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VISCO-ELASTIC STRAIN CAUSED BY A MONOTONOUSLY INCREASING STRESS

by

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Abstract

The purpose of this investigation is to present a differential equation for the calculation of the compression of a centrally loaded straight bar where time is the argument and the compressive stress is a monotonously changing function of time. The solution of this differential equation is given in the case where the stress increases linearly. Likewise, the solution of the differential equation is given when the stress is constant. The comparison between the results of the tests performed on bars of Finnish pinewood and the theoretically calculated results is finally presented. The theory set forth in this publication may be applicable to any material the visco-elastic characteristic of which likes that of wood *e.g.* to ice (see SALA and OLKKONEN [4]).

1. *Introduction*

Irreversibility is characteristic of the visco-elastic behaviour of a material. It is not possible to present the dependency of the strain on the stress as a single valued function where time is the independent variable. The corresponding values of the stress and the strain essentially depend on the stress-strain history.

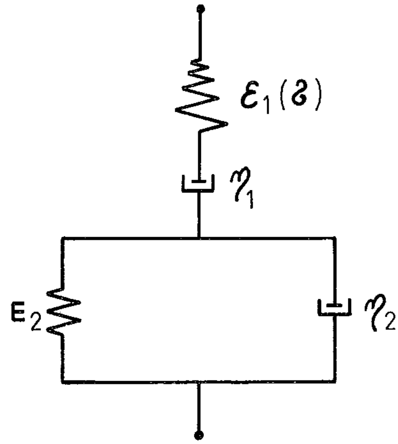

Fig. 1. A nonlinear visco-elastic model.

2. *Monotonously increasing stress*

The nonlinear visco-elastic model presented in Fig. 1 is taken for the basis of this investigation. This was obtained from the well-known linear Maxwell—Kelvin model by replacing the linear spring in the Maxwell unit with a nonlinearly functioning device which gives the compression for each value of the compression stress σ at the time $t = 0$. In Fig. 1 this initial compression has the symbol $\varepsilon_1(\sigma)$ and in this investigation it is represented by the actual compressive stress-strain curve of the material in question or by the mathematical expression corresponding to it.

The differential equation of the strain in the visco-elastic model given in Fig. 1 is

$$\dot{\varepsilon} + \frac{\varepsilon}{\tau} = \sigma(t)\left(\frac{1}{\eta_1} + \frac{1}{\tau E_2}\right) + \frac{\varepsilon_1(\sigma)}{\tau} + \varepsilon_1'(\sigma)\dot{\sigma} + \frac{1}{\tau\eta_1}\int_0^t \sigma(t)dt. \qquad (1)$$

Here $\tau = \frac{\eta_2}{E_2}$, $\sigma = \sigma(t)$ is a monotonously changing function of time t, $\varepsilon_1'(\sigma)$ is the derivative of $\varepsilon_1(\sigma)$ while the dot represents the differentiation with respect to time.

3. *Linearly increasing stress*

Consider $\sigma = kt$ where k is a positive constant. The deformation law of YLINEN [5, 6] can be used as the mathematical representative of the initial compression $\varepsilon_1(\sigma)$:

$$\varepsilon_1 = \frac{1}{E}\left[c\sigma - (1-c)\ln\left(1 - \frac{\sigma}{\sigma_y}\right)\right], \tag{2}$$

where E is the modulus of elasticity, σ_y is the yielding stress and c is a material constant whose numeric value *e.g.* for Finnish pinewood is ca. 0.9 (see *e.g.* SALA [3]).

For the sake of less complicated mathematical calculations the law presented by NEUBER [2]

$$\varepsilon_1 = \frac{1}{E}\frac{\sigma}{\sqrt{1 - \left(\frac{\sigma}{\sigma_y}\right)^2}} \tag{3}$$

was, however, used. Expression (3) may be replaced by the first three terms of its Taylor series provided that σ does not come too near to σ_y

$$\frac{E}{\sigma_y}\varepsilon_1 = \frac{\sigma}{\sigma_y} + \frac{1}{2}\left(\frac{\sigma}{\sigma_y}\right)^3 + \frac{3}{8}\left(\frac{\sigma}{\sigma_y}\right)^5, \tag{4}$$

when

$$E\varepsilon_1' = 1 + \frac{3}{2}\left(\frac{\sigma}{\sigma_y}\right)^2 + \frac{15}{8}\left(\frac{\sigma}{\sigma_y}\right)^4 .$$

Thus the differential equation of the strain in question is

$$\tau\varepsilon + \varepsilon = A\tau + (A + B + 2G\tau)t + (3C\tau + G)t^2 + Ct^3 + 5D\tau t^4 + Dt^5 \tag{5}$$

which with the initial condition $\varepsilon = 0$ when $t = 0$ gives

$$\varepsilon = B\tau(e^{-t/\tau} - 1) + (A + B)t + Gt^2 + Ct^3 + Dt^5 \tag{6}$$

where $A = \frac{k}{E}$, $B = \frac{k}{E_2}$, $C = \frac{k^3}{2E\sigma_y^2}$, $D = \frac{3k^5}{8E\sigma_y^4}$ and $G = \frac{k}{2\eta_1}$.

4. *The visco-elastic creep curve caused by constant stress*

For the determination of the numeric values of the coefficients E_2, η_1 and η_2 the differential equation (1) is solved in the case of a constant

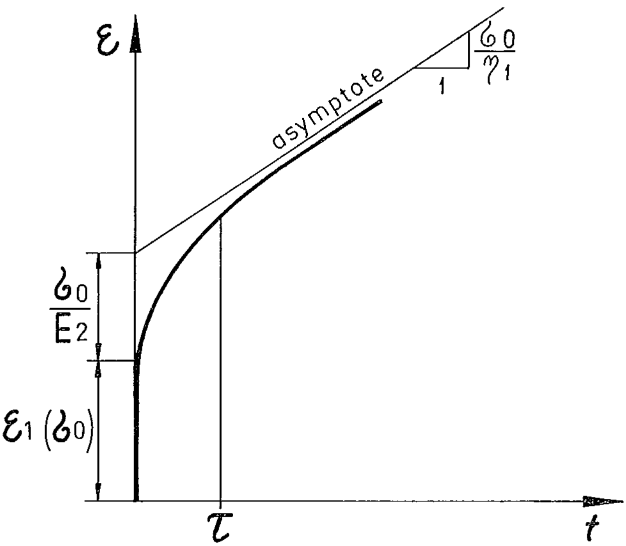

Fig. 2. The visco-elastic creep curve caused by constant stress.

stress, consequently $\sigma = \sigma_0$ and $\dot{\sigma} = 0$. Thus a solution with the initial condition $\varepsilon = \varepsilon_1$ when $t = 0$ is obtained

$$\varepsilon = \varepsilon_1(\sigma_0) + \frac{\sigma_0}{\eta_1} t + \frac{\sigma_0}{E_2} (1 - e^{-t/\tau}) . \tag{7}$$

Expression (7) could have been directly obtained as a sum of the strains obtained by the Maxwell unit and the Kelvin unit, respectively. The graph of equation (7) is shown in Fig. 2. That value of time t, which gives for the expression in parenthesis in the last term of expression (7) the value $1 - 1/e = 0.632$, may be held as τ (see *e.g.* CALCOTE [1]).

5. *Tests with pinewood bars*

The creep curve tests with a constant stress: The tests were performed with compression bars of Finnish pinewood whose specific weight was 0.56, the percentage of moisture ca. 8 and the summerwood content ca. 30%. The test bars were parallelepipeds with a cross-sectional area of 20×20 mm^2 and with a length of 100 mm. There were altogether 6 test bars and the modulus of elasticity E and the yielding stress σ_y were determined for each bar separately. The test results are given in Table 1.

Table 1.

N:o	E kp/cm²	σ_y kp/cm²	σ_0 kp/cm²	$1/\eta_1$ cm²/kp h	$1/E_2$ cm²/kp	τ h	ε_1	t_∞ h
1	$10.5 \cdot 10^4$	710	550	$4.0 \cdot 10^{-8}$	$1.09 \cdot 10^{-6}$	0.5	$7.0 \cdot 10^{-3}$	5
2	10.5	600	600	9.6	1.82	1.0	8.0	8
3	9.3	722	600	8.9	5.9	5.0	7.9	35
4	8.3	700	500	0	5.9	13.0	7.1	70
5	8.2	695	450	0	3.3	5.0	6.7	70
6	8.3	720	550	0	3.5	8.0	7.5	70

There is reason to observe that for each of the bars numbered 4, 5 and 6 the coefficient $1/\eta_1 = 0$. This signifies that the asymptote of the creep curve is parallel to the argument axle for these bars. This appears to have a connection with the fact that the difference $\sigma_y - \sigma_0$ for these bars is greater than that for the bars 1, 2 and 3. In fact this is very self-evident. May it further be mentioned that t_∞ gives the time point at which the test curve reaches the asymptote at the test accuracy.

The test with a linearly growing stress: For the rate of increase of the stress the numeric value $k = 43$ kp/cm² in an hour was chosen. As an arithmetical mean the values $1/\eta_1 = 0$, $\tau = 8.7$ h and $E_2 = 2.6 \cdot 10^5$ kp/cm², based on the bars N:os 4, 5 and 6, and the values $\sigma_y = 690$ kp/cm² and $E = 9.2 \cdot 10^4$ kp/cm², based on all the bars were used. The compressions caused by the linearly increasing stress in the bar were calculated from equation (6) for some discrete time values corresponding to the same range of size as the σ-values in Table 1 and they appear in Table 2 in the column ε_c. The corresponding test values ε_t were taken from the test curve. The values of the difference $\varepsilon_t - \varepsilon_c$ presented in the last column of Table 2 may show that the accordance between the theory and the test results is good.

Table 2.

t h	σ kp/cm²	ε_t 10^{-4}	ε_c 10^{-4}	$\varepsilon_t - \varepsilon_c$
9.3	400	59.8	58.6	+1.2
10.0	430	65.9	65.2	+0.7
10.7	460	72.1	72.5	−0.4
11.4	490	79.4	80.2	−0.8
12.1	521	88.0	88.9	−0.9
12.8	550	98.5	97.8	+0.7

6. *Conclusions*

Unfortunately the possibilities of the authors in the performance of the tests were so restricted that *e.g.* the linearly increasing stress tests had to be reduced to one single test. Besides, the visco-elastic creep tests could not be performed with low constant stress values for the reason that the equipment could only deficiently register so small compression values. All that must be borne in mind when one takes an attitude towards our test results.

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REFERENCES

1. CALCOTE, LEE R., 1968: *Introduction to continuum mechanics.* U.S.A. p. 180 etc.
2. NEUBER, H., 1958: *Kerbspannungslehre.* Zweite erweiterte Auflage, Berlin, S. 188 etc.
3. SALA, I., 1951: Über die unelastische Knickung eines verjüngten Stabes. (English abstract: On the inelastic buckling of a tapered bar.) *Finland's Institute of Technology. Scientific Researches N:o 3.* Helsinki.
4. —»— and OLKKONEN, E., 1971: On the visco-elastic properties of ice and wood. *Geophysica,* **11**, 249—260.
5. YLINEN, A., 1948: Eräs aksiaalisen jännitystilan muodonmuutosfunktio ja sitä vastaava nurjahduskaava. (Finnish original.) *Tekn. Aikl.* **38**, 9—14. Helsinki.
6. —»— 1956: A method for determining the buckling stress and the required cross-sectional area for centrally loaded straight columns in elastic and inelastic range. *Publ. IABSE,* **16**, 529—550.