cent of the yielding stress, *i.e.*, σ_c is nearly the same as σ_y.

In the latter group of the Otaniemi tests the visco-elastic creep began with $\sigma = 0$ and continued with a low constant rate of the increasing stress up to a certain σ-value, after which the increase of the stress continued with a higher rate. In one of these tests in which the rate

Table 1. Buckling test results with pinewood bars. λ is the slenderness ratio of the bar. σ_c is the critical compressive stress on an average of the values obtained in several tests with bars of the same slenderness ratio, paranthesized the largest of the participants in the average value. σ_0 is the compressive stress under which the visco-elastic creep took place. t is the lasting time of the creep period. σ_c' is the critical compressive stress obtained with each bar after the creep period.

Test place	λ	σ_c kp/cm²	σ_0 kp/cm²	t / d	σ_c' kp/cm²
Otaniemi	17	624 (650)	450	3.7	695
			500	4	700
			550	3	720
			550	4.8	713
			600	3.1	722
Oulu	40	523 (558)	400	4	597
			450	4	641
			440	14	620
Oulu	60	248 (273)	200	3	358
			200	3	311
			150	4	355
			200	4	301

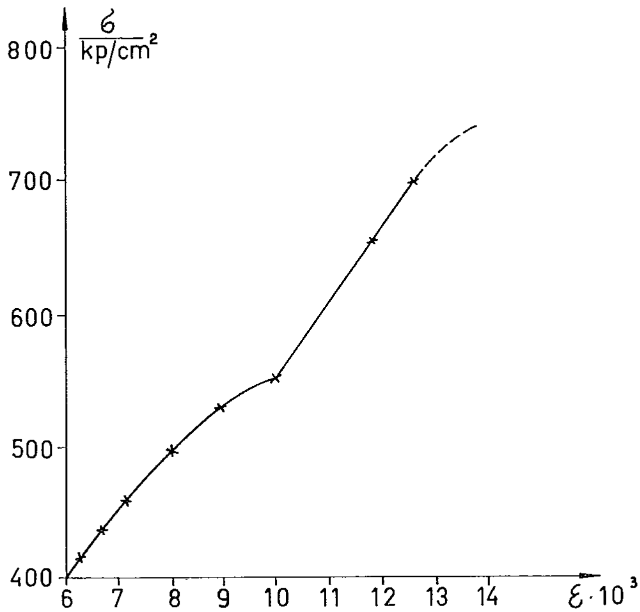

Fig. 5. The upper part of the compressive stress-strain curve of a centrally loaded prismatic wooden bar when the stress increased at first with a rate of 42 kp/cm² in an hour and from the value of 550 kp/cm² on with a rate of 0.25 kp/cm² in a second.

of the increasing stress was 42 kp/cm² in an hour the slope of the beginning $\sigma(\varepsilon)$ curve pertaining to the faster phase of the increasing stress after the slow period at $\sigma = 550$ kp/cm², was as great as ca. $5.6 \cdot 10^4$ kp/cm² (Fig. 5) when the slope at the point $\sigma = 0$ was ca. $9 \cdot 10^4$ kp/cm², and, on the other hand, the tangent of the actual compressive stress-strain diagram at $\sigma = 550$ kp/cm² would be much more gently sloping, if not nearly parallel to the ε-axis.

The bars in the tests performed by Mr. Kangas in Oulu were made of Finnish pinewood, which was mainly similar to the wood used in the Otaniemi tests treated above. A part of the test results are presented in Table 1.

6. *Some conclusions on the buckling of a prismatic wooden bar, or an ice bar, respectively, in the visco-elastic range*

The range of the validity of the above theory: Above the buckling load, or the critical load of a centrally loaded prismatic bar was defined as the axial load, which is sufficient to keep the bar in a slightly bent form. In consequence of this definition the critical stress of a bar with hinged ends can be calculated from the Euler—Engesser formula

$$\sigma_c = \frac{\pi^2 E_t}{\lambda^2}\,, \quad E_t = \frac{d\sigma}{d\varepsilon}\,.$$

Under all the test circumstances treated above the thus defined tangent modulus E_t, when it was determined after a visco-elastic creep period of a wooden bar, appeared to be greater than that determined before the visco-elastic deformation at the same stress value. Thence the critical load of a centrally loaded prismatic bar of Finnish pinewood seems to increase with the visco-elastic compression, at least when $\lambda < \lambda_p$, as it is qualitatively shown in Fig. 3 (dotted curve), and as it appears quantitatively from the test results in Table 1.

Mr. Kangas presents in his work that the tangent modulus E_t of pinewood after a visco-elastic creep period may be distinctly greater than the elastic modulus E of the same wood when σ_0 is small enough, which seems to confirm the supposition that the visco-elastic deformation would cause an increase of the critical compressive stress in the Euler regime, too. The construction of the test equipments in the Oulu tests was presumably such that the test results concerning the elastic modulus and the tangent modulus of pinewood are dependable to a less degree than those having reference to the values of the critical stress, wherefore the author abstains from taking an attitude in the matter in question, in particular when the statement lacks the experimental verification by the aid of the buckling tests in the Euler range.

Some restraints in regard to time and stress on the use of the above theory: According to an earlier investigation mentioned in point 1. the above may be applied to the ice bars on the conditions presented in the said publication, too. Besides the restrictions to time and temperature on ice there is perhaps a more general one to the magnitude of the constant compressive stress which causes the visco-elastic creep, at any rate on wood, as it may be apparent from the following.

When the source of the visco-elastic creep, the constant compressive stress σ_0, is very close to the yielding stress σ_y the rate of the creep does not decrease with time towards zero, but it approaches to a certain constant. The consequence of this is that, because of a too great compressive strain, the bar breaks before long. In one of the tests performed

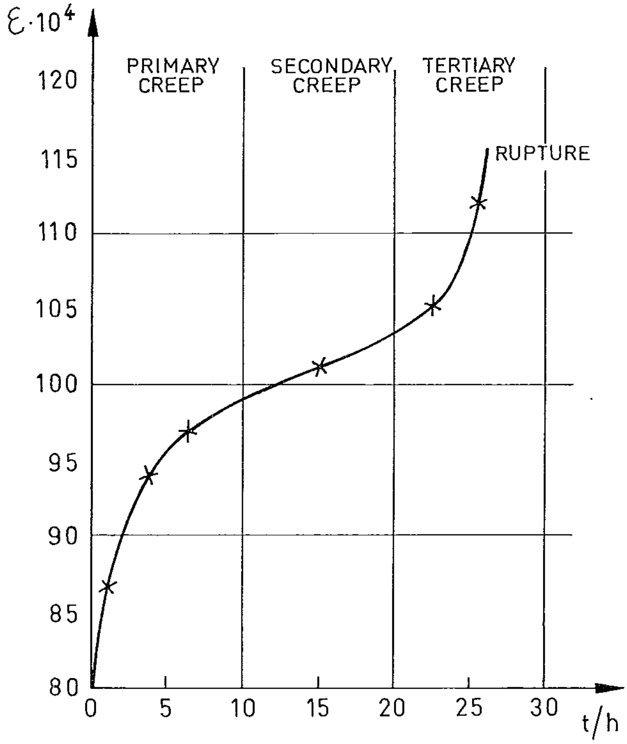

Fig. 6. A complete visco-elastic creep curve caused by a constant compressive stress of 600 kp/cm² on a centrally loaded prismatic wooden bar, when the predictive value for the yielding stress was ca. 620 kp/cm².

in this way in Otaniemi the bar was compressed with a stress of $\sigma_0 = 600$ kp/cm², when the predictive value for the yielding stress was at an average of ca. 620 kp/cm². The visco-elastic creep attained a rate of ca. $5.9 \cdot 10^{-5}$ in an hour, and the creep lasted ca. 1.15 days before the rupture of the bar occurred. A short time prior to the rupture a noticeable growth in the rate of the increase of the creep was realized (the tertiary creep phase in Fig. 6). When the constant compressive stress σ_0 was not so close to the yielding stress the slope of the $\varepsilon(t)$ curve approached with time towards zero, as it is seen in Fig. 7. To these latter test bars the theory presented above is applicable with full right, because of the fact that their creep process lacks the tertiary phase and the latter part of the secondary phase. For ice there obviously exists both a

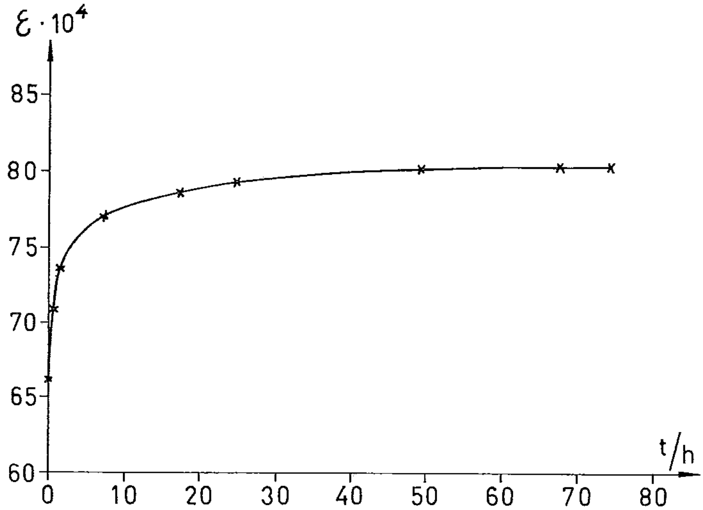

Fig. 7. A visco-elastic creep curve which lacks the tertiary phase and the latter part of the secondary creep phase. A prismatic wooden bar is loaded centrally with a constant compressive stress of 450 kp/cm². The predictive value for the yielding stress is the same as in Fig. 6.

secondary and a tertiary phase in its creep curve, and thus the restriction of the lasting time of the creep process in the ice tests is inevitable when one wants to apply to ice the theory of buckling in the visco-elastic range presented in this paper.

Many additional investigations are, however, requisite for the complete clearing up of the course of the buckling process which appears in connection with a visco-elastic creep.

Acknowledgements: The author's thanks are due to Mrs. Impi Haulio M.A. for the linguistic form of this publication, as well as to Mr. A. A. Ranta, Mech.E. for the graphs.

REFERENCES

1. Sala, I. and E. Olkkonen, 1971: On the visco-elastic properties of ice and wood. *Geophysica,* **11**, 249–260.