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ABOUT BOTTOM FRICTION CAUSED BY AN ALTERNATING WIND

by

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Abstract

A theoretical approach has been made to determine the coefficient in a bottom friction formula often used in numerical works concerning the one layer model. A uniform viscosity and a sinusoidally varying wind shear are assumed. The friction law adopted is shown to be rather good for shallow waters but poor for deep ones.

1. *Introduction*

In numerical works concerning the motion of water in sea areas, friction against the bottom plays an important role. Because no exact friction law is known, a number of experimental laws are used. Many investigations have been performed using a one-layered sea, *i.e.* the currents are thought to move with the mean velocity between the bottom and the surface. Hansen [2] and others assumed that friction were proportional to the square of the mean velocity, but otherwise, dependence on the depth was not considered. They obtained a reasonable good agreement with observation. A direct comparison of four different friction laws has been made by Svansson [3] by using the method of least squares. No distinct preference between the different laws can be made, but to obtain the best results, different coefficients for areas of different depths must be chosen. One of the laws investigated is of the form

$$\tau_b = r_0 U \tag{1}$$

with

$$r_0 = \frac{k}{h^2}, \tag{2}$$

where $\varrho\tau_b$ is the bottom friction, U the volume transport, h the depth and k a constant. This law, which is practical in numerical solution of vertically integrated hydrodynamical equations, has been used in a few works with $k = \pi^2\nu/4$ or $k = 2\nu$. (See [1], [5].) The former value is obtained assuming a uniform viscosity and a current profile of cosine form and letting the current slow down by the frictional forces only. At the latter value it is arrived simply by assuming a uniform viscosity together with a linear current profile. Our aim is to show that the friction law (1), (2) with $k = 2\nu$ can also be used to determine the bottom friction of a current, which is called for by an alternating wind.

2. *Model*

A sea with an infinite horizontal upper surface, a constant depth h and a uniform turbulent eddy viscosity $\mu = \varrho\nu$ is assumed. In a one-dimensional case the complex velocity w (the true velocity being the real part of w) is governed by the equation

$$\frac{\partial w}{\partial t} = \nu \frac{\partial^2 w}{\partial z^2}, \tag{3}$$

where t is the time and z the vertical coordinate. No attention has been paid on the geostrophic acceleration, on active pressure forces or on extraneous forces. The corresponding terms are thus subdued. As boundary conditions, it is assumed that the velocity at the bottom is zero,

$$w_{-h} = 0, \tag{4}$$

and that the complex wind stress at the free surface is

$$\varrho\tau_s = \varrho\tau_0 e^{i\sigma t} \tag{5}$$

with a constant maximum wind stress $\varrho\tau_0$ and a constant wind speed frequency $\sigma/2\pi$.

3. *Solution*

By usual methods a solution of (3) is found to be

$$w = Ae^{-ct} \sin [(z - z_0) \sqrt{c/\nu}] \tag{6}$$

with arbitrary constants A, c, z_0. Boundary condition (4) yields

$$z_0 = -h , \tag{7}$$

h being the depth. Boundary condition (5) becomes according to the Newton's friction law

$$\nu \left(\frac{\partial w}{\partial z}\right)_0 = \tau_0 e^{i\sigma t} . \tag{8}$$

From this and Eqn. (6), c and A are obtained and introduced together with z_0 from Eqn. (7) back into Eqn. (6).

$$w = \frac{\tau_0}{\sqrt{2\sigma\nu}} \frac{(1+i) \sin [(1-i)(z+h) \sqrt{\sigma/(2\nu)}]}{\cos [(1-i)h \sqrt{\sigma/(2\nu)}]} e^{i\sigma t} . \tag{9}$$

4. *Determination of the coefficient for the friction law*

Friction $\varrho\tau_b$ against the bottom may be obtained from Eqn. (9) according to the Newton's friction law.

$$\tau_b = \nu \left(\frac{\partial w}{\partial z}\right)_{-h} = \frac{\tau_0}{\cos [(1-i)h \sqrt{\sigma/(2\nu)}]} e^{i\sigma t} . \tag{10}$$

The complex volume transport is

$$W = \int_{-h}^{0} w dz = \frac{\tau_0 i}{\sigma} \left(\frac{1}{\cos [(1-i)h \sqrt{\sigma/(2\nu)}]} - 1\right) e^{i\sigma t} , \tag{11}$$

and r is thus found according to Eqns. (10), (11)

$$r = \frac{\tau_b}{W} = \frac{\sigma}{i\{1 - \cos [(1-i)h \sqrt{\sigma/(2\nu)}]\}} . \tag{12}$$

The limiting value for small h is thus

$$r_0 = \frac{2\nu}{h^2} , \tag{13}$$

which is in accordance with (2), when $k = 2\nu$.

5. *Numerical evaluations*

In Fig. 1 the function

$$u' = \mathscr{R}_e \frac{\sqrt{2\sigma\nu}}{\tau_0} w \tag{14}$$

is plotted for two depths:

$$h' = h\sqrt{\sigma/(2\nu)} = 1.2 \text{ or } 2.4 \tag{15}$$

In Fig. 2 the moduli and arguments of certain functions are plotted as a function of the »depth»

$$h' = h\sqrt{\sigma/(2\nu)} . \tag{16}$$

The functions are

$$w_0' = \frac{\sqrt{2\sigma\nu}}{\tau_0} w , \quad t = 0 , \quad z = 0 , \tag{17}$$

$$w'_{-\frac{h}{2}} = \frac{\sqrt{2\sigma\nu}}{\tau_0} w , \quad t = 0 , \quad z = -\frac{h}{2} , \tag{18}$$

$$W' = \frac{\sigma}{\tau_0} W , \quad t = 0 , \tag{19}$$

$$\tau' = \frac{\tau_b}{\tau_0} , \quad t = 0 . \tag{20}$$

Furthermore, the modulus of a function $\varkappa$ is included. This function is defined by

$$r = \frac{\varkappa\nu}{h^2} = \frac{\tau_b}{W} \tag{21}$$

for all values of the »depth» (16). The argument of $\varkappa$ is given by

$$\arg(\varkappa) = \arg(\tau') - \arg(W') . \tag{22}$$

6. *Discussion*

The curves in Fig. 1 represent consecutive current profiles for two depths covering half of the wind period in each case. The separation between a dot and a cross is a measure for the error committed, when

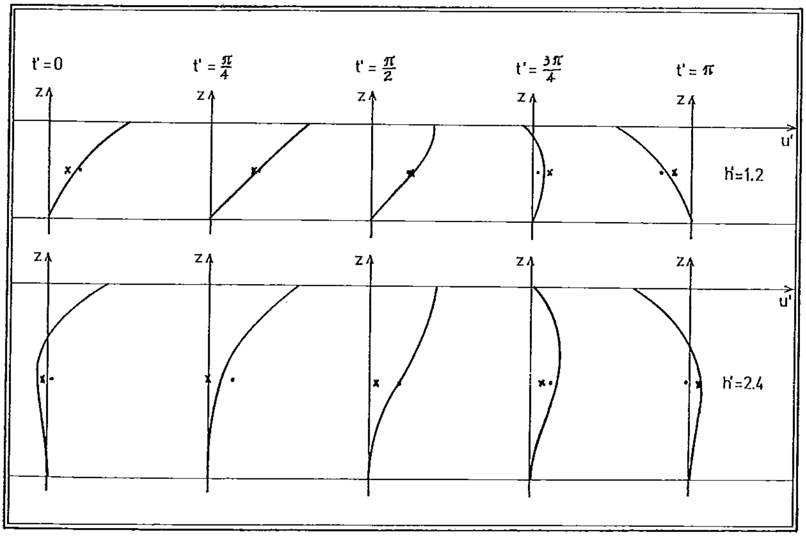

Fig. 1. Consecutive velocity profiles of currents caused by an alternating wind. When the kinematic viscosity of the water is 0.02 m²/s and the frequency of the alternating wind 0.88 d⁻¹, then the depth of the upper figure is 30 m, and the depth of the lower one 60 m.

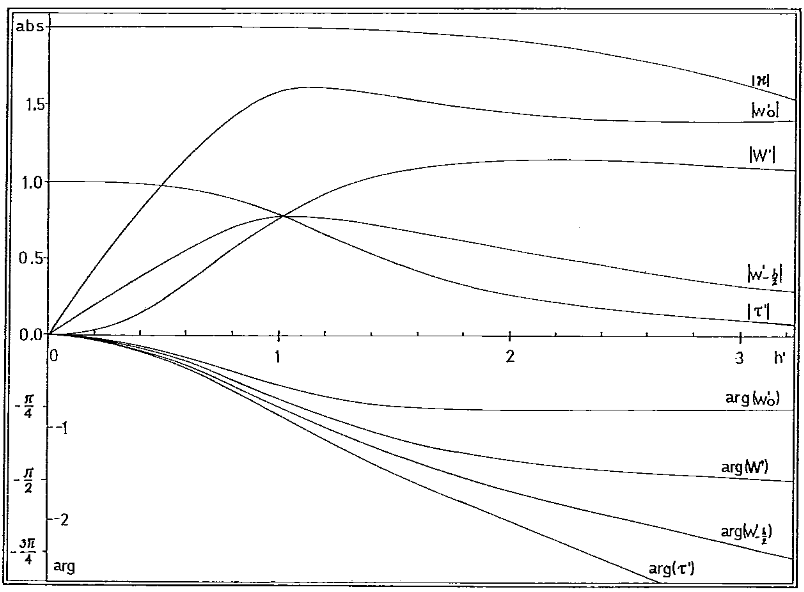

Fig. 2. Moduli and arguments of the functions w_0', $w' - \frac{h}{2}$, W', τ' as functions of h'. In addition, the modulus of $\varkappa$ is included.

the friction law (1) is used. (See [4].) This error grows larger, when the depth increases. (*Cf.* [3].) Sometimes the friction may have a wrong sign, too.

The moduli of the functions in Fig. 2, excluding $\varkappa$, give the amplitudes of quantities in question, whereas the arguments imply the time lags with regard to the »time»

$$t' = \sigma t \, . \tag{23}$$

It can be seen from the figure, or deduced from the corresponding equations that with increasing depth the current speed on the surface and the volume transport approach constant values, different from zero, whereas the current speed below the surface and the bottom friction tend to zero. The argument values in the latter cases grow toward $-\infty$. This means especially that the function $\varkappa$ in Eqn. (14) becomes inproper, because its time lag in reference to the volume transport grows toward infinity with increasing depth causing the friction term τ_b alternatively either to assume a correct or a wrong sign. It is true that the friction is then small and its influence thus minimal. But with small depths, $\varkappa$ behaves properly and gives thus justification to the use of the law investigated with the specific value (13) derived for the case of a simple water motion induced by an alternating wind.

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