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# ON THE STABILITY OF A SELF-GRAVITATING INHOMOGENEOUS FLUID IN THE FORM OF TWO CONFOCAL ELLIPSOIDS CARRYING DEDEKIND-TYPE INTERNAL CURRENTS

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RESUMEN

Mediante la técnica del virial a segundo armónico se analiza, en primera aproximación, la estabilidad de un fluido autogravitante estático, tipo Dedekind, que consiste de dos elipsoides confocales de diferente densidad. Estas figuras, que mantienen su equilibrio en base a corrientes internas de vorticidad diferencial, resultan debido a que algunos de los miembros de una *serie* de esferoides inhomogéneos rotantes son de frecuencia nula, de donde se bifurcan en secuencias de $\varepsilon$ (la densidad relativa del cuerpo) fija. Se encuentra que tales secuencias tienen un régimen de inestabilidad, el cual es tanto más amplio mientras menor sea $\varepsilon$, pero que se estrecha al incrementar $\varepsilon$. Para $\varepsilon$ muy grande la inestabilidad persiste en la porción final de las secuencias, en donde se hallan las figuras cuyo elipsoide interno tiene la excentricidad ecuatorial más prominente.

ABSTRACT

The second order virial equations are employed to analyze, in a first approximation, the stability of a self-gravitating fluid made up of two confocal ellipsoids carrying internal currents of differential vorticity, which allow their equilibrium. These Dedekind-type figures result because some of the members of a *series* of inhomogeneous rotating spheroids have null frequencies, from which they bifurcate in sequences of fixed $\varepsilon$, the body's relative density. We find that such sequences have each an instability regime, which is wide at low $\varepsilon$, and becomes gradually narrower as $\varepsilon$ increases. Instability persists—even for very large $\varepsilon$—at the final portion of the sequences, where the figures whose internal ellipsoid has the most prominent equatorial flattening are located.

*Key Words:* **GRAVITATION — HYDRODYNAMICS — STARS: ROTATION**



## 1. INTRODUCTION

Referring to a past paper (Montalvo, Martínez & Cisneros 1983) on the hydrostatic relative equilibrium of a self-gravitating, perfect, incompressible, fluid made up of two confocal spheroids, endowed with solid-body differential rotation we recall that, provided the internal spheroid (the "nucleus") is of higher density, rotates faster, and is flatter than the external spheroid (the "atmosphere"), such a model satisfies the equilibrium conditions, the solution comprising a wide spectrum, or *series*, of figures; the also revised case of common angular velocity leads to a negative result (Tassoul 1978). In a more recent paper (Cisneros, Martínez, & Montalvo 2000; hereafter Paper I), the study of the stability of the spheroidal series revealed the existence of tri-axial, static, figures carrying internal currents of differential vorticity, a work that relied heavily on a paper by Tassoul & Ostriker (1968; hereafter, T&O). These Dedekind-type inhomogeneous figures arise only in the event that the vorticity of the nucleus is larger than the vorticity of the atmosphere, the general solution

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encompassing both spheroids and ellipsoids, the stability of the latter being our current interest. We hasten to add that the ellipsoidal shape of the model must be taken with some caution (see § 3). Anyway, ellipsoidal inhomogeneous figures are by no means commonly found. It is not redundant to point out here that figures of equilibrium are impossible for inhomogeneous ellipsoidal masses rotating, either with $\omega_n = \omega_a$, or with $\omega_n \neq \omega_a$, where $\omega$ refers to angular velocity about the smallest axis. The first case is the substance of Hamy's theorem (Hamy 1887), which states that no stratification made up of *any number* of confocal ellipsoids, all of them rotating with the same angular velocity, is an equilibrium figure. For the second case we have no reference but, *a priori*, its impossibility looks sound if one considers, as a plausible argument, that the velocity field of the nose of the nucleus would induce unwanted dynamical effects. The static ellipsoids of differential vorticity, on the other hand, originate because, among the members of the quoted series of inhomogeneous rotating spheroids, there are some whose pulsating frequency vanishes, from which they branch off, or bifurcate. Let us mention, by the way—though it will not concern us in the present context—that, under the stated conditions of differential vorticity, these ellipsoids can even rotate (with a common, though quite low, angular velocity) as a solid body, as long as the atmosphere develops a retrograde internal current (Cisneros, Martínez, & Montalvo 1995); an analysis of the stability of these last figures could be the subject of another work. Our current analysis of stability—actually dynamical stability, since frictional effects will be ignored—will be carried out, as in Paper I, by means of the virial method but, in contrast, Chandrasekhar's approach will be used instead of that of T&O, since the later assumes axial symmetry, and so is not adequate for our current purposses. On this account, our procedure necessarily implies that the virial equilibrium conditions hold.

It must be pointed out that in none of our previous works on the equilibrium of inhomogeneous masses the virial method was invoked. In fact, those other works were based on the fulfillment of the hydrostatic conditions, namely, the continuity of pressure at the external surface of the whole body, and the continuity of pressure at the interface nucleus-atmosphere. Yet, the frequencies of the Dedekind ellipsoids—which should correspond to those obtained in the limit as $\varepsilon$ goes to 0—are recovered intact from the current work and, more important, this is also true for the frequencies of the source figures, namely the rotating spheroids of differential angular velocity. In spite of its fluid nature and its somewhat geometrical restriction, the model contains an element of truth, namely that, as in real stars, the cores are of higher density and bear internal motions more important than their envelopes, which invests our model of a certain astrophysical interest.

## 2. THE VIRIAL METHOD

The well-known results on the equilibrium and the stability of the various classical ellipsoidal series (Maclaurin's, Jacobi's, Dedekind's, etc.,) were recovered in the 1960's by means of the virial method—which, as regards to fluids, is not restricted to homogeneous masses—in a series of individual papers by Chandrasekhar & Lebovitz, a work that was later compiled by the first author in a book (hereafter Ch 1969). In the reminder of this work, the equations from that book quoted herein will come from to its Chapter 7, unless explicitly stated otherwise. In essence, the virial method consists in replacing the equation of motion by its moments with respect to the coordinates. Thus to second order—the current approximation—the linearized virial equations are obtained by multiplying the $j$ equation of motion by $x_i$, followed by an integration over the volume instantaneously occupied by the fluid. The resulting equation is then sligthly disturbed by means of a Lagrangian displacement $\vec{\xi}(\vec{x}, t)$ which, in a first approximation, can be given by

$$\xi_i = L_{i;k} x_k, \tag{1}$$

where $L_{i;k}$ denotes nine constants. Let

$$\vec{\xi}(\vec{x}, t) = e^{\lambda t} \vec{\xi}(\vec{x}), \tag{2}$$

be the time dependence of the perturbation, where $\lambda$ is a characteristic value (a kind of frequency). Our task will be to derive two dispersion relations similar, at any rate, to those derived from Eqs. (115) and (124), from which the frequencies of the Riemann S-type ellipsoids can be obtained. These equations—which are already free of terms related to the variation in pressure and, hence, fit well to our incompressible fluid—contain, we remark, the equilibrium conditions when they are stated in the virial form, and are valid for a Riemann fluid. We must, of course, set $\Omega = 0$ in both determinants in order that they should correspond, in the first place, to the Dedekind problem. In this section we give a brief account of the main steps required to obtain these determinants, and postpone till § 3 the treatment of the inhomogeneous mass itself.

### 2.1. *The Homogeneous Ellipsoids*

The stability of a Riemann fluid with respect to oscillations belonging to the second harmonics can be handled by considering the linearized version of the second-order virial equation. The pertinent equation to be considered in this case is Eq. (152), with the various terms in it having the values given in Eqs. (143), (144), and (145) (Chapter 2, Ch 1969). If, in considering these equations, the Lagrangian displacement $\vec{\xi}(\vec{x},t)$ is given by Eq. (2), the virial equation gives

$$\lambda^2 V_{i;j} - 2\lambda Q_{jl} V_{i;l} - 2\lambda\Omega\varepsilon_{il3} V_{l;j} - 2\Omega\varepsilon_{il3}(Q_{lk}V_{j;k} - Q_{jk}V_{l;k}) + \mathbf{Q}^2_{jl}V_{i;l} + \mathbf{Q}^2_{il}V_{j;l} = \Omega^2(V_{i;j} - \delta_{i3}V_{3j}) + \delta W_{ij} + \delta_{ij}\delta\Pi, \quad (3)$$

where the last two terms refer to the variation of the tensor of potential energy and the variation of pressure, respectively; the variation of the tensor of kinetic energy is already implicit in the terms involving the $Q$'s, and the variation of the moment of inertia tensor is proportional to the coefficients $V_{i;j}$'s. The meaning of the $V_{i;j}$'s and of the matrices $\mathbf{Q_{jl}}$ is given in the next section, but we shall be interested in the special case when the velocity field in an initial steady state is a linear function of the coordinates, and is of the form

$$u_i = Q_{ij}x_j. \quad (4)$$

In order to obtain the determinants (115) and (124) following from Eq. (3) we must, in the first place, note that Eqs. (98)–(106) are obtained after replacing the pair $(i,j)$ in Eq. (3)—where summation over repeated indices is understood—by (3,3), (1,1), (2,2), (1,2), (2,1), (1,3), (2,3), (3,1), and (3,2). These nine virial equations can be separated into two non-combining groups, according to their parity with respect to the axis about which the internal motions occur, namely the $x_3$ axis: an equation is odd if one, and only one, of the indices in the $V_{ij}$'s that attend it, is always 3, whereas it is even if there are two indices, or none, with the value 3. Later, when we consider these determinants as applied to our inhomogeneous *static* fluid, we shall set in them $\Omega = 0$.

#### 2.1.1. *The Even Modes*

Now, Eq. (115) is obtained after Eqs. (113) and (114) which, along with the condition

$$L_{k;k} = \frac{\partial \xi_k}{\partial x_k}, \quad (5)$$

which is zero, as required by the solenoidal character of $\vec{\xi}$ [see Eq. (96)], form a homogeneous system of three equations in the variables $V_{11}$, $V_{22}$, and $V_{33}$, where

$$V_{ij} = L_{ik}I_{jk} + L_{jk}I_{ik} \quad (6)$$

and $I_{ik}$ are the moments of inertia. The second equality of Eqs. (113) and (114) is obtained after expressing the differences $\delta W_{11} - \delta W_{33}$, and $\delta W_{22} - \delta W_{33}$, respectively, in accordance with Eq. (149) (Chapter 3, Ch 1969), where $\delta W_{ij}$ refers to the variation of potential energy. The tensor of potential energy is given by

$$\mathbf{W}_{sk;ij}(x) = \int_V \rho x_s \partial \mathbf{B}_{ij}(x)/\partial x_k dV, \quad (7)$$

and the tensor of gravitational potential by

$$\mathbf{B}_{ij} = G\int_V \rho(x')\frac{(x_i - x'_i)(x_j - x'_j)}{|\vec{x} - \vec{x}'|^3} dV, \quad (8)$$

so that Eq. (7), which depends on four, rather than two indices, is usually called the supertensor of potential energy; here, the first pair of indices refers to the coordinates $x$, while the second pair refers to the tensor $\mathbf{B}_{ij}$ (Chandrasekhar & Lebovitz 1962). The expression for $\mathbf{B}_{ij}$ at an interior point of a homogeneous ellipsoid is given by

$$\mathbf{B}_{ij}^{int.} = \pi G\rho[2B_{ij}x_ix_j + a_i^2\delta_{ij}(A_i - \sum_{l=1}^{3} A_{il}x_l^2)]^{int.}, \tag{9}$$

where $A_i$, $A_{il}$, $B_{ij}$ are the so called "index symbols", $a_i$ are the semiaxes of the ellipsoid, and $G$ is the gravitational constant. The index symbols are elliptic integrals running from 0 ($\lambda$) to $\infty$ if we are to compute interior (exterior) contributions to $\mathbf{B}_{ij}$, where $\lambda$ is the ellipsoidal coordinate of the considered point [Ch 1969, Eqs. (103) and (104), Chapter 3]. The $B_{ij}$'s should not be confused with the potential tensor $\mathbf{B}_{ij}$. We thus have the determinant

$$\begin{vmatrix} \frac{1}{2}\lambda^2 - Q_1Q_2 + 3B_{11} - B_{13} & B_{12} - B_{23} & -\frac{1}{2}\lambda^2 - 3B_{33} + B_{13} \\ B_{12} - B_{13} & \frac{1}{2}\lambda^2 - Q_1Q_2 + 3B_{22} - B_{23} & -\frac{1}{2}\lambda^2 - 3B_{33} + B_{23} \\ 1/a_1^2 & 1/a_2^2 & 1/a_3^2 \end{vmatrix} = 0, \tag{10}$$

from which the dispersion relation for the even modes follows. The showing up of the elements $-Q_1Q_2$ in the first column, first row, and in the second column, second row, instead of the $2B_{12}$'s of Chandrasekhar's determinant, is a result of using relation (29). These quantities are elements of the matrix (97) [see also Eq. (146), Chapter 6, Ch 1969], and are given by

$$Q_1 = \frac{-a_1^2}{(a_1^2 + a_2^2)}\zeta, \qquad Q_2 = \frac{+a_2^2}{(a_1^2 + a_2^2)}\zeta, \tag{11}$$

where $Q_1 = Q_{12}$, and $Q_2 = Q_{21}$, and from which we obtain

$$Q_1Q_2 = \frac{-a_1^2a_2^2}{(a_1^2 + a_2^2)^2}\zeta^2, \tag{12}$$

where it is assumed that

$$a_1 \geq a_2 \geq a_3. \tag{13}$$

The components of the internal motion having an assigned (uniform) vorticity, $\zeta$, about the $x_3$ direction, can be written in the form

$$u_1 = Q_1x_2, \qquad u_2 = Q_2x_1, \qquad u_3 = 0. \tag{14}$$

At this point, it is convenient to define an average of the vorticity [see Eq. (8) of Paper I for a similar definition of an average of the angular velocity],

$$\langle Z\rangle = \frac{\int_V Z(\bar\omega)\bar\omega^2\rho(x)dV}{\int_V \bar\omega^2\rho(x)dV}. \tag{15}$$

Similarly, we define an average of the squared vorticity [see Eq. (9) of Paper I for a similar definition of an average of the squared angular velocity]

$$\langle Z^2\rangle = \frac{\int_V Z^2(\bar\omega)\bar\omega^2\rho(x)dx}{\int_V \bar\omega^2\rho(x)dV}. \tag{16}$$

#### 2.1.2. *The Odd Modes*

The determinant (124) is obtained after Eqs. (119)–(121), which constitute a homogeneous system of four equations in the unknowns $V_{1;3}$, $V_{3;1}$, $V_{2;3}$, and $V_{3;2}$, where $V_{i;j} = L_{i;j}I_{ij}$. From the above considerations there results

$$\begin{vmatrix} \lambda^2 + 2B_{13} & 2B_{13} + Q_1Q_2 & 0 & 0 \\ 2B_{13} & \lambda^2 + 2B_{13} + Q_1Q_2 & 0 & -2\lambda Q_1 \\ 0 & 0 & \lambda^2 + 2B_{23} & 2B_{23} + Q_1Q_2 \\ 0 & -2\lambda Q_2 & 2B_{23} & \lambda^2 + 2B_{23} + Q_1Q_2 \end{vmatrix} = 0, \tag{17}$$

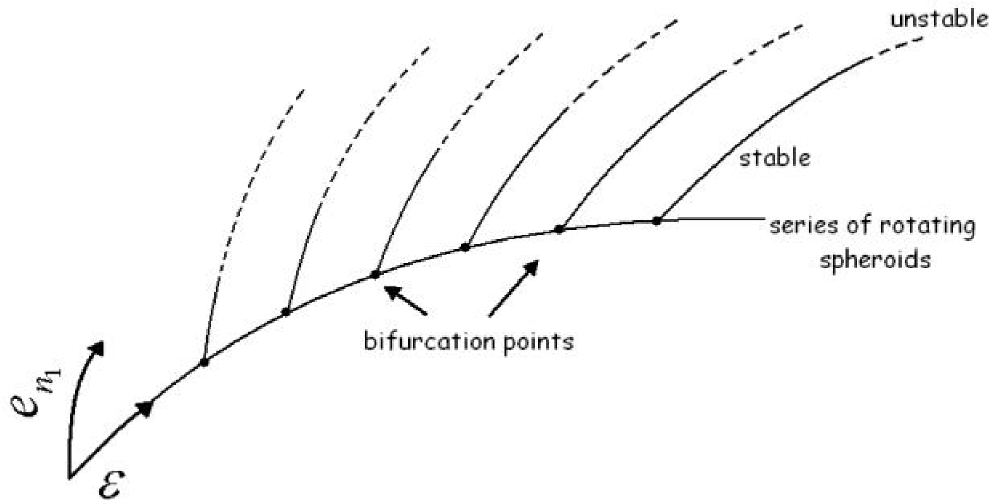


Fig. 1. Highly schematic diagram depicting the sequences of ellipsoids bifurcating from the series of rotating spheroids of Paper I. The portion of the curves shown as dashed lines are the domains of instability.



from which the dispersion relation for the odd modes follows. The showing up of the terms in the second column, first row, and fourth column, third row, instead of the respective elements in Chandrasekhar's determinant, is a result of using Eqs. (122) and (123). The elements in the second column, fourth row, and in the fourth column, second row, can be handled by means of the definition (15). The determinants (10) and (17) provide the means to obtain the dispersion relations for a Dedekind homogeneous mass; they still need to be modified if we intend to use them for an inhomogeneous mass. We pause here on the question regarding the homogeneous ellipsoids to start our own study.

## 3. THE INHOMOGENEOUS MODEL

The ellipsoids—of *exact* equilibrium, after Paper I—of the current work were obtained on the basis of the continuity of the potential—and so of the pressure—across the border between the nucleus and the atmosphere; this interface is assumed to be free from surface tension, and by hypothesis no flow of mass can occur across it (Landau & Lifshitz 1959). A given ellipsoid can be specified by the body's relative density $\varepsilon[=(\rho_n-\rho_a)/\rho_a]$, where $\rho$ is the density, along with the squared vorticities, $Z_n^2$, $Z_a^2$, in units $4\pi G\rho_a$, and the equatorial (1) and meridional (2) eccentricities, $\epsilon_{n_1}$, $\epsilon_{a_1}$, $\epsilon_{n_2}$, and $\epsilon_{a_2}$, of the nucleus ($n$) and the atmosphere ($a$). The families which these figures conform can be arranged in sequences of fixed $\varepsilon$ if we proceed as follows. Since mathematically there is no difference between a rotating spheroid and a static spheroid the latter, if it is a bifurcation spheroid, can be isolated, and then made to evolve progressively until the ellipsoidal shape appears neatly. To this end, we let $\epsilon_{n_1}$ go through increasing values, which brings on an increase of $Z_n^2$. The atmosphere, however, responds to this increase of $Z_n^2$ by a decreasing $Z_a^2$, and a corresponding decrease of $\epsilon_{a_2}$ occurs. We recall that, for any figure of the sequences, $\epsilon_{a_1}$ is very low. This fact, together with the confocality condition

$$\epsilon_{n_1}/\epsilon_{a_1} = \epsilon_{n_2}/\epsilon_{a_2}, \tag{18}$$

constrains the geometry of the figures. We use the confocality condition to simplify as much as possible the mathematical treatment (but see, for example, the reasons given by Chambat as to the *necessity* of such condition) (Chambat 1994). Thus, the more accomplished is the ellipsoidal shape for the nucleus, the less it is so for the atmosphere —more accurately its surface of zero pressure—which actually becomes nearly spherical. So, strictly speaking, the ovoidal shape for *both* nucleus and atmosphere is never attained in equilibrium. For instance, the overall shape of one of the figures of the last portion of a sequence is like a small massive seed embeded within a tenuous, almost spherical, atmosphere, a result of the fact that for these figures the vorticity of the nucleus is very much larger than that of the atmosphere which, in addition, is close to zero [for the general appearance of one of these figures see Figure 3(b) (Cisneros et al. 1995)]. For easy reference, we shall refer to all the figures of a sequence simply as ellipsoids, which will be generated from a spheroid of similar flattening as that at which the Maclaurin and Jacobi series meet. Now, there are other spheroids of null frequency, so nuclei not as flattened at the poles as 0.8127 are—provided that their equatorial flattening is no higher than about 0.01— also possible, the atmospheres having then a noticeable polar flattening; but, in any case, the shapes of the last ellipsoids are only slightly different from one sequence to another.

### 3.1. *Generalities*

Turning now to the current problem, we assume for simplicity that the $L_{i;j}$'s in Eq. (1) are continuous across the border between the nucleus and the atmosphere, so they need not be labeled with a superscript $n$ or $a$, used here to make a distinction between quantities pertaining to the nucleus or the atmosphere. Now, the set of nine individual virial equations for the model is not particularly helpful for our current purposes, and so they will not be fully displayed, as they were in Paper I. Rather, we shall begin our study directly from Eq. (10), for the even modes, and from Eq. (17), for the odd modes. Any element of these determinants is an integral, which is not only inhomogeneous *per se*, but also because of the inhomogeneous character of its integrand, namely the tensor of potential energy $\mathbf{B}_{ij}$, and this last quantity depends on points of both the nucleus and the atmosphere. These quantities may all be expressed in terms of the standard elliptic integrals of the first and second kind (MacMillan 1958), and can be evaluated in a similar fashion as the integrals pertaining to the inhomogeneous spheroids of Paper I [on this account, see Figure 1, Figure 2, and Eq. (10) of that paper]. We recall that the total $\mathbf{B}_{ij}$ for the atmosphere is made up of an interior plus an exterior contribution:

$$\mathbf{B}_{ij}(x_a) = \mathbf{B}_{ij}^{int.}(x_a) + \mathbf{B}_{ij}^{ext.}(x_a), \tag{19}$$

and it follows that the total tensor of potential energy of the nucleus is the sum of two interior contributions. Similarly, the total supertensor of potential energy, in the unit $\pi G\rho_a$, reads

$$\mathbf{W}_{sk;ij} = \int_{V_n} \rho_n x_s \partial\mathbf{B}_{ij}(x_n)/\partial x_k dV + \int_{V_a} \rho_a x_s \partial\mathbf{B}_{ij}(x_a)/\partial x_k dV \ . \tag{20}$$

We define

$$J_{11} \equiv I_{11}^n + (I_{11}^b - rI_{11}^n), \tag{21}$$

$$J_{22} \equiv I_{22}^n + (I_{22}^b - rI_{22}^n), \tag{22}$$

and

$$J_{33} \equiv I_{33}^n + (I_{33}^b - rI_{33}^n), \tag{23}$$

as the new quantities that replace the usual moments of inertia $I_{ij}$, where $r = (\varepsilon + 1)^{-1}$.

#### 3.1.1. *The Even Modes*

Note that, since the $V_{11}$, $V_{22}$, and $V_{33}$ correspond to the first, second, and third columns of the determinant (10), the elements of the first, second, and third columns must be averaged over $J_{11}$, $J_{22}$, and $J_{33}$, respectively. Correspondingly, the average defined in Eq. (16), becomes

$$\langle Z^2\rangle J_{11} \equiv Z_n^2 I_{11}^n + Z_a^2(I_{11}^b - rI_{11}^n), \tag{24}$$

if it is placed in the first column, and

$$\langle Z^2 \rangle J_{22} \equiv Z_n^2 I_{22}^n + Z_a^2 (I_{22}^b - r I_{22}^n), \tag{25}$$

if it is placed in the second column. For brevity, we call these terms $Z_1^2$, or,

$$Z_1^2 = \frac{Z_n^2 + Z_a^2 (J_{11} - 1) r}{J_{11}}, \tag{26}$$

and $Z_2^2$, or,

$$Z_2^2 = \frac{Z_n^2 + Z_a^2 (J_{22} - 1) r}{J_{22}}. \tag{27}$$

The numerical values of $Z_n^2$ and $Z_a^2$ can be taken directly from a previous calculation (Cisneros et al. 1995). With this notation, the evaluation of the index symbols is straigthforward. For example, if we wish to evaluate the symbol $B_{33}$ placed, say, in the third column of the determinant (10), we write

$$B_{33(3)} = \frac{B_{33}^{ext.} + B_{33}^{int.} (J_{33} - 1) r}{J_{33}}, \tag{28}$$

where $B_{33}^{ext.}$ and $B_{33}^{int.}$ mean the exterior and interior contributions of the potential for this particular index symbol [see Eq. (19)]. The extra (3) in $B_{33(3)}$ has no mathematical meaning, it serves only as a guide that reminds us that this element is placed in the third column, and so it must be averaged over $J_{33}$, and the like. It remains only to give proper consideration to the third elements of each column [which are not index symbols, see Eq. (10)]. Provided they are constructed for our inhomogeneous mass and calling these elements $T_{1(1)}$, $T_{2(2)}$, and $T_{3(3)}$, we can calculate them in a similar way. Reducing the determinant (7) according to our conventions, and substituting $\sigma = -i\lambda$, where $\sigma$ is the frequency, the resulting equation can be written

$$ax^2 + bx + c = 0, \tag{29}$$

where

$$a = 4(T_{1(1)} + T_{2(2)} + T_{3(3)}),$$

$$\begin{aligned} b = -4(((Z_1^2 + 3B_{11(1)} - B_{13(1)}) + (Z_2^2 + 3B_{22(2)} - B_{23(2)}))T_{3(3)} - \\ ((-3B_{33(3)} + B_{23(3)} + (-Z_1^2 - 3B_{11(1)} + B_{13(1)}))T_{2(2)} + \\ ((B_{23(2)} - B_{12(2)}))T_{1(1)} + ((B_{13(1)} - B_{12(1)})T_{2(2)} - \\ ((-3B_{33(3)} + B_{13(3)} + (-Z_2^2 - 3B_{22(2)} + B_{23(2)}))T_{1(1)})), \end{aligned}$$

and

$$\begin{aligned} c = 4(((Z_1 + 3B_{11(1)} - B_{13(1)})(Z_2 + 3B_{22(2)} - B_{23(2)})T_{3(3)} - \\ ((Z_1 + 3B_{11(1)} - B_{13(1)})(-3B_{33(3)} + B_{23(3)}))T_{2(2)} - \\ ((B_{12(2)} - B_{23(2)})(B_{12(1)} - B_{13(1)}))T_{3(3)} + \\ ((B_{12(2)} - B_{23(2)})(-3B_{33(3)} + B_{23(3)}))T_{1(1)} + \\ ((-3B_{33(3)} + B_{13(3)})(B_{12(1)} - B_{13(1)}))T_{2(2)} - \\ ((-3B_{33(3)} + B_{13(3)})(Z_2 + 3B_{22(2)} - B_{23(2)}))T_{3(3)})). \end{aligned}$$

### 3.1.2. *The Odd Modes*

Next, the inhomogeneous version of the determinant (17) will be considered. We first note that the unknowns $V_{1;3}$, $V_{3;1}$, $V_{2;3}$, and $V_{3;2}$, correspond to the first, second, third, and fourth columns respectively, so the elements of the first and third columns must be averaged over $J_{33}$, those of the second column over $J_{11}$, and those of the fourth column over $J_{22}$. If, say, $B_{13}$ is placed in the second column, following our previous convention we write it as

$$B_{13(2)} = \frac{B_{13}^{ext.} + B_{13}^{int.}(J_{22} - 1)r}{J_{22}}, \tag{30}$$

and the like. The average of vorticity is of the form

$$\langle Z \rangle J_{11} \equiv Z_n I_{11}^n + Z_a(I_{11}^b - r I_{11}^n), \tag{31}$$

if it happens to be placed in the second column, and the like. The determinant (17) can be reduced to a cubic equation of the form

$$dx^3 + ex^2 + fx + g = 0, \tag{32}$$

where

$$d = 8,$$

$$\begin{aligned} e = -4(2B_{23(2)} - D_2 + 2B_{23(3)} + 2B_{131} - D_1 + 2B_{13(3)} - 4D), \\ f = 2((2B_{23(3)}(2B_{23(2)} - D_2) + 2B_{13(3)}(2B_{13(1)} - D_1) + \\ (2B_{13(1)} - D_1 + 2B_{13(3)})(2B_{23(2)} - D_2 + 2B_{23(3)}) - \\ (2B_{23(2)} - D_2)2B_{23(3)} - (2B_{23(3)} + 2B_{13(3)})(4D) - \\ (2B_{13(1)} - D_1)2B_{13(3)})), \end{aligned}$$

and

$$\begin{aligned} g = -((2B_{13(1)} - D_1 + 2B_{13(3)})2B_{23(3)}(2B_{23(2)} - D_2) + \\ (2B_{23(2)} - D_2 + 2B_{23(3)})2B_{13(3)}(2B_{13(1)} - D_1) - \\ (2B_{13(1)} - D_1 + 2B_{13(3)})(2B_{23(2)} - D_2)2B_{23(3)} - 2B_{13(3)}2B_{23(3)}(4D) - \\ (2B_{23(2)} - D_2 + 2B_{23(3)})(2B_{13(1)} - D_1)2B_{13(3)}), \end{aligned}$$

where $D$ stands for the negative of the product of the averages given by the inhomogeneous version of Eq. (15); calling $d_1$ and $d_2$ these terms when they refer to the element of the fourth column, second row, and the second column, fourth row, respectively, we have

$$D = -d_1 d_2, \tag{33}$$

where

$$d_1 = \frac{Z_n + Z_a(J_{11} - 1)r}{J_{11}}, \tag{34}$$

and

$$d_2 = \frac{Z_n + Z_a(J_{22} - 1)r}{J_{22}}. \tag{35}$$

## 4. NUMERICAL RESULTS

The numerical solution of Eq. (29) and Eq. (32), namely the body's characteristic frequencies of oscillation, is given in Table 1, for several values of $\varepsilon$; this table is similarly constructed as that of an earlier work (Cisneros et al. 1995), with the addition of five new columns giving the oscillation frequencies. The values of the parameters corresponding to the equilibrium have been recalculated on the basis that the figures are now known to be in exact equilibrium, so that more significant digits than those given for brevity in Table 1 are certainly needed. Table 1 is segmented in sequences, each sequence listing the parameters that describe the equilibrium (Columns 1 to 6), and the stability (Columns 7 to 11) of its members. The first three columns give $Z_n^2$, $\epsilon_{n_1}$ (this last eccentricity is increased selectively), and $\epsilon_{n_2}$; the next three columns give $Z_a^2$, $\epsilon_{a_1}$, and $\epsilon_{a_2}$; the other columns give the squared oscillation modes, $\sigma^2$, normalized to $\sqrt{\pi G \rho_a}$; Columns 7 to 9 give the odd modes, $\sigma_1^2$, $\sigma_2^2$, and $\sigma_3^2$; Columns 10 and 11 are for the even modes, $\sigma_4^2$, and $\sigma_5^2$. The sequence for $\varepsilon = 2$ is provided with three models, all of them with $\epsilon_{n_1} = 0.0005$: one is the model which fits naturally into the sequence, and is stable; another has a nucleus relatively flatter at the poles than the first, and is unstable; this exemplifies how sensitive can stability be around this region, where the nucleus is still far from being truly ellipsoidal; the third model has $\epsilon_{n_2}$ about 0.7, clearly below the value belonging to the bifurcation spheroid itself; this last model has a nucleus certainly not as ellipsoidal as desired, but is stable, and similar figures for other sequences also prove to be stable.

### 4.1. *The Instability Regimes*

The discernment of instability along a sequence is provided by the existence of negative squared frequencies among the general solution of either Eq. (29), or Eq. (32) which, in the current case, occurs via the even mode $\sigma_4^2$. Let us remark that in Chandrasekhar's approach for the Dedekind figures, instability does not show up through the even modes, which is just contrary to what was found here. Any figure of equilibrium—any row of Table 1—which does not have a negative $\sigma_4^2$ is therefore stable. A sequence of low $\varepsilon$ is—except for a narrow region about the bifurcation spheroid— made up of unstable ellipsoids, the domain of instability becoming confined, for high $\varepsilon$, to a small segment at the end of the sequence. Even for $\varepsilon = 500$ (not included in Table 1), instability persists beyond $\epsilon_{n_1} \approx 0.7$. Figure 1 is a (highly schematic) diagram showing the sequences of static ellipsoids branching off the series of rotating spheroids from the bifurcation points; it can be seen that the domain of instability—dashed lines—becomes narrower as $\varepsilon$ increases.

## 5. SUMMARY

We have carried out a stability study of an inhomogeneous static ellipsodal mass carrying internal currents of differential vorticity. The results indicate that, if the figures are arranged in sequences as in Table 1, regimes of instabilty are clearly distinguished in every sequence. Now, since in passing, on a given sequence, from a regime of stability to one of instability, the squared frequency changes sign, a null frequency is clearly implied. However, the mathematical form of our equations does not permit to find out easily where exactly this change—which signals the onset of instability—occurs, as we were able to do with the spheroids of Paper I; in considering a null frequency, then, we must content ourselves with a gross trial calculation; we will not pursue on this point any further.

TABLE 1

PROPERTIES AND CHARACTERISTIC FREQUENCIES OF OSCILLATION OF OUR MODELS[a]

| $Z_n^2$ | $\epsilon_{n_1}$ | $\epsilon_{n_2}$ | $10^4 Z_a^2$ | $10^4\,\epsilon_{a_1}$ | $\epsilon_{a_2}$ | $10^4\,\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $\sigma_4^2$ | $\sigma_5^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\varepsilon = 0.1$ | | | | | |
| 0.4116 | 0.00001 | 0.8130 | 4114.95 | 0. 10 | 0.8126 | 2058 | 0.3534 | 1.0986 | 0.8230 | 0.8470 |
| 0.3855 | 0.0001 | 0.8127 | 3765.55 | 0. 96 | 0.7822 | 1905 | 0.3526 | 1.0675 | 0.7947 | 0.8012 |
| 0.3396 | 0.0002 | 0.8270 | 3119.07 | 1. 75 | 0.7223 | 1612 | 0.3676 | 1.0202 | 0.7054 | 0.7836 |
| 0.3063 | 0.0003 | 0.8427 | 2644.53 | 2. 39 | 0.6722 | 1374 | 0.3889 | 0.9918 | 0.6519 | 0.7839 |
| 0.2650 | 0.0004 | 0.8643 | 2062.05 | 2. 78 | 0.6008 | 1064 | 0.4205 | 0.9563 | 0.5959 | 0.7932 |
| 0.1962 | 0.0005 | 0.8250 | 1334.13 | 2. 96 | 0.4887 | 683 | 0.4500 | 0.8755 | 0.5739 | 0.7191 |
| 0.1959 | 0.0006 | 0.8271 | 1324.47 | 3. 53 | 0.4871 | 677 | 0.4510 | 0.8752 | 0.5725 | 0.7207 |
| 0.1885 | 0.0007 | 0.8275 | 1232.33 | 3. 98 | 0.4706 | 629 | 0.4567 | 0.8659 | 0.5673 | 0.7172 |
| 0.1740 | 0.0008 | 0.8273 | 1056.03 | 4. 23 | 0.4371 | 537 | 0.4680 | 0.8470 | 0.5578 | 0.7093 |
| 0.1496 | 0.0009 | 0.8308 | 750.89 | 4. 01 | 0.3706 | 379 | 0.4905 | 0.8118 | 0.5404 | 0.6991 |
| 0.1245 | 0.0010 | 0.8300 | 449.86 | 3. 50 | 0.2884 | 226 | 0.5169 | 0.7691 | 0.5251 | 0.6843 |
| 0.1219 | 0.0050 | 0.9875 | 7.74 | 1. 93 | 0.0381 | 4 | 0.5667 | 0.8233 | −0.5803 | 2.8775 |
| 0.1224 | 0.0100 | 0.9933 | 3.75 | 2. 67 | 0.0265 | 2 | 0.5604 | 0.8480 | −1.6427 | 5.0106 |
| 0.1220 | 0.0500 | 0.9931 | 0.34 | 4. 03 | 0.0080 | 0.02 | 0.5612 | 0.8461 | −1.5476 | 4.8193 |
| 0.1221 | 0.1000 | 0.9931 | 0.24 | 6. 75 | 0.0067 | 0.01 | 0.5612 | 0.8461 | −1.5615 | 4.8471 |
| 0.1221 | 0.2000 | 0.9932 | 0.12 | 9. 26 | 0.0046 | 0.006 | 0.5615 | 0.8458 | −1.6000 | 4.9238 |
| 0.1223 | 0.3000 | 0.9934 | 0.08 | 11. 20 | 0.0037 | 0.004 | 0.5620 | 0.8452 | −1.6610 | 5.0458 |
| 0.1230 | 0.4000 | 0.9937 | 0.06 | 12. 90 | 0.0032 | 0.003 | 0.5628 | 0.8443 | −1.7590 | 5.2416 |
| 0.1246 | 0.5000 | 0.9941 | 0.06 | 15. 10 | 0.0030 | 0.003 | 0.5638 | 0.8430 | −1.9065 | 5.5663 |
| 0.1282 | 0.6000 | 0.9946 | 0.07 | 19. 30 | 0.0032 | 0.004 | 0.5653 | 0.8410 | −2.1366 | 5.9961 |
| 0.1364 | 0.7000 | 0.9952 | 0.08 | 22. 51 | 0.0032 | 0.004 | 0.5674 | 0.8382 | −2.5244 | 6.7715 |
| 0.1566 | 0.8000 | 0.9962 | 0.08 | 24. 10 | 0.0030 | 0.004 | 0.5704 | 0.8338 | −3.2866 | 8.2957 |
| 0.2270 | 0.9000 | 0.9976 | 0.09 | 27. 10 | 0.0030 | 0.003 | 0.5754 | 0.8259 | −5.4601 | 12.6432 |

| $Z_n^2$ | $\epsilon_{n_1}$ | $\epsilon_{n_2}$ | $10^4 Z_a^2$ | $10^4\,\epsilon_{a_1}$ | $\epsilon_{a_2}$ | $10^4\,\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $\sigma_4^2$ | $\sigma_5^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\varepsilon = 0.5$ | | | | | |
| 0.5610 | 0.00001 | 0.8128 | 5600.93 | 0.10 | 0.8122 | 2803 | 0.4813 | 1.4970 | 1.1210 | 1.5232 |
| 0.5600 | 0.0001 | 0.8126 | 5579.19 | 0.99 | 0.8112 | 2796 | 0.4803 | 1.4947 | 1.1186 | 1.1492 |
| 0.5210 | 0.0002 | 0.8130 | 4482.66 | 1.87 | 0.7600 | 2438 | 0.4519 | 1.3910 | 0.9241 | 1.0270 |
| 0.4802 | 0.0003 | 0.8301 | 2913.89 | 2.38 | 0.6601 | 1726 | 0.5159 | 1.3348 | 0.6255 | 1.0052 |
| 0.4521 | 0.0004 | 0.8370 | 1966.75 | 2.70 | 0.5660 | 1136 | 0.5994 | 1.3341 | 0.5167 | 1.0199 |
| 0.4329 | 0.0005 | 0.8470 | 1198.13 | 2.69 | 0.4566 | 648 | 0.6861 | 1.3358 | 0.4496 | 1.0628 |
| 0.4138 | 0.0006 | 0.8487 | 660.40 | 2.44 | 0.3457 | 340 | 0.7460 | 1.3103 | 0.4198 | 1.0756 |
| 0.4007 | 0.0007 | 0.8499 | 297.44 | 1.93 | 0.2346 | 150 | 0.7850 | 1.2815 | 0.3990 | 1.0840 |
| 0.3946 | 0.0008 | 0.8530 | 77.38 | 1.13 | 0.1203 | 39 | 0.8070 | 1.2623 | 0.3798 | 1.0990 |
| 0.3957 | 0.0009 | 0.8553 | 54.60 | 1.06 | 0.1011 | 27 | 0.8089 | 1.2642 | 0.3736 | 1.1099 |
| 0.4457 | 0.0010 | 0.9076 | 126.13 | 1.69 | 0.1534 | 63 | 0.7951 | 1.4189 | 0.1921 | 1.5226 |
| 0.4796 | 0.0050 | 0.9564 | 7.03 | 1.90 | 0.0363 | 3.5 | 0.7717 | 1.6440 | −0.4434 | 2.8341 |
| 0.4797 | 0.0100 | 0.9566 | 5.71 | 3.42 | 0.0327 | 2.8 | 0.7715 | 1.6454 | −0.4498 | 2.8470 |
| 0.4797 | 0.0500 | 0.9570 | 0.43 | 4.72 | 0.0090 | 0.2 | 0.7715 | 1.6470 | −0.4615 | 2.9703 |

TABLE 1 (CONTINUED)

| $Z_n^2$ | $\epsilon_{n_1}$ | $\epsilon_{n_2}$ | $10^4 Z_a^2$ | $10^4\,\epsilon_{a_1}$ | $\epsilon_{a_2}$ | $10^4\,\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $\sigma_4^2$ | $\sigma_5^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 0.5$ | | | | | | | | | | |
| 0.4797 | 0.1000 | 0.9572 | 0.25 | 7.11 | 0.0068 | 0.1 | 0.7720 | 1.6463 | −0.4663 | 2.8797 |
| 0.4799 | 0.2000 | 0.9579 | 0.14 | 10.44 | 0.0050 | 0.07 | 0.7742 | 1.6428 | −0.4843 | 2.9153 |
| 0.4807 | 0.3000 | 0.9591 | 0.10 | 12.93 | 0.0041 | 0.05 | 0.7781 | 1.6368 | −0.5164 | 2.9788 |
| 0.4832 | 0.4000 | 0.9608 | 0.08 | 15.16 | 0.0036 | 0.04 | 0.7838 | 1.6276 | −0.5669 | 3.0791 |
| 0.4894 | 0.5000 | 0.9631 | 0.09 | 18.56 | 0.0036 | 0.04 | 0.7918 | 1.6143 | −0.6445 | 3.2333 |
| 0.5032 | 0.6000 | 0.9663 | 0.08 | 21.12 | 0.0034 | 0.04 | 0.8029 | 1.5995 | −0.7661 | 3.4763 |
| 0.5343 | 0.7000 | 0.9705 | 0.10 | 25.68 | 0.0036 | 0.05 | 0.8185 | 1.5680 | −0.9714 | 3.8879 |
| 0.6115 | 0.8000 | 0.9764 | 0.11 | 28.86 | 0.0035 | 0.06 | 0.8411 | 1.5256 | −1.3766 | 4.7026 |
| 0.8777 | 0.9000 | 0.9849 | 0.12 | 32.07 | 0.0035 | 0.06 | 0.8780 | 1.4502 | −2.5405 | 7.0451 |

| $Z_n^2$ | $\epsilon_{n_1}$ | $\epsilon_{n_2}$ | $10^4 Z_a^2$ | $10^4\,\epsilon_{a_1}$ | $\epsilon_{a_2}$ | $10^4\,\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $\sigma_4^2$ | $\sigma_5^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | | | | | | | | | | |
| 0.7480 | 0.00001 | 0.8127 | 7462.46 | 0.10 | 0.8121 | 3737 | 0.6416 | 1.9959 | 1.4946 | 1.5376 |
| 0.7234 | 0.0001 | 0.8128 | 6297.77 | 0.96 | 0.7799 | 3452 | 0.5949 | 1.8920 | 1.3311 | 1.3902 |
| 0.7036 | 0.0002 | 0.8280 | 4089.20 | 1.69 | 0.7000 | 2670 | 0.6204 | 1.7864 | 0.8636 | 1.2932 |
| 0.7067 | 0.0003 | 0.8330 | 3807.53 | 2.47 | 0.6870 | 2529 | 0.6386 | 1.7969 | 0.7822 | 1.2721 |
| 0.7170 | 0.0004 | 0.8433 | 3431.82 | 3.17 | 0.6687 | 2316 | 0.6716 | 1.8342 | 0.6803 | 1.2946 |
| 0.7144 | 0.0005 | 0.8466 | 2918.22 | 3.74 | 0.6346 | 1976 | 0.7156 | 1.8654 | 0.5873 | 1.2979 |
| 0.7440 | 0.0006 | 0.8849 | 455.10 | 1.95 | 0.2880 | 233 | 1.0458 | 2.1574 | 0.1976 | 1.6595 |
| 0.7414 | 0.0007 | 0.8849 | 258.93 | 1.73 | 0.2188 | 130 | 1.0655 | 2.1494 | 0.1843 | 1.6680 |
| 0.7405 | 0.0008 | 0.8849 | 190.35 | 1.70 | 0.1880 | 95 | 1.0715 | 2.1454 | 0.1794 | 1.6707 |
| 0.7448 | 0.0009 | 0.8888 | 77.61 | 1.22 | 0.1204 | 39 | 1.0786 | 2.1599 | 0.1491 | 1.7230 |
| 0.7761 | 0.0010 | 0.9102 | 77.85 | 1.32 | 0.1206 | 39 | 1.0671 | 2.3003 | −0.0080 | 2.0643 |
| 0.7979 | 0.0050 | 0.9271 | 10.22 | 2.36 | 0.044 | 5.1 | 1.0556 | 2.4306 | −0.2092 | 2.4830 |
| 0.7980 | 0.0100 | 0.9274 | 2.85 | 2.49 | 0.023 | 1.4 | 1.0557 | 2.4318 | −0.2128 | 2.4896 |
| 0.7982 | 0.0500 | 0.9277 | 1.08 | 7.67 | 0.014 | 0.54 | 1.0558 | 2.4329 | −0.2172 | 2.4984 |
| 0.7983 | 0.1000 | 0.9280 | 0.35 | 8.73 | 0.008 | 0.17 | 1.0569 | 2.4312 | −0.2213 | 2.5065 |
| 0.7986 | 0.2000 | 0.9291 | 0.14 | 10.71 | 0.005 | 0.07 | 1.0610 | 2.4231 | −0.2359 | 2.5348 |
| 0.7999 | 0.3000 | 0.9311 | 0.10 | 13.33 | 0.0041 | 0.05 | 1.0690 | 2.4089 | −0.2619 | 2.5857 |
| 0.8039 | 0.4000 | 0.9340 | 0.09 | 16.14 | 0.0038 | 0.05 | 1.0815 | 2.3875 | −0.6303 | 2.6667 |
| 0.8138 | 0.5000 | 0.9380 | 0.06 | 16.11 | 0.0030 | 0.03 | 1.0982 | 2.3567 | −0.3662 | 2.7921 |
| 0.8359 | 0.6000 | 0.9434 | 0.09 | 21.95 | 0.0034 | 0.05 | 1.1213 | 2.3128 | −0.4655 | 2.9909 |
| 0.8855 | 0.7000 | 0.9506 | 0.10 | 25.94 | 0.0035 | 0.05 | 1.1535 | 2.2491 | −0.6337 | 3.3309 |
| 1.0082 | 0.8000 | 0.9605 | 0.11 | 29.34 | 0.0035 | 0.06 | 1.2004 | 2.1507 | −0.9681 | 4.0112 |
| 1.4280 | 0.9000 | 0.9748 | 0.12 | 32.50 | 0.0035 | 0.06 | 1.2769 | 1.9761 | −1.9421 | 5.9942 |

| $Z_n^2$ | $\epsilon_{n_1}$ | $\epsilon_{n_2}$ | $10^4 Z_a^2$ | $10^4\,\epsilon_{a_1}$ | $\epsilon_{a_2}$ | $10^4\,\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $\sigma_4^2$ | $\sigma_5^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 2$ | | | | | | | | | | |
| 1.1223 | 0.00001 | 0.8128 | 11173.71 | 0.10 | 0.8120 | 5607 | 0.9618 | 2.9933 | 2.2424 | 2.3057 |
| 1.0918 | 0.0001 | 0.8127 | 8496.27 | 0.94 | 0.7680 | 5110 | 0.8970 | 2.8135 | 1.9544 | 2.0530 |
| 1.1194 | 0.0002 | 0.8270 | 7733.10 | 1.84 | 0.7610 | 4995 | 0.8628 | 2.7813 | 1.7827 | 1.9984 |

TABLE 1 (CONTINUED)

| $Z_n^2$ | $\epsilon_{n_1}$ | $\epsilon_{n_2}$ | $10^4 Z_a^2$ | $10^4\,\epsilon_{a_1}$ | $\epsilon_{a_2}$ | $10^4\,\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $\sigma_4^2$ | $\sigma_5^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\varepsilon = 2$ | | | | | |
| 1.1096 | 0.0003 | 0.8330 | 3227.11 | 3.00 | 0.6100 | 2739 | 1.0496 | 2.8010 | 0.8271 | 1.6271 |
| 1.1158 | 0.0004 | 0.8350 | 2457.10 | 2.69 | 0.5610 | 2021 | 1.1474 | 2.9084 | 0.6566 | 1.5599 |
| 1.1343 | 0.0005 | 0.8365 | 750.79 | 2.18 | 0.3561 | 434 | 1.5127 | 3.2121 | 0.3633 | 1.4807 |
| 0.7861 | 0.0005 | 0.6988 | 563.10 | 2.18 | 0.3044 | 340 | 1.4615 | 2.4266 | 0.5497 | 0.9375 |
| 1.3042 | 0.0005 | 0.9880 | 526.10 | 2.18 | 0.3104 | 267 | 1.1937 | 6.3645 | −7.9687 | 18.3905 |
| 1.2536 | 0.0006 | 0.8790 | 230.28 | 1.41 | 0.2059 | 117 | 1.6176 | 3.6665 | 0.1063 | 1.9532 |
| 1.2542 | 0.0007 | 0.8790 | 191.25 | 1.50 | 0.1880 | 96 | 1.6228 | 3.6669 | 0.1021 | 1.9559 |
| 1.2546 | 0.0008 | 0.8789 | 153.63 | 1.54 | 0.1688 | 77 | 1.6274 | 3.6662 | 0.0983 | 1.9579 |
| 1.2551 | 0.0009 | 0.8790 | 132.44 | 1.61 | 0.1569 | 66 | 1.6297 | 3.6667 | 0.0956 | 1.9603 |
| 1.2554 | 0.0010 | 0.8790 | 114.04 | 1.66 | 0.1457 | 70 | 1.6307 | 3.6686 | 0.0937 | 1.9615 |
| 1.2935 | 0.0050 | 0.8941 | 14.43 | 2.91 | 0.0520 | 7.2 | 1.6269 | 3.8374 | −0.0292 | 2.2168 |
| 1.2952 | 0.0100 | 0.8948 | 3.20 | 2.74 | 0.0245 | 1.6 | 1.6267 | 3.8447 | −0.0361 | 2.2301 |
| 1.2953 | 0.0500 | 0.8949 | 0.32 | 4.33 | 0.0077 | 0.16 | 1.6275 | 3.8433 | −0.0374 | 2.2323 |
| 1.2953 | 0.1000 | 0.8953 | 0.24 | 7.42 | 0.0066 | 0.12 | 1.6298 | 3.8389 | −0.0403 | 2.2379 |
| 1.2958 | 0.2000 | 0.8971 | 0.11 | 10.06 | 0.0045 | 0.06 | 1.6390 | 3.8211 | −0.0525 | 2.2609 |
| 1.2978 | 0.3000 | 0.9000 | 0.08 | 12.50 | 0.0037 | 0.04 | 1.6551 | 3.7899 | −0.0742 | 2.3026 |
| 1.3039 | 0.4000 | 0.9043 | 0.08 | 15.43 | 0.0035 | 0.04 | 1.6790 | 3.7426 | −0.1085 | 2.3692 |
| 1.3188 | 0.5000 | 0.9102 | 0.08 | 19.26 | 0.0035 | 0.04 | 1.7126 | 3.6747 | −0.1610 | 2.4730 |
| 1.3523 | 0.6000 | 0.9181 | 0.07 | 20.58 | 0.0031 | 0.04 | 1.7589 | 3.5783 | −0.2436 | 2.6392 |
| 1.4273 | 0.7000 | 0.9286 | 0.08 | 23.72 | 0.0031 | 0.04 | 1.8233 | 3.4383 | −0.3839 | 2.9268 |
| 1.6115 | 0.8000 | 0.9430 | 0.09 | 26.73 | 0.0031 | 0.04 | 1.9173 | 3.2225 | −0.6654 | 3.5111 |
| 2.2316 | 0.9000 | 0.9638 | 0.10 | 29.44 | 0.0031 | 0.05 | 2.0707 | 2.8402 | −1.5018 | 5.2464 |

| $Z_n^2$ | $\epsilon_{n_1}$ | $\epsilon_{n_2}$ | $10^4 Z_a^2$ | $10^4\,\epsilon_{a_1}$ | $\epsilon_{a_2}$ | $10^4\,\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $\sigma_4^2$ | $\sigma_5^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\varepsilon = 5$ | | | | | |
| 2.2452 | 0.0001 | 0.8127 | 22335.77 | 0.999 | 0.8120 | 11220 | 1.9255 | 5.9898 | 4.4864 | 4.6172 |
| 2.2149 | 0.0002 | 0.8130 | 15401.91 | 1.522 | 0.7610 | 10476 | 1.7962 | 5.7213 | 4.1209 | 4.1877 |
| 2.2055 | 0.0003 | 0.8136 | 10684.58 | 2.618 | 0.7099 | 9435 | 1.7905 | 5.5175 | 3.5351 | 3.8570 |
| 2.2167 | 0.0004 | 0.8165 | 8030.19 | 3.287 | 0.6710 | 8431 | 1.8565 | 5.4330 | 3.0212 | 3.5599 |
| 2.1948 | 0.0005 | 0.7975 | 378.87 | 1.586 | 0.2530 | 222 | 3.2180 | 6.5641 | 0.4287 | 1.4562 |
| 2.3460 | 0.0006 | 0.8255 | 344.23 | 1.773 | 0.2440 | 193 | 3.2450 | 7.0715 | 0.3346 | 1.6502 |
| 2.3499 | 0.0007 | 0.8257 | 286.19 | 1.902 | 0.2244 | 156 | 3.2709 | 7.0994 | 0.3167 | 1.6518 |
| 2.3571 | 0.0008 | 0.8266 | 229.13 | 1.959 | 0.2024 | 121 | 3.2947 | 7.1351 | 0.2981 | 1.6600 |
| 2.5009 | 0.0009 | 0.8527 | 143.09 | 1.710 | 0.1620 | 73 | 3.3108 | 7.6922 | 0.1655 | 1.9280 |
| 2.5120 | 0.0010 | 0.8541 | 64.54 | 1.283 | 0.1096 | 32 | 3.3286 | 7.7346 | 0.1416 | 1.9523 |
| 2.5253 | 0.0050 | 0.8564 | 42.04 | 5.173 | 0.0886 | 21 | 3.3302 | 7.7889 | 0.1250 | 1.9835 |
| 2.5298 | 0.0100 | 0.8570 | 7.66 | 4.421 | 0.0379 | 4 | 3.3339 | 7.8015 | 0.1166 | 1.9936 |
| 2.5317 | 0.0500 | 0.8574 | 0.47 | 5.49 | 0.0094 | 0.17 | 3.3358 | 7.8043 | 0.1132 | 1.9995 |
| 2.5318 | 0.1000 | 0.8580 | 0.25 | 7.90 | 0.0068 | 0.12 | 3.3415 | 7.7927 | 0.1107 | 2.0040 |
| 2.5325 | 0.2000 | 0.8604 | 0.12 | 10.80 | 0.0046 | 0.06 | 3.3644 | 7.7448 | 0.1005 | 2.0225 |
| 2.5361 | 0.3000 | 0.8645 | 0.08 | 12.63 | 0.0036 | 0.04 | 3.4040 | 7.6609 | 0.0822 | 2.0562 |
| 2.5466 | 0.4000 | 0.8704 | 0.06 | 13.99 | 0.0030 | 0.03 | 3.4630 | 7.5340 | 0.0535 | 2.1104 |

TABLE 1 (CONTINUED)

| $Z_n^2$ | $\epsilon_{n_1}$ | $\epsilon_{n_2}$ | $10^4 Z_a^2$ | $10^4\,\epsilon_{a_1}$ | $\epsilon_{a_2}$ | $10^4\,\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $\sigma_4^2$ | $\sigma_5^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\varepsilon = 5$ | | | | | |
| 2.5724 | 0.5000 | 0.8786 | 0.09 | 20.85 | 0.0037 | 0.05 | 3.5458 | 7.3521 | 0.0099 | 2.1956 |
| 2.6302 | 0.6000 | 0.8894 | 0.09 | 23.15 | 0.0034 | 0.05 | 3.6598 | 7.0939 | −0.0582 | 2.3336 |
| 2.7589 | 0.7000 | 0.9039 | 0.08 | 23.33 | 0.0030 | 0.04 | 3.8187 | 6.7195 | −0.1739 | 2.5760 |
| 3.0707 | 0.8000 | 0.9235 | 0.12 | 30.68 | 0.0035 | 0.06 | 4.0506 | 6.1433 | −0.4080 | 3.0778 |
| 4.0864 | 0.9000 | 0.9516 | 0.14 | 35.07 | 0.0037 | 0.07 | 4.4303 | 5.1247 | −1.1225 | 4.6052 |

| $Z_n^2$ | $\epsilon_{n_1}$ | $\epsilon_{n_2}$ | $10^4 Z_a^2$ | $10^4\,\epsilon_{a_1}$ | $\epsilon_{a_2}$ | $10^4\,\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $\sigma_4^2$ | $\sigma_5^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\varepsilon = 10$ | | | | | |
| 4.1135 | 0.0001 | 0.8127 | 39700.27 | 0.99 | 0.8081 | 20527 | 3.5162 | 10.9595 | 8.2040 | 8.4346 |
| 4.1241 | 0.0002 | 0.8176 | 14090.42 | 1.99 | 0.6771 | 17412 | 3.4369 | 10.2798 | 6.6044 | 7.0642 |
| 4.2689 | 0.0003 | 0.8301 | 5683.94 | 2.05 | 0.5677 | 11623 | 3.9949 | 10.5754 | 4.0176 | 5.2983 |
| 4.4284 | 0.0004 | 0.8430 | 2247.11 | 2.17 | 0.4568 | 4138 | 4.7659 | 12.3819 | 1.7273 | 3.3537 |
| 4.5414 | 0.0005 | 0.8483 | 128.05 | 0.90 | 0.1523 | 67 | 6.1076 | 14.6492 | 0.1680 | 2.0034 |
| 4.6054 | 0.0006 | 0.8548 | 82.75 | 0.87 | 0.1234 | 42 | 6.1168 | 14.9573 | 0.1135 | 2.0896 |
| 4.6386 | 0.0007 | 0.8583 | 65.27 | 0.90 | 0.1099 | 33 | 6.1160 | 15.1216 | 0.0866 | 2.1390 |
| 4.5629 | 0.0008 | 0.8501 | 54.05 | 0.94 | 0.1002 | 27 | 6.1372 | 14.7566 | 0.1300 | 2.0308 |
| 4.4192 | 0.0009 | 0.8350 | 35.02 | 0.87 | 0.0808 | 18 | 6.1682 | 14.1351 | 0.1970 | 1.8603 |
| 4.4194 | 0.0010 | 0.8350 | 35.03 | 0.97 | 0.0808 | 18 | 6.1682 | 14.1359 | 0.1969 | 1.8605 |
| 4.4480 | 0.0050 | 0.8378 | 16.81 | 3.35 | 0.0561 | 8.4 | 6.1691 | 14.2511 | 0.1798 | 1.8923 |
| 4.4506 | 0.0100 | 0.8380 | 3.12 | 2.89 | 0.0242 | 1.6 | 6.1713 | 14.2586 | 0.1763 | 1.8956 |
| 4.4512 | 0.0500 | 0.8383 | 0.25 | 4.05 | 0.0068 | 0.12 | 6.1750 | 14.2526 | 0.1749 | 1.8976 |
| 4.4513 | 0.1000 | 0.8390 | 0.17 | 6.72 | 0.0056 | 0.09 | 6.1862 | 14.2286 | 0.1726 | 1.9016 |
| 4.4523 | 0.2000 | 0.8417 | 0.11 | 10.66 | 0.0045 | 0.06 | 6.2315 | 14.1300 | 0.1631 | 1.9182 |
| 4.4583 | 0.3000 | 0.8463 | 0.08 | 13.01 | 0.0037 | 0.04 | 6.3101 | 13.9575 | 0.1462 | 1.9482 |
| 4.4752 | 0.4000 | 0.8531 | 0.08 | 16.72 | 0.0036 | 0.04 | 6.4270 | 13.6969 | 0.1198 | 1.9976 |
| 4.5166 | 0.5000 | 0.8625 | 0.06 | 17.39 | 0.0030 | 0.03 | 6.5909 | 13.3233 | 0.0798 | 2.0749 |
| 4.6092 | 0.6000 | 0.8749 | 0.07 | 21.07 | 0.0031 | 0.04 | 6.8168 | 12.7934 | 0.0177 | 2.2010 |
| 4.8142 | 0.7000 | 0.8914 | 0.09 | 24.92 | 0.0032 | 0.04 | 7.1315 | 12.0256 | −0.0873 | 2.4239 |
| 5.3058 | 0.8000 | 0.9137 | 0.09 | 27.94 | 0.0032 | 0.05 | 7.5911 | 10.8446 | −0.3003 | 2.8903 |
| 6.8596 | 0.9000 | 0.9456 | 0.10 | 29.99 | 0.0031 | 0.05 | 8.3452 | 8.7589 | −0.9607 | 4.3286 |

| $Z_n^2$ | $\epsilon_{n_1}$ | $\epsilon_{n_2}$ | $10^4 Z_a^2$ | $10^4\,\epsilon_{a_1}$ | $\epsilon_{a_2}$ | $10^4\,\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $\sigma_4^2$ | $\sigma_5^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\varepsilon = 25$ | | | | | |
| 9.6270 | 0.00015 | 0.8127 | 94418.84 | 1.49 | 0.8090 | 48602 | 8.3405 | 25.9488 | 19.4360 | 20.0013 |
| 10.0840 | 0.0002 | 0.8267 | 3761.04 | 1.08 | 0.4454 | 11375 | 11.5932 | 33.6104 | 6.8376 | 9.4239 |
| 11.0082 | 0.0003 | 0.8672 | 791.85 | 1.06 | 0.3054 | 1255 | 13.4165 | 36.2369 | 1.0958 | 3.4440 |
| 11.0126 | 0.0004 | 0.8670 | 552.26 | 1.25 | 0.2710 | 644 | 13.7669 | 36.9169 | 0.6936 | 2.9447 |
| 11.0185 | 0.0005 | 0.8671 | 426.11 | 1.42 | 0.2471 | 406 | 13.9569 | 37.2825 | 0.4904 | 2.7189 |
| 7.2235 | 0.0005 | 0.6988 | 1278.58 | 1.42 | 0.3044 | 4441 | 12.1491 | 22.0692 | 3.9631 | 5.1139 |
| 10.3549 | 0.0006 | 0.8360 | 322.83 | 1.57 | 0.2190 | 276 | 14.2141 | 34.0576 | 0.5535 | 2.1992 |
| 10.3700 | 0.0007 | 0.8366 | 273.23 | 1.72 | 0.2054 | 211 | 14.2947 | 34.2471 | 0.4699 | 2.1281 |

TABLE 1 (CONTINUED)

| $Z_n^2$ | $\epsilon_{n_1}$ | $\epsilon_{n_2}$ | $10^4 Z_a^2$ | $10^4\,\epsilon_{a_1}$ | $\epsilon_{a_2}$ | $10^4\,\sigma_1^2$ | $\sigma_2^2$ | $\sigma_3^2$ | $\sigma_4^2$ | $\sigma_5^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $\varepsilon = 25$ | | | | | |
| 10.1198 | 0.0008 | 0.8254 | 204.93 | 1.76 | 0.1821 | 139 | 14.4254 | 33.3235 | 0.4221 | 1.9361 |
| 10.1303 | 0.0009 | 0.8257 | 138.70 | 1.68 | 0.1540 | 82 | 14.5271 | 33.4946 | 0.3289 | 1.8759 |
| 10.1377 | 0.0010 | 0.8258 | 73.92 | 1.40 | 0.1154 | 39 | 14.6116 | 33.6139 | 0.2559 | 1.8464 |
| 10.0944 | 0.0050 | 0.8237 | 0.54 | 0.04 | 0.0101 | 0.3 | 14.6745 | 33.4745 | 0.2168 | 1.8249 |
| 10.1019 | 0.0100 | 0.8240 | 0.16 | 0.08 | 0.0055 | 0.08 | 14.6739 | 33.5038 | 0.2154 | 1.8281 |
| 10.0969 | 0.0500 | 0.8240 | 0.11 | 2.73 | 0.0045 | 0.06 | 14.6863 | 33.4643 | 0.2156 | 1.8272 |
| 10.0971 | 0.1000 | 0.8247 | 0.03 | 2.95 | 0.0024 | 0.02 | 14.7110 | 33.4031 | 0.2133 | 1.8309 |
| 10.0996 | 0.2000 | 0.8277 | 0.03 | 6.13 | 0.0025 | 0.02 | 14.8235 | 33.1522 | 0.2043 | 1.8463 |
| 10.1115 | 0.3000 | 0.8328 | 0.04 | 9.73 | 0.0027 | 0.02 | 15.0185 | 32.7134 | 0.1882 | 1.8744 |
| 10.1468 | 0.4000 | 0.8403 | 0.05 | 12.79 | 0.0027 | 0.02 | 15.3085 | 32.0504 | 0.1632 | 1.9199 |
| 10.2329 | 0.5000 | 0.8505 | 0.05 | 15.33 | 0.0026 | 0.02 | 15.7151 | 31.1007 | 0.1256 | 1.9920 |
| 10.4249 | 0.6000 | 0.8642 | 0.05 | 17.26 | 0.0025 | 0.02 | 16.2753 | 29.7538 | 0.0675 | 2.1100 |
| 10.8481 | 0.7000 | 0.8822 | 0.06 | 21.08 | 0.0027 | 0.03 | 17.0562 | 27.8031 | −0.0302 | 2.3198 |
| 11.8510 | 0.8000 | 0.9065 | 0.07 | 23.45 | 0.0026 | 0.03 | 18.1970 | 24.8046 | −0.2286 | 2.7619 |
| 14.9171 | 0.9000 | 0.9412 | 0.07 | 25.47 | 0.0027 | 0.04 | 19.5131 | 20.0724 | −0.8513 | 4.1396 |

[a] Notation: $\varepsilon+1$ is the density ratio $\rho_n\rho_a^{-1}$; $Z^2$ refers to squared vorticity normalized to $4\pi G\rho_a$; $\epsilon_{n_1}$, $\epsilon_{a_1}$, $\epsilon_{n_2}$, and $\epsilon_{a_2}$ are the equatorial (1) and meridional (2) eccentricities of the nucleus ($n$) and the atmosphere ($a$); $\sigma^2$ is the squared frequency normalized to $\sqrt{\pi G\rho_a}$; subscripts in $\sigma^2$ are used as follows: 1, 2, for the even modes, 3, 4, 5 for the odd modes.

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