Topology Knowledge Versus Fault Tolerance:The Case of Probabilistic Reliable Communication Or: How Far Must You See to Hear Reliably

[Extended Abstract]

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Abstract. We consider the problem of probabilistic reliable communication (PRC) over synchronous networks modeled as directed graphs in the presence of a Byzantine adversary when players' knowledge of the network topology is not complete. We show that possibility of PRC is extremely sensitive to the changes in players' knowledge of the topology. This is in complete contrast with earlier known results on the possibility of perfectly reliable communication over undirected graphs where the case of each player knowing only its neighbours gives the same result as the case where players have complete knowledge of the network. Specifically, in either case, $(2t+1)$-vertex connectivity is necessary and sufficient, where t is the number of nodes that can be corrupted by the adversary [2, 9]. We introduce a novel model for quantifying players' knowledge of network topology, denoted by $\mathcal{TK}$. Given a directed graph G, influenced by a Byzantine adversary that can corrupt up to any t players, we give a necessary and sufficient condition for possibility of PRC over G for any arbitrary topology knowledge $\mathcal{TK}$.

Keywords: reliable communication, topology knowledge, synchronous networks, directed graphs, Byzantine adversary

1 Introduction

A number of non-trivial fault-tolerant distributed computing problems may be abstracted as : For a network $\mathcal{N}$ (modeled as a graph), a problem π, and an adversary $\mathcal{A}$, given any two of them, give a relation of the third with respect to the other two. For example, supposing problem π and adversary $\mathcal{A}$ are given, give the condition on $\mathcal{N}$ such that π is solvable. In all such cases, it is *generally* assumed that the players in $\mathcal{N}$ have complete knowledge of the network topology, that is, they know the graph G, which makes up $\mathcal{N}$. However, this assumption is not true in most practical circumstances such as the Internet. Moreover, theoretically, it is curious to consider the effect of players with varying amounts of knowledge about the network topology in fault-tolerant distributed computation. In this paper, we refer to the notion of knowledge possessed by a node about the graph G by Topology Knowledge $\mathcal{TK}$ at that node. Considering $\mathcal{TK}$ as another parameter in our abstraction, we ask: For a network $\mathcal{N}$, a problem π, an adversary $\mathcal{A}$, and topology knowledge $\mathcal{TK}$ given for each node in $\mathcal{N}$, give the condition on $\mathcal{N}$ such that π is solvable.

To the best of our knowledge, there has not been any *focused* study in the literature on how $\mathcal{TK}$ affects distributed computability *in the presence of an adversary*. This is probably the first such attempt. We give the first results showing that including $\mathcal{TK}$ as a parameter affects fault-tolerant distributed computation. We show our results by taking the problem of probabilistic reliable communication (PRC) between a sender S and a receiver R as an instance.

In the extant literature, there are studies such as the one on *Sense of Direction*, which is a topological labeling property, studied as a part of larger study on structural knowledge. The literature indicates that this property is a fundamental requirement for distributed computability (See [4, 3]). These studies also have results on distributed complexity for various topologies. Our

work has no current relation with the extant studies in structural knowledge. In this paper, we consider labeling of the network as fixed, similar to that in [9].

One can easily note that the notion of $\mathcal{TK}$ is relevant only in the presence of a suitable adversary. In the absence of a suitable adversary, each node can let the rest of the players (of course only as far as the connectivity permits!) know its view of the network in order to end up with a global common picture of the topology for each of the connected components respectively. Consequently, while the distributed *complexity* may increase, the distributed *computability* is unaffected. However, in the presence of the adversary, any amount of communication would entail only an (adversary controlled) approximation of the actual topology, thereby perhaps affecting even the distributed computability. Intuitively though, the adversary can at best completely hide the edges between two faulty players and if lucky, succeed in partially hiding even the edges with one faulty end-node. However, useful messages are seldom transmitted via the aforementioned edges. This gives a feeling that distributed *computability* may not be affected — in fact, for the case of perfectly reliable communication it has been proved that the knowledge of one's neighbors alone is as sufficient as the knowledge of the global topology; specifically, $(2t+1)$-connectivity is necessary and sufficient irrespective of $\mathcal{TK}$, where up to t players are Byzantine faulty (We refer to this as a t-adversary)[9]. Counter-intuitively, for the case of probabilistic reliable communication, we show that the optimal fault-tolerance heavily *depends* on $\mathcal{TK}$.

We now explain as to what we mean by PRC. We consider PRC in synchronous networks (modeled as a directed graph) in the presence of a Byzantine adversary. In the problem of PRC over a synchronous network $\mathcal{N} = (\mathbb{P}, \mathcal{E})$ where $\mathbb{P}$ is the set of vertices and $\mathcal{E}$ denotes the set of arcs/edges in the network, the sender $\mathbf{S} \in \mathbb{P}$ wishes to send a message m to the receiver $\mathbf{R} \in \mathbb{P}$ in a robust manner such that the message is correctly received by $\mathbf{R}$ with a very high probability, in spite of the presence of up to t Byzantine-faulty nodes in $\mathcal{N}$.

The problem of PRC sits well with the direction of study that we take on topology knowledge ($\mathcal{TK}$) versus fault-tolerance ($\mathcal{FT}$). More specifically, our study should be termed $\mathcal{TK}$ versus Byzantine $\mathcal{FT}$, as we work with a Byzantine adversary. The techniques that we adopt (See Section 3,Section 4) to work with an arbitrary $\mathcal{TK}$ in our attempt to find the condition for the possibility of PRC give us a hint at a possible classification (See Section 3) of distributed computing problems w.r.t. $\mathcal{TK}$ versus $\mathcal{FT}$. We have a strong feeling that PRC would be a primitive in such a classification, if ever it is possible. Moreover, the problem of reliable communication is a fundamental primitive for any non-trivial distributed protocol. Specifically, if reliable communication were possible between all pairs of players in the network (with some arbitrary topology knowledge $\mathcal{TK}$) then all the non-faulty players can *simulate* and overlay a complete subgraph among them thereby alleviating the issues posed by the adversary as well as the (lack of) topology knowledge! Thus, in some sense, the problem of probabilistic reliable communication is at the core of understanding the effect of topology knowledge on distributed computing in general.

Related Work: The problem of perfectly reliable communication tolerating a t-adversary over undirected graphs is introduced by Dolev *et al.* [2]. PRC over undirected graphs was introduced by Franklin and Wright [5, 6]. Desmedt and Wang [1] were the first to study the problem of reliable communication over directed networks. $(2t+1)$-connectivity between $\mathbf{S}$ and $\mathbf{R}$ is necessary and sufficient for all the problems given in the prequel. Shankar *et al.* [7] show that the results for the case of possibility of PRC in directed graphs are markedly different from the earlier results, which appear consistent. This difference is because their results show that connectivity requirements on the graph for the possibility of PRC over directed graphs as being the receiver $\mathbf{R}$-specific (See Necessity proof of Theorem 3). In this sequence of study, we introduce and solve the problem of PRC in directed graphs with an additional parameter $\mathcal{TK}$. Our results in this paper subsume all the results mentioned above and generalize it to include $\mathcal{TK}$ as a parameter. The line of study that

we take originated in the paper by Srinathan and Pandurangan in [8], where the authors show the conditions for the possibility of PRC over directed graphs for a non-threshold adversary.

1.1 A Motivating Example

Through an example, we share in brief as to what we do in this paper. Consider the network in Figure 1 influenced by a 1-adversary. Over this network, we show in Lemma 1 that if each player begins with the (topology) knowledge of only his neighbours then PRC from **S** to **R** is impossible. We also show in Lemma 2 that if each player begins with the knowledge of up to his neighbours' neighbours then PRC from **S** to **R** is possible.

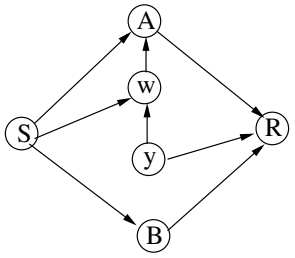

Fig. 1. Graph G

Lemma 1. *PRC from* **S** *to* **R** *in* G *(given in Figure 1) with error probability less than* $\frac{1}{2}$ *is impossible tolerating a* 1*-adversary if each player begins with the knowledge of only his neighbours (both in-neighbours and out-neighbours).*

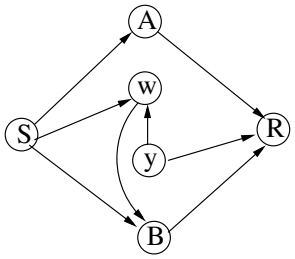

Fig. 2. Graph G'

Proof: Assume that a protocol Π solves PRC (with error probability less than $\frac{1}{2}$) from **S** to **R** in G tolerating 1-adversary. Imagine a graph G' as shown in the Figure 2. If one considers Π as a collection of programs to be run by each of the players in the graph G, then define another protocol Π' such that in Π', the players **S**, **R**, y run the same programs as they run in the protocol Π, the players A and B swap the programs that they run in Π, and the player w runs the program in Π with one change: wherever it is to send its message to A in Π, it sends its message to B in Π'. Notice that this clearly is a PRC protocol from from **S** to **R** in G' tolerating 1-adversary. Consider two executions E_a of Π in G and E_b of Π' in G'. In execution E_a, let **S** transmit a message m_a while the player A is corrupt. In execution E_b, let **S** transmit a message m_b ($m_b \neq m_a$), while the player B is corrupt. Note that initially **R** cannot distinguish between E_a and E_b because the neighbourhood of **R** is identical in G and G' and **R** knows only its neighbourhood.

Using the following strategy, the adversary can ensure indistinguishable views for receiver **R** in each of the executions E_a and E_b. Assuming that all messages are equally likely to be transmitted, **R**'s output would equal m_a with probability at most $\frac{1}{2}$ thereby contradicting the existence of the assumed protocol Π. Here's the strategy: In the execution E_a of Π in G, A forwards to **R** in any round r, a message α_r, where α_r is what an honest A would have sent to **R** in the execution E_b of Π' in G' in round r. Note that this is possible because the adversary knows what an honest A would receive from **S** in any round in execution E_b; this is straightforward since **S** has no incoming edges, and the only incoming edge to A is from **S** and hence adversary can locally simulate Π' with input m_b to arrive at α_r. Correspondingly, execution E_b of Π' in G', B forwards to **R** in any round r, a message β_r, where β_r is what an honest B would have sent to **R** in the execution E_a of Π in G in round r. Note that this is possible because the adversary knows what an honest B would receive from **S** in any round in execution E_a; this is straightforward since **S** has no incoming edges, and the only incoming edge to B is from **S** and hence adversary can locally simulate Π with input m_a to arrive at β_r. Notice that, in the above strategy, all the messages that are sent by w are dropped by the adversary and render no impact on the output of **R**, for, **R** never receives them.

Now, the player y cannot distinguish between E_a and E_b. This is because it has the same neighbourhood in both the graphs and y knows only its neighbours. Since y has no incoming edges,

he can never distinguish between E_a and E_b in any round. Therefore whatever y sends to $\mathbf{R}$ in executions E_a and E_b has to be identically distributed (if the protocol were deterministic it'd have been identical).

Notice that, in any round $\mathbf{R}$ receives messages only from A, B and y. Therefore $\mathbf{R}$ gets identical messages (or identically distributed messages) from each of A, B and y in both the executions E_a and E_b. Thus, under the aforesaid adversary strategy, $\mathbf{R}$ can never distinguish between E_a and E_b. □

Lemma 2. *PRC from* $\mathbf{S}$ *to* $\mathbf{R}$ *in* G *(given in Figure 1) with arbitrarily small error probability is possible tolerating a* 1*-adversary if each player begins with the knowledge of up to his neighbours' neighbours (both in-neighbours and out-neighbours).*

Proof: Notice that, in G, if the players know their neighbours' neighbours, they would get to know the entire topology of the graph G. Consider the following protocol:

Let us assume that the message m to be transmitted is an element of a finite field $\mathbb{F}$. Throughout the protocol, any unreceived messages are stored as $\perp$.

- In round 1, $\mathbf{S}$ sends m to A, B and w. In the same round, y chooses two elements K_1 and K_2 uniformly at random from $\mathbb{F}$ and sends these two elements to both w and $\mathbf{R}$. This completes round 1 and $\mathbf{S}$ and y halt. At the end of round 1, let w receive M_w from $\mathbf{S}$, and W_1, W_2 from y.
- In round 2, Both A and B forward to $\mathbf{R}$ the corresponding received messages from $\mathbf{S}$. In the same round, w sends the following pair to A: $(M_w, W_1M_w + W_2)$. This completes round 2, and both w and B halt.
- In round 3, A forwards the pair received from w to $\mathbf{R}$. This completes round 3 and A halts.

$\mathbf{R}$ receives the following elements at the end of each round:

- At the end of round 1, $\mathbf{R}$ receives two elements from y, say ρ_1, ρ_2.
- At the end of round 2, $\mathbf{R}$ receives X_A from A and X_B from B.
- At the end of round 3, $\mathbf{R}$ receives a pair, say (P,Q) from A.

Finally, $\mathbf{R}$ recovers m as follows:

- If $X_A = X_B$, then $\mathbf{R}$ sets $m \leftarrow X_A$ and halts.
- Else if $(P \neq \rho_1^{-1}(Q - \rho_2))$ OR $(P = \perp)$ OR $(Q = \perp)$ then $\mathbf{R}$ sets $m \leftarrow X_B$ and halts.
- Else $\mathbf{R}$ sets $m \leftarrow P$ and halts.

We give the proof of correctness for the above protocol now:
Case 1: If $\mathbf{R}$ finds that $X_A = X_B$, then, it is certain that neither A nor B has corrupted the message sent by $\mathbf{S}$. Hence $\mathbf{R}$ recovers m correctly and the protocol succeeds. Case 2: If $\mathbf{R}$ finds that $X_A \neq X_B$,then one of A or B is corrupt. Which in turn means that both w and y are honest. So, the data that y sent to w and $\mathbf{R}$ can be treated as a shared secret key between them. Hence, the pair that w sends through A is the message m from $\mathbf{S}$ and its information-theoretic signature. If the signature verification fails, or data not received (check Step 2 above), then $\mathbf{R}$ knows with a high probability, that A is corrupt. Therefore, B is honest,and X_B is the message, which $\mathbf{R}$ recovers and the protocol succeeds. Else if the verification succeeds, then with very high probability, the received message P is the same as m sent by $\mathbf{S}$. Thus $\mathbf{R}$ recovers m correctly and the protocol succeeds. □

Discussion: One can verify that PRC between $\mathbf{S}$ and $\mathbf{R}$ is possible in both the graphs G and G' tolerating a 1-adversary when each player has the knowledge of the entire topology. The protocol given in proof of Lemma 2 for the graph G can be modified suitably to get a protocol for the graph G'. Indeed, both the graphs G and G' satisfy the conditions on graphs for the possibility of PRC as detailed in [7]. We have shown that there exist graphs where unless one has complete knowledge of

the topology, PRC protocols do not exist. As regards this specific example, by showing that PRC is not possible when players have just the knowledge of their neighbours,and it is possible when they have knowledge of up to their neighbours' neighbours - we wish to point out that this could be turned to a distance-measure - where knowledge of one's neighbours is Level 1 knowledge, and knowledge of up to one's neighbours' neighbours as Level 2 knowledge and so on. Clearly, for the graphs G and G', how far must you see to hear reliably? Up to Level 2.

In this paper, we give the complete characterization for the possibility of PRC over any given network modeled as directed graphs under the influence of a (static) t-Byzantine adversary when the players have knowledge of partial topology of the network. Our focus in this paper is on finding the possibility of PRC protocols, and we do not focus on the complexity of the protocols. Our constructions to prove the possibility of PRC protocols lead us to protocols with super-polynomial complexity. We remark that finding polynomial-time solutions for the problem is quite challenging and we do not rule out the possibility of an exponential lower bound. Our characterization theorem implicitly gives the answer to the question of "how far must you see to hear reliably?" - but we found it difficult to give a closed form for the knowledge of up to which Level is needed for the possibility of PRC protocol. We leave that as an open problem.

Organization of the Paper: In Section 2, we give a formal characterization of what $\mathcal{TK}$ entails. In Section 3, we reduce the task of deciding the computability of any problem π with an arbitrary and complex $\mathcal{TK}$ to the task of deciding the computability of π with well-defined and simple $\mathcal{TK}$. We use a reduction from $\mathcal{TK}$ to a uniform $\mathcal{TK}$ so as to solve for possibility of PRC with ease, this is shown in Section 4. Subsequently, we apply our reduction to the problem of PRC and in Section 3 prove a necessary and sufficient condition on $\mathcal{TK}$ for the existence of protocols for PRC with a specified fault-tolerance. Implicit in our results is the fact that optimal fault-tolerance increases as knowledge of the network topology improves.

2 Model and Definitions

The network is modeled as a directed graph $\mathcal{N} = (\mathbb{P}, \mathcal{E})$ where $\mathbb{P}$ is the set of vertices and $\mathcal{E}$ denotes the set of arcs/edges in the directed graph. The system is assumed to be synchronous, that is, the protocol is executed in a sequence of *rounds* wherein in each round, a player can perform some local computation, send new messages to his out-neighbors, receive the messages sent in that round by his in-neighbors (and if necessary perform some more local computation), in that order. In the graph, we assume that the channels are *secure*. In other words, if $(u, v) \in \mathcal{E}$ then *the player u can securely send a message to player v in one round*. During the execution, the adversary may corrupt up to any t players. We work with a (static) Byzantine adversary that may completely control all the corrupted players and make them behave in arbitrary fashion. Every honest player that receives a message from its in-neighbor knows the sender as it can identify the channel along which the message is received. We give a couple of definitions before detailing our model further.

Definition 1 (Topology Knowledge of *player i* ($\mathcal{TK}_i$)). *In a given network $\mathcal{N} = (\mathbb{P}, \mathcal{E})$, we define* Topology Knowledge of player i $\mathcal{TK}_i$ *as the knowledge possessed by* player i *about the network $\mathcal{N}$. It is represented by a set of graphs,i.e. $\mathcal{TK}_i = \{G^i_{k_i}\}$, with a condition that one of the graphs from the set $\mathcal{TK}_i$ is the actual graph $\mathcal{N}$ where $1 \leq k_i < 2^{|\mathbb{P}|^2}$ and $1 \leq |\mathcal{TK}_i| < 2^{|\mathbb{P}|^2}$*

Definition 2 (Topology Knowledge $\mathcal{TK}$). *We define the notion Topology Knowledge $\mathcal{TK}$ as the collection of all the individual $\mathcal{TK}_i$'s (for all the players in $\mathbb{P}$). Specifically, $\mathcal{TK} = \{\mathcal{TK}_i | i \in \mathbb{P}\}$.*

We assume each *player i* knows the following at the outset: (a) Set of vertices in $\mathcal{N}, i.e., \mathbb{P}$; (b) Topology Knowledge of *player i*, $\mathcal{TK}_i$. We also assume the worst-case scenario where the adversary knows all the $\mathcal{TK}_i$'s for $i \in \mathbb{P}$ as well as is aware of the actual graph G.

We explain now, as to why $\mathcal{TK}_i$ is so defined. In order that a set of graphs be considered a model of the notion of *knowledge possessed by* player i *about the network* $\mathcal{N}$, we require that the set contains the actual graph $\mathcal{N}$. The intuition behind this way of modeling is based on the assumption (a) above. Given that a *player i* knows the set of players, he also knows that the actual graph G must be one of the $2^{|\mathbb{P}|^2}$ directed graphs that are ever possible among the players. Note that if $\mathcal{TK}_i$ contains all possible graphs it captures the situation where the player i is completely oblivious of the edges in G (including *his own neighbors*). At the other extreme, if $\mathcal{TK}_i$ contains just a single graph G it captures the scenario where the player i is completely knowledgeable about the topology of the network (including the edges between two faulty players!). Note that neither of these extreme cases are likely to occur at runtime since one usually knows at least (the number) of his neighbors and one usually can never be sure about the presence/absence of edges between faulty players.

Note that it is not necessary to represent $\mathcal{TK}_i$ by physically enumerating all the graphs that $\mathcal{TK}_i$ contains. This would certainly be an impractical representation. Rather, it may be enough if the player i has a program/circuit which on input a graph H can decide whether or not $H \in \mathcal{TK}_i$. For instance, if player i is aware of his neighbors, then $\mathcal{TK}_i$ would contain only those graphs which are consistent with respect to player i's neighborhood. This of course can easily be represented by a program that on input a graph H, tests the neighborhood of i in H and decides whether or not H belongs to $\mathcal{TK}_i$. However, we remark that a simple counting argument suggests that there exists a $\mathcal{TK}_i$ that cannot be decided via any *efficient* circuit (or for that matter, inefficiency is inexorable irrespective of what representation is used). Such a $\mathcal{TK}_i$ may entail a super-polynomial surge in the complexity of any protocol wishing to use its knowledge fully. While a practical approach may be to approximate $\mathcal{TK}_i$ to the "nearest" efficiently decidable $\mathcal{TK}'_i$ (thereby losing some information), we stress that this work is about *computability* in the presence of the adversary and we therefore, without loss of generality, take the liberty of not focussing on issue regarding the representation of $\mathcal{TK}_i$. In other words, we model the players (as well as the adversary) as interactive Turing machines with unbounded computing power and *not* as probabilistic polynomial-time Turing machines (PPT). We leave the task of studying the effects of topology knowledge in a model where the players are PPT's as an interesting open problem.

Note 1. When each of the graphs is written in $G = (V, E)$ form, we note that $G^i_k = (\mathbb{P}, \mathcal{E}^i_{k_i})$, and only the edge-set keeps changing across the graphs for each of the players. Therefore, we can replace every graph $G^i_{k_i}$ in the set $\mathcal{TK}_i$ with the set of its edge-sets $\{\mathcal{E}^i_{k_i}\}$ for each *player i* to make matters more convenient. We will be using $\mathcal{TK}_i$ synonymously with $\{\mathcal{E}^i_{k_i}\}$ from now on.

A comment on the above modeling is due: The notion of knowledge of the network topology has a number of elements in it – such as the size of the network, the labeling of the network, edge-sets, location awareness etc. Modeling all the elements of the topology knowledge so as to make it as general as possible is not the focus of this paper. We assume that all other parameters are published but for the edge-set of the network. Thus, our focus is on the effect of the knowledge of edge-set (with other topological aspects as public information) on fault-tolerance in distributed computing. Notice that our model offers flexibility for one to choose between various levels of edge information that may be provided to the players. This concludes our discussion on modeling $\mathcal{TK}$.

Definition 3 (k-sized $\mathcal{TK}_i$ and k-sized $\mathcal{TK}$). *A k-sized $\mathcal{TK}_i$ is defined as $\mathcal{TK}_i$ where $|\mathcal{TK}_i| = k$. If there exists an integer k such that every $\mathcal{TK}_i$ in the collection $\mathcal{TK}$ is ℓ-sized for some $\ell \leq k$, then the topology knowledge $\mathcal{TK}$ is defined as a k-sized $\mathcal{TK}$.*

Note 2. The edge knowledge of a player with 2-sized $\mathcal{TK}_i$ is greater than the edge knowledge of a player with k-sized $\mathcal{TK}_i$, $k > 2$. This follows from the fact that a player with 2-sized $\mathcal{TK}_i$ has

only 2 edge-sets one of which is the correct one as opposed to a player with k-sized $\mathcal{TK}_i$ who has k edge-sets with the knowledge that one of them is the correct edge-set. That is, the respective probabilities of choosing the correct edge-set is much higher at 1/2 for the former case as opposed to $1/k$ for the latter.

In the next two sections, we present techniques to handle arbitrary $\mathcal{TK}$. We use these techniques to show the existence of protocols for probabilistic reliable communication. In Section 3, we show that the solvability of problems with certain properties in the presence of an N-sized $\mathcal{TK}$ can be reduced to the solvability of the problems in the presence of several 2-sized $\mathcal{TK}$s. We give the properties that are to be satisfied by the problems and also show that the problem of PRC satisfies the same. Working with a 2-sized $\mathcal{TK}$ involves working with different players having different views of the topology, and this can get cumbersome. We further reduce the solvability of the problems with a 2-sized $\mathcal{TK}$ to the solvability of the problems with all possible 2-sized $\mathcal{TK}_i$s, each of which is globally declared, so as to result in the network possessing nodes that either know the graph or are confused between the corresponding graphs in the globally declared 2-sized $\mathcal{TK}_i$. This uniformity in the knowledge possessed by the nodes makes it convenient to work with.

3 From N-sized $\mathcal{TK}$ to 2-sized $\mathcal{TK}$

In this section, we show that those problems π that satisfy certain conditions have a property that, over a network N working with an N-sized $\mathcal{TK}$ is equivalent to working with (several) 2-sized $\mathcal{TK}$ with respect to the solvability of π. This equivalence works only for those problems that satisfy the conditions in the Note 3.These conditions are a prerequisite for the correctness of majority voting which we employ to show the equivalence. Majority voting is applied to the outputs obtained on solving π for various different-sized $\mathcal{TK}$s.

Note 3. The conditions and the corresponding equivalence between solvability of problems with N-sized $\mathcal{TK}$ and the solvability of problems using some k-sized $\mathcal{TK}$. where $k <= N$, are as follows:

- **IO Condition:** The problem should be such that its Input-Output relation must be a function, that is, for a given input, there is only a unique output.
- **Input Honesty Condition:** Is there a requirement for the input givers to be honest?
  - If yes, then there is an equivalence that can be shown from N-sized $\mathcal{TK}$ to 2-sized $\mathcal{TK}$.
  - If not,**Output Agreement Condition:** Is there a requirement for agreement amongst the outputs?
    - If yes, **Output Secrecy Condition:**Is there a requirement for the secrecy of outputs?
      - If yes, then there is an equivalence that can be shown from N-sized $\mathcal{TK}$ to k-sized $\mathcal{TK}$, where $k > 2$.
      - If not, then there is an equivalence that can be shown from N-sized $\mathcal{TK}$ to 2-sized $\mathcal{TK}$.
    - If not,then the equivalence is dependent on the Input-Output relation. In other words, it is sensitive to input-output relation and equivalence is not generically obtained.

Definition 4. *Given a graph $G = (\mathbb{P}, E)$ with N-sized $\mathcal{TK} = \{\mathcal{TK}_1, \mathcal{TK}_2, \ldots, \mathcal{TK}_{|\mathbb{P}|}\}$, $N > 2$, where each $\mathcal{TK}_i = \{\mathcal{E}_1^i, \mathcal{E}_2^i, \ldots, \mathcal{E}_k^i\}$, $k \leq N$, and a 2-sized topology knowledge $\mathcal{Z} = \{\mathcal{Z}_1, \mathcal{Z}_2, \ldots, \mathcal{Z}_{|\mathbb{P}|}\}$ with each $\mathcal{Z}_i = \{A_0^i, A_1^i\}$, $1 \leq i \leq |\mathbb{P}|$, such that one of the A_j^i's is exactly the edge-set E (the edge-set of G) while the other is an edge-set (different from E) that is present in $\mathcal{TK}_i$. In other words, there exists a $j \in \{0, 1\}$ such that $E = A_j^i$ and $A_{\bar{j}}^i \in \mathcal{TK}_i$, we say that $\mathcal{Z}$ is derived from $\mathcal{TK}$.*

Note that an N-sized $\mathcal{TK}$ can have up to $(N-1)^{|\mathbb{P}|}$ distinct 2-sized topology knowledge sets derived from it.

Theorem 1. *A protocol Π that solves a problem π, satisfying either IO Condition and Input Honesty Condition or IO Condition and Output Agreement Condition in Note 3, over a network $\mathcal{N}$ with N-sized $\mathcal{TK}$ ($N > 2$) influenced by an adversary $\mathcal{A}$ exists if and only if, for each of the 2-sized topology knowledge ($\mathcal{Z}$) that can be derived from $\mathcal{TK}$, there exists a protocol solving π with topology knowledge $\mathcal{Z}$.*

Proof: The proof follows an induction on the size of $\mathcal{TK}$, making use of majority voting. Refer to Appendix B for the exact proof. □

We wish to remark that the problem of PRC satisfies both the cases of IO Condition and Input Honesty Condition or IO Condition and Output Agreement Condition in Note 3. Even if we assume that the sender **S** were dishonest, since there is only one output from the receiver **R**, the Output Agreement condition is satisfied. Our proof for Theorem 1 gives the glimpse as to how the equivalences in Note 3 can be obtained. When there is a requirement of secrecy of outputs, note that, majority voting would reveal the output, hence, there is a need to consider ways in which you hide the output, and this could be done by an equivalence to a k-sized $\mathcal{TK}$ where $k > 2$. If there is no input-output relation, then we cannot use this approach. Note that if problems could be classified based on the conditions in Note 3, then PRC seems to be primitive, as it has an equivalence to the minimum sized $\mathcal{TK}$ possible w.r.t. $\mathcal{TK}$ versus $\mathcal{FT}$.

4 Towards a More Uniform $\mathcal{TK}$

From the prequel, it is enough to characterize the possibility of PRC for a 2-sized $\mathcal{TK}$. For a set of players $\mathbb{P}$, a 2-sized $\mathcal{TK} = \{\mathcal{TK}_1, \mathcal{TK}_2, \ldots, \mathcal{TK}_{|\mathbb{P}|}\}$, where each $\mathcal{TK}_i = \{\mathcal{E}_0^i, \mathcal{E}_1^i\}$, $\forall i \in \mathbb{P}$. Notice that each player can have a different $\mathcal{TK}_i$, and working with varying $\mathcal{TK}_i$s can become cumbersome. In our characterization, we avoid working directly with 2-sized $\mathcal{TK}$ for this reason. In this section, we show that for any problem π, working with a 2-sized $\mathcal{TK}$ can be brought down to working with a modified $\mathcal{TK}$, say, $\mathcal{TK}'$ which is formed by the intersection of each of the $\mathcal{TK}_i$s in 2-sized $\mathcal{TK}$ and a set denoted by 2-sized $\mathcal{TK}_G$ which in all respects has the properties of a 2-sized $\mathcal{TK}_i$ and is known globally to all the players. There are $2^{|\mathbb{P}|^2}$ possibilities for 2-sized $\mathcal{TK}_G$. We show that solving π using a 2-sized $\mathcal{TK}$ is possible if and only if π can be solved for all the $2^{|\mathbb{P}|^2}$ possibilities for $\mathcal{TK}'$. Following this, we give a necessary and sufficient condition for the possibility of PRC given $\mathcal{TK}'$.

4.1 From 2-sized $\mathcal{TK}$ to $\mathcal{TK}'$

We begin with the following definitions with which we work in this paper:

Definition 5 (Global $\mathcal{TK}_G$ and k-sized $\mathcal{TK}_G$). *For a given network $\mathcal{N} = (\mathbb{P}, \mathcal{E})$, a global $\mathcal{TK}_G$ is defined as a globally published topology information accessible to all the players in the network $\mathcal{N}$. Its representation and properties are similar to that of $\mathcal{TK}_i$. It is represented as a set of graphs, i.e. $\mathcal{TK}_G = \{G_i\}$ with the condition that one of the graphs in $\mathcal{TK}_G$ is the actual graph $\mathcal{N}$, where $1 \leq |\mathcal{TK}_G| < 2^{|\mathbb{P}|^2}$. If $|\mathcal{TK}_G| = k$, then the $\mathcal{TK}_G$ shall be called k-sized $\mathcal{TK}_G$. Similar to Note 1, $\mathcal{TK}_G$ can be represented using the corresponding edge-sets of the graphs in it.*

Definition 6 ($\mathcal{TK}_i'$ and $\mathcal{TK}'$). *Given a network $\mathcal{N} = (\mathbb{P}, \mathcal{E})$, a $\mathcal{TK}_G$ and a $\mathcal{TK}$, then we define $\mathcal{TK}_i'$ as the updated topology knowledge of* player i *from knowing $\mathcal{TK}_G$. Collection of all such $\mathcal{TK}_i's$ forms $\mathcal{TK}'$,i.e., $\mathcal{TK}' = \{\mathcal{TK}_i \cap \mathcal{TK}_G\}$ $\forall i \in \mathbb{P}$. Similar to Note 1, $\mathcal{TK}'$ can be represented using the corresponding edge-sets of the graphs in it.*

Definition 7 (Strong Path). *A sequence of vertices $v_1, v_2, v_3, \ldots, v_k$ is said to be a strong path from v_1 to v_k in the network $\mathcal{N} = (\mathbb{P}, \mathcal{E})$ if for each $1 \leq i < k$, $(v_i, v_{i+1}) \in \mathcal{E}$. Furthermore, we assume that there vacuously exists a strong path from a node to itself.*

Definition 8 (t-(S,R,)-strong-connectivity). *A digraph is said to be t-(**S**,**R**)-strong-connected if the graph is such that there exists at least t vertex disjoint strong paths from **S** to **R**.*

Definition 9 (Semi-Strong Path). *A sequence of vertices $v_1, v_2, v_3, \ldots, v_k$ is said to be a semi-strong path from v_1 to v_k in the network $\mathcal{N} = (\mathbb{P}, \mathcal{E})$ if there exists a $1 \leq j \leq k$ such that the sequence v_j to v_1 as well as v_j to v_k both are strong paths in the network. We call the vertex v_j as the head of the semi-strong path.*

Definition 10 (Weak Path). *A sequence of vertices $v_1, v_2, v_3, \ldots, v_k$ is said to be a weak path from v_1 to v_k in the network $\mathcal{N} = (\mathbb{P}, \mathcal{E})$ if for each $1 \leq i < k$, either $(v_i, v_{i+1}) \in \mathcal{E}$ or $(v_{i+1}, v_i) \in \mathcal{E}$. Furthermore, we assume that there vacuously exists a weak path from a node to itself.*

Definition 11 (PRC Protocol). *Let $\mathcal{N} = (\mathbb{P}, \mathcal{E})$ be a network, with topology knowledge $\mathcal{TK}$, under the influence of a Byzantine adversary that may corrupt up to any t players. We say that a protocol for transmitting a message from **S** to **R** is (t, δ)-reliable if for any valid adversary strategy, the probability that **R** outputs **m** given that **S** has sent **m**, is at least δ where the probability is over the random inputs of all the players and random inputs of the adversary.*

Definition 12 ((Player p)-Group). *(Player p)-group is defined with respect to two t-sized subsets of $\mathbb{P}$, say B_1 and B_2. A p-group in graph G is the set of all players q (including p) such that q has a strong path from it to p not passing through any node in $(B_1 \cup B_2)$.*

Note 4. Following Definition 13, network $\mathcal{N}$ can be divided into four components for a given two t-sized sets, B_1 and B_2 as follows: **R**-group, B_1, B_2 and the rest of the players together as one, say $C = \mathbb{P} \setminus (B_1 \cup B_2 \cup$ **R**-group). Since the sender **S** and the receiver **R** are honest, we consider the case when $\mathbf{S} \in C$. The only other possibility is $\mathbf{S} \in$ **R**-group, in which case the protocol for PRC is obvious. **R**-group has no knowledge of the actual graph $\mathcal{N}$, and each player in **R**-group has the same topology knowledge as that of the globally declared 2-sized $\mathcal{TK}_G$, made of two edge-sets $\{E_0, E_1\}$. Set $X \subset C$. Set $Z \subset C$. Players in $\mathbb{P} \setminus (X \cup$ **R**-group) all have the same topology knowledge $\{E_0, E_1\}$.

Theorem 2. *A protocol Π that solves a problem π over a network $\mathcal{N}$ with 2-sized $\mathcal{TK}$ influenced by an adversary $\mathcal{A}$ exists if and only if, for each of the $\mathcal{TK}'$ sets formed from the $2^{|\mathbb{P}|^2}$ possibilities for the 2-sized $\mathcal{TK}_G$, there exists a protocol solving π with $\mathcal{TK}'$.*

Proof: Only-If part: This is the easier part. It is evident that the topology knowledge $\mathcal{TK}'$ is higher than the topology knowledge $\mathcal{TK}$, since every $\mathcal{TK}_i$ is a superset of $\mathcal{TK}_i' = \mathcal{TK}_i \cap \mathcal{TK}_G$. Therefore, if a protocol Π works correctly over $\mathcal{TK}$, it would vacuously be a correct protocol over $\mathcal{TK}'$ too.
If part: Let there exist protocols for problem π for each of the valid 2-sized $\mathcal{TK}_G$s. We now show that this implies that there exists protocols for π for any valid 3-sized $\mathcal{TK}_G$. Specifically, let a 3-sized $\mathcal{TK}_G$ be $\mathcal{T} = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$. Consider the three 2-sized subsets, namely, $X_1 = \{\mathcal{E}_1, \mathcal{E}_2\}$, $X_2 = \{\mathcal{E}_2, \mathcal{E}_3\}$ and $X_3 = \{\mathcal{E}_1, \mathcal{E}_3\}$. Note that at least two of the above three X_i's are valid global $\mathcal{TK}_G$'s (that is they contain the actual edge set). Suppose the players execute the protocol for solving π for each of these two $\mathcal{TK}_G$s, their outputs must exactly match since the problem is solved over the same network with the same inputs! However, the players are unaware of which of the two X_i's are valid

Definition 13 (Critical Combination). *Given the following:*

- *A network $\mathcal{N} = (\mathbb{P}, \mathcal{E})$, where $\mathbb{P}$ is the set of players, and $\mathcal{E}$, the set of edges between the players in $\mathbb{P}$.*
- *Two players identified as $\mathbf{S}$ (denoting sender) and $\mathbf{R}$ (denoting receiver) such that $\{\mathbf{S}, \mathbf{R}\} \in \mathbb{P}$.*
- *Two t-sized sets - B_1 and B_2, such that $B_1 \subset \mathbb{P}$, $B_2 \subset \mathbb{P}$ in the network $\mathcal{N}$.*
- *A 2-sized $\mathcal{TK}_G = \{E_0, E_1\}$ for the network $\mathcal{N}$.*
- *A 2-sized $\mathcal{TK}$ for the network $\mathcal{N}$.*
- *A set X of players, $X = \{i | \mathcal{TK}'_i = \mathcal{N}$ and $i \in (\mathbb{P} \setminus B_1 \cup B_2 \cup \mathbf{R}\text{-group})\}$; Let $C = (\mathbb{P} \setminus B_1 \cup B_2 \cup \mathbf{R}\text{-group})$; X is a set of players in C which know the actual graph $\mathcal{N}$ after knowing $\mathcal{TK}_G$.*
- *$X \cap \mathbf{R}\text{-group} = \emptyset$.*
- *A set $Z \subset C$ of players that are part of all weak paths from $\mathbf{S}$ to $\mathbf{R}$ and those players that have a semi-strong path from itself to $\mathbf{R}$.*

Network $\mathcal{N}$ is said to be in a Critical Combination if any of the following hold:

- *Either B_1 or B_2 cut across all strong paths between $\mathbf{S}$ and $\mathbf{R}$s.*
- *$B_1 \cup B_2$ cut across all weak paths between $\mathbf{S}$ and $\mathbf{R}$.*
- *$\exists W$ such that every weak path p that avoids both B_1 and B_2 between $\mathbf{S}$ and $\mathbf{R}$ has a node, say w, that has both its adjacent edges (along p) directed inwards and $w \in W$ and the following hold:*
  - *Both B_1 and B_2 are vertex cut-sets between w and $\mathbf{R}$. In other words, every strong path from w to $\mathbf{R}$ passes through both B_1 and B_2.*
  - *For $\alpha \in \{0, 1\}$, B_1 is a vertex cut-set between all nodes in $(W \cup (Z \cap X))$ and $\mathbf{R}$ in the edge-set $E_\alpha \in \mathcal{TK}_G$ and B_2 is a vertex cut-set between all nodes in $(W \cup (Z \cap X))$ and $\mathbf{R}$ in the edge-set $E_{\overline{\alpha}} \in \mathcal{TK}_G$.*

global $\mathcal{TK}_G$s. It turns out that this does not matter since the players could execute protocols for all the three X_i's and perform a majority voting on the outputs to obtain the correct output for problem π for the 3-sized $\mathcal{TK}_G$, namely $\mathcal{T}$.

By induction, one can now solve the problem π for any given 4-sized $\mathcal{TK}_G$ (since any 4-sized $\mathcal{TK}_G$ can be split into three subsets of size ≤ 3 such that at least two of them valid and yield exactly the same output). Continuing further, we find that solving π with any m-sized $\mathcal{TK}_G$ (where $2 < m \leq 2^{|\mathbb{P}|^2}$) is possible if and only if *pi* is solvable for each of its 2-sized subset $\mathcal{TK}_G$s. Notice that the case of $m = 2^{|\mathbb{P}|^2}$ is nothing but the case where $\mathcal{TK}$ is exactly equal to $\mathcal{TK}'$. Hence the theorem. □

5 Complete Characterization of PRC given $\mathcal{TK}'$

We begin with a few definitions and lemmas.

Theorem 3 (Main Theorem). *For $\delta \geq \frac{1}{2}$, a (t,δ)-reliable PRC protocol between $\mathbf{S}$ and $\mathbf{R}$ in the network $\mathcal{N} = (\mathbb{P}, \mathcal{E})$ with a 2-sized $\mathcal{TK}$ tolerating a t-adversary exists if and only if for every B_1, B_2 of size at most t, such that $B_1 \in \mathbb{P}$, $B_2 \in \mathbb{P}$ in the network, Critical Combination [Definition 13] does not occur in $\mathcal{N}$.*

Proof. Necessity: For Necessity,we must show that a network $\mathcal{N}$ that is in Critical Combination guarantees that no PRC protocol from $\mathbf{S}$ to $\mathbf{R}$ exists in $\mathcal{N}$. We now take each of the conditions described for a network to be in Critical Combination (Definition 13) and show that no PRC protocol can exist between S and R when the condition is true.

Note that if the $\mathcal{TK}'_i$ of all the players, following the inputs given in the Definition 13, is a singleton set, that is all the players have identified the actual graph $\mathcal{N}$ of the network, then the requirement for the PRC protocol is, as per the conditions in [7]. The results in [7] have a strong

receiver **R**-specificity. With our results we gain a better insight into this **R**-specificity. In fact, we show that the knowledge of the actual graph or the absence of it for the receiver **R** is the key for the existence or non-existence of the PRC protocol. The following four lemmas shall capture the requirement of our necessity proof.

Lemma 3. *Following Definition 13, if either B_1 or B_2 cut across all strong paths between* **S** *and* **R** *in $\mathcal{N}$, then a PRC protocol between* **S** *and* **R** *does not exist.*

Proof: It is obvious to see that in this case, an adversary that can corrupt up to any t players can corrupt the set B_1 or B_2 that cuts across all strong paths between **S** and **R**, and thereby disconnect the two in which case no PRC protocol can ever exist. □

Lemma 4. *Following Definition 13,if $B_1 \cup B_2$ cut across all weak paths between* **S** *and* **R**, *then a (t,δ)-reliable PRC protocol ($\delta \geq \frac{1}{2}$) between* **S** *and* **R** *does not exist.*

Lemma 5. *Following Definition 13, if $\exists W$ such that every weak path p that avoids both B_1 and B_2 between* **S** *and* **R** *has a node, say w, that has both its adjacent edges (along p) directed inwards and $w \in W$ and both B_1 and B_2 are vertex cut-sets between w and* **R**, *then a PRC protocol between* **S** *and* **R** *does not exist. In other words,if every strong path from w to* **R** *passes through both B_1 and B_2, then a (t,δ)-reliable PRC protocol ($\delta \geq \frac{1}{2}$) between* **S** *and* **R** *does not exist.*

Proofs of Lemmas 4 and 5 are given in Appendix A.

Lemma 6. *Following Definition 13, if $\exists W$ such that every weak path p that avoids both B_1 and B_2 between* **S** *and* **R** *has a node, say w, that has both its adjacent edges (along p) directed inwards and $w \in W$ and for $\alpha \in \{0,1\}$, B_1 is a vertex cut-set between all nodes in $(W \cup (Z \cap X))$ and* **R** *in the edge-set $E_\alpha \in \mathcal{TK}_G$ and B_2 is a vertex cut-set between all nodes in $(W \cup (Z \cap X))$ and* **R** *in the edge-set $E_{\overline{\alpha}} \in \mathcal{TK}_G$. then a ($t$,$\delta$)-reliable PRC protocol ($\delta \geq \frac{1}{2}$) between* **S** *and* **R** *does not exist.*

Proof Outline: The proof of this lemma gives the topology knowledge effect on the possibility of (t,δ)-reliable PRC protocol for a given network modeled as a directed graph. The proof is by contradiction. Our proof follows in the same lines as that of the proof of Lemma 1, and generalizes it to all directed graphs. The complete proof is given in Appendix A.

This completes the necessity part of the proof for Theorem 3.

Proof. Sufficiency: For Sufficiency, Network $\mathcal{N}$ that is not in Critical Combination guarantees the existence of a PRC protocol from S to R in $\mathcal{N}$. So, we prove by giving a protocol and its proof of correctness. Note 4 gives us the glimpse of the state of the network $\mathcal{N}$ following Definition 13. Owing to space constraints, we move the discussion of the protocol to Appendix C. We use a sub-protocol using which we achieve our task. The design of the same is here below.

Designing the sub-protocol $\Pi_{\alpha\beta}$: Critical Combination does not occur in network $\mathcal{N}$ and this implies all that all the conditions that cause critical combination are falsified. We see the consequences of the same here, and use this to design our sub-protocol. Neither of B_α or B_β cut across all strong paths between **S** and **R**. Since B_α and B_β are the sets chosen by adversary one of which it can corrupt, there must be at least one honest strong path from **S** and **R** that does not pass through either B_α or B_β in $\mathcal{N}$. The deletion of both the sets B_α and B_β from the network $\mathcal{N}$ does not cut across all weak paths between **S** and **R**. There must exist at least one honest weak path from **S** to **R** in $\mathcal{N}$ that avoids both the sets B_α and B_β. We start with this honest weak path, say p. We consider the following two cases in the design of the sub-protocol $\Pi_{\alpha\beta}$:

Case (1) :*The path p is such that $w =$* **S***:* In this case, **S** can establish a shared key with the receiver **R** because it has a semi-strong path between it and **R**. Using the key it transmits the

message reliably. The detailed proof as to how it does is given in the Appendix C. Case (2): *The path p is such that there are $k > 0$ players like w ($w \neq \mathbf{S}$), say $w_1, \ldots, w_k$ along p:* We will first consider the case when $k = 1$. For each of the subsequent cases ($k > 1$), we repeat the appropriate protocols given below on all w_i's ($1 \leq i \leq k$) and in the sequel succeed in establishing reliable communication between $\mathbf{S}$ and $\mathbf{R}$ with a high probability. Detailed proofs are given in Appendix C. □

6 Conclusion

We have provided a complete characterization for the problem of Probabilistic Reliable Communication given topology knowledge as a parameter. The significance of our results can be understood from the following implications: (a) *Generalization:* Our results are a strict generalization of the existing results for probabilistic reliable communication [7]; (b) *The "Randomization-effect"*: It is well known that for perfectly reliable communication, fault-tolerance is independent of nodes' knowledge of the network topology [9]; we show that in the case of probabilistic reliable communication, fault-tolerance is extremely sensitive to changes in the knowledge of network topology. (c) *Optimization*: Our results may be used to answer the question: *what is the optimal fault-tolerance that is achievable in reliable communication for a specified $\mathcal{TK}$ and vice-versa.*

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A Proof of Lemma 4, Lemma 5, Lemma 6

Lemma 7. *Following Definition 13,if $B_1 \cup B_2$ cut across all weak paths between* **S** *and* **R**, *then a PRC protocol between* **S** *and* **R** *does not exist.*

Note 4 gives us the glimpse of the state of the network $\mathcal{N}$ following Definition 13.

It is evident from the definition of **R**-group that there do not exist vertices $u \in C$ and $v \in$ **R**-group, such that the edge (u, v) is in $\mathcal{N}$. When $B_1 \cup B_2$ cut across all weak paths between **S** and **R**, there do not exist vertices $u \in C$ and $v \in$ **R**-group, such that the edge (v, u) is in $\mathcal{N}$.

We prove the impossibility even for the best case where every other edge (other than those between C and R-group) exists and when every player knows the actual graph.

Define two executions $\mathbf{E}_1$ and $\mathbf{E}_2$ as follows. In both executions the vertices in **R**-group hold the random inputs $\{\rho_u | u \in$ **R**-group $\}$. In the execution $\mathbf{E}_\alpha \in \{\mathbf{E}_1, \mathbf{E}_2\}$, the Byzantine set B_α is corrupt and the message m_α is transmitted by **S**, the random inputs of the vertices in $(C \cup B_{\overline{\alpha}})$[1] are $\{\rho_u | u \in (C \cup B_{\overline{\alpha}})\}$. The behavior of the Byzantine set

B_α in the execution $\mathbf{E}_\alpha$ is to send no message whatsoever to $C \cup B_{\overline{\alpha}}$ and to send to **R**-group exactly the same messages that are sent to **R**-group by the honest B_α in the execution $\mathbf{E}_{\overline{\alpha}}$. In order for the Byzantine set B_α to behave as specified in the execution $\mathbf{E}_\alpha$, the adversary needs to simulate the behavior of $(C \cup B_\alpha)$ in the execution $\mathbf{E}_{\overline{\alpha}}$. To achieve this task, the adversary simulates round-by-round the behavior of the vertices in $(C \cup B_\alpha)$ for the execution $\mathbf{E}_{\overline{\alpha}}$ using $\{\rho_u | u \in (C \cup B_\alpha)\}$ as the random inputs for the vertices in $(C \cup B_\alpha)$. At the beginning of each round, each simulated player has a history of messages that it got in the simulation of the previous rounds and its simulated local random input. The simulated player sends during the simulation the same messages that the honest player would send in the original protocol in the same state. The simulated messages that (players in) B_α sends to **R** are really sent by the players. All other messages are used only to update the history for the next round. The messages which are added to the history of each simulated vertex are the real messages that are sent by players in **R**-group and the simulated messages that are sent by the vertices in $(C \cup B_\alpha)$. No messages from $B_{\overline{\alpha}}$ are added to history. The history of messages of each simulated vertex in execution $\mathbf{E}_\alpha$ is the same as the history of the vertex in execution $\mathbf{E}_{\overline{\alpha}}$. Therefore, the messages sent by B_1 and B_2 to members of **R**-group in both executions are exactly the same and the members of **R**-group and in particular the receiver **R** receive and send the same messages in both executions. Thus, the receiver **R** cannot distinguish whether the set B_1 is corrupt and the message transmitted by **S** is m_1 or the set B_2 is corrupt and the message transmitted by **S** is m_2. Now, consider all the pairs of executions where the random inputs range over all possible values. In each pair of executions, whenever **R** accepts the correct message in one execution it commits an error in the other. Thus, for any strategy by **R** for choosing whether to receive m_1 or m_2 there is some α such that when m_α is transmitted, the receiver accepts m_α with probability at most $\frac{1}{2}$. □

Lemma 8. *Following Definition 13, if $\exists W$ such that every weak path p that avoids both B_1 and B_2 between* **S** *and* **R** *has a node, say w, that has both its adjacent edges (along p) directed inwards and $w \in W$ and both B_1 and B_2 are vertex cut-sets between w and* **R**, *then a PRC protocol between* **S** *and* **R** *does not exist. In other words,if every strong path from w to* **R** *passes through both B_1 and B_2, then a PRC protocol between* **S** *and* **R** *does not exist.*

[1] We denote $\overline{1} = 2$ and vice-versa;

Proof: Note 4 gives us a glimpse of the state of the network $\mathcal{N}$ following Definition 13.

In Lemma 4, we proved that when there are no weak paths between **S** and **R** that avoid B_1 and B_2, PRC protocol does not exist. We now show that in spite of the presence of multiple such weak paths between **S** and **R** that avoid B_1 and B_2, if they have a node of the type w, with both its edges inwards towards w along the path, PRC protocol does not exist when both B_1 and B_2 are vertex cut-sets between w and **R**. We take the case where every player complete knowledge of the topology in spite of which PRC is shown to be impossible.

At least one edge from these weak paths must be from a node in **R**-group to another node in C (since these are paths outside $(B_1 \cup B_2)$ and from **S** to **R**). We will show that removing that edge has no effect on the possibility of PRC thereby proving the required result.

Firstly, how can these edges be useful? The answer is that they can be used by players in **R**-group to send some secret messages to the players in C such that the adversary, oblivious of these messages, cannot simulate the messages of C without being distinguished by **R**-group. However, if we are able to show that no such secret information can help PRC from **S** to **R**, then we are through. We do the same now.

A node x is said to have no influence on **R** if the output of **R** is independent of values sent by x. Otherwise x is said to influence **R**. Consider an edge (y, x) in $\mathcal{N}$ such that $y \in$ **R**-group and $x \in C$. We need to know whether x can influence **R** by using the data received from y. Suppose we manage to show that it cannot then we are through since what it means is that data sent along the edge (y, x) has no effect on **R** and hence can be ignored. We now proceed to prove the same.

Suppose that the node **R** can be influenced by x. This (at least) means that there must be a path $x, w_1, w_2, \ldots, w_q, \mathbf{R}$ in $\mathcal{N}$ such that x transmits some information to w_1, then w_1 transmits some information to w_2 that depends on the information it got from x and so on until some information gets to **R**.[2]

Given that every path from x to **R** passes through some node(s) in B_α *followed* by some node(s) in $B_{\overline{\alpha}}$ for some $\alpha \in \{1, 2\}$, the adversary if it corrupts the α^{th} set in $\mathcal{A} = \{B_1, B_2\}$, does the following: let w_j be the first vertex in B_α on a path from x to **R**. The corrupt w_j ignores the real messages that it gets from the players in $C \cup B_{\overline{\alpha}}$ and thus the messages that it sends do not depend on the message sent by x. Similarly, the messages sent by x when B_α simulates the players in C do not influence the messages it sends to **R** since the path from x to **R** passes through at least one vertex from $B_{\overline{\alpha}}$ and no messages are sent by players in $B_{\overline{\alpha}}$ during the simulation. Thus even if **R** may know that the correct secret (that was exchanged using the edge (y, x)) was not used, he will not know which set in $\mathcal{A}$ to blame. Thus the simulated messages of x have no influence on the messages received by **R** and can be ignored. Hence, the impossibility of PRC proved in Lemma 4 is not altered by using the edges from **R**-group to C. □

Lemma 9. *Following Definition 13, if $\exists W$ such that every weak path p that avoids both B_1 and B_2 between* **S** *and* **R** *has a node, say w, that has both its adjacent edges (along p) directed inwards and $w \in W$ and for $\alpha \in \{0, 1\}$, B_1 is a vertex cut-set between all nodes in $(W \cup (Z \cap X))$ and* **R** *in*

[2] Since the network is synchronous, it may be possible to transmit information without actually sending message bits. However, even such transmissions are possible only between nodes that can actually exchange some message-bits as well. Thus, an information-path is necessarily a physical path too.

the edge-set $E_\alpha \in \mathcal{TK}_G$ and B_2 is a vertex cut-set between all nodes in $(W \cup (Z \cap X))$ and $\mathbf{R}$ *in the edge-set $E_{\overline{\alpha}} \in \mathcal{TK}_G$. then a (t,δ)-reliable PRC protocol ($\delta \geq \frac{1}{2}$) between* $\mathbf{S}$ *and* $\mathbf{R}$ *does not exist.*

Proof: Notice that, in the network $\mathcal{N}$, $X \cap \mathbf{R}$-group $= \emptyset$, and therefore the entire $\mathbf{R}$-group has the same 2-sized $\mathcal{TK}'_i$. Further, the state of the network $\mathcal{N}$ following Definition 13 is such that players either know the actual edge-set or those players that have a doubleton set as their updated topology knowledge shall all have the same $\mathcal{TK}'_i$ which is set to the edge-sets in $\mathcal{TK}_G$. All nodes in $Z \cap X$ have complete knowledge of the topology of the network, and since they have semi-strong paths from themselves to $\mathbf{R}$, each of these players can share a common secret key with $\mathbf{R}$, through the head of the semi-strong paths, that is unknown to the adversary. Nodes in W can have strong paths passing through one of B_1 or B_2, and since they are part of $\mathbf{S}$ to $\mathbf{R}$ weak paths, of which some may be semi-strong paths, those that have semi-strong paths can share a key with $\mathbf{R}$ through the corresponding head of the semi-strong path. Each of these nodes can potentially *influence* $\mathbf{R}$ by exchanging information in secret without the knowledge of the adversary, making use of their shared key. Please note the sense in which *influence* is used is the same as expressed in the proof of Lemma 5. These nodes belonging to $(W \cup (Z \cap X))$ do not make an impact on the output of $\mathbf{R}$ in the same manner as shown in the strategy of the adversary in the proof of Lemma 5.

Assume that a protocol Π solves PRC (with error probability less than $\frac{1}{2}$) from $\mathbf{S}$ to $\mathbf{R}$ in the edge-set E_0 tolerating t-adversary. If one considers Π as a collection of programs to be run by each of the players in the graph G, then define another protocol Π' such that in Π', the players $\mathbf{S}$, $\mathbf{R}$, and players in $\mathbf{R}$-group run the same programs as they run in the protocol Π, the players in sets B_1 and B_2 swap the programs that they run in Π, and the players in the set $(W \cup (Z \cap X))$ runs the program in Π with one change: wherever it is to send its message to B_1 in Π, it sends its message to B_2 in Π'. This protocol Π' is clearly a protocol to solve PRC from $\mathbf{S}$ to $\mathbf{R}$ in the edge-set E_1 tolerating t-adversary.

Consider two executions F_1 of Π in the edge-set E_0 and F_2 of Π' in edge-set E_1. In both executions the vertices in $\mathbf{R}$-group hold the random inputs $\{\rho_u | u \in \mathbf{R}\text{-group }\}$. In the execution $\mathbf{F}_\alpha \in \{F_1, F_2\}$, the Byzantine set B_α is corrupt and the message m_α is transmitted by $\mathbf{S}$, the random inputs of the vertices in $(C \cup B_{\overline{\alpha}})$[3] are $\{\rho_u | u \in (C \cup B_{\overline{\alpha}})\}$. The behavior of the Byzantine set B_α in the execution F_α is to send no message whatsoever to $C \cup B_{\overline{\alpha}}$ and to send to $\mathbf{R}$-group exactly the same messages that are sent to $\mathbf{R}$-group by the honest B_α in the execution $F_{\overline{\alpha}}$. In order for the Byzantine set B_α to behave as specified in the execution F_α, the adversary needs to simulate the behavior of $(C \cup B_\alpha)$ in the execution $\mathbf{E}_{\overline{\alpha}}$. To achieve this task,the adversary simulates round-by-round the behavior of the vertices in $(C \cup B_\alpha)$ for the execution $F_{\overline{\alpha}}$ using $\{\rho_u | u \in (C \cup B_\alpha)\}$ as the random inputs for the vertices in $(C \cup B_\alpha)$. At the beginning of each round, each simulated player has a history of messages that it got in the simulation of the previous rounds and its simulated local random input. The simulated player sends during the simulation the same messages that the honest player would send in the original protocol in the same state. The simulated messages that (players in) B_α sends to $\mathbf{R}$ are really sent by the players. All other messages are used only to update the history for the next round. The messages which are added to the history of each simulated vertex are the real messages that are sent by players in $\mathbf{R}$-group and the simulated messages that are sent by the vertices in $(C \cup B_\alpha)$. No messages from $B_{\overline{\alpha}}$ are added to history. The history of messages of each simulated vertex in execution F_α is the same as the history of the vertex in execution $F_{\overline{\alpha}}$. Therefore, the messages sent by B_1 and B_2 to members of $\mathbf{R}$-group in both executions are exactly the same and the members of $\mathbf{R}$-group and in particular the receiver $\mathbf{R}$ receive and send the same messages in both executions. Thus, the receiver $\mathbf{R}$ cannot distinguish whether the set B_1 is corrupt and the message transmitted by $\mathbf{S}$ is m_1 or the set B_2 is corrupt and the message transmitted by

[3] We denote $\overline{1} = 2$ and vice-versa;

S is m_2. Now, consider all the pairs of executions where the random inputs range over all possible values. In each pair of executions, whenever **R** accepts the correct message in one execution it commits an error in the other. Thus, for any strategy by **R** for choosing whether to receive m_1 or m_2 there is some α such that when m_α is transmitted, the receiver accepts m_α with probability at most $\frac{1}{2}$. □

B Proof of Theorem 1

Theorem 4. *A protocol Π that solves a problem π, satisfying either IO Condition and Input Honesty Condition or IO Condition and Output Agreement Condition in Note 3, over a network $\mathcal{N}$ with N-sized $\mathcal{TK}$ ($N > 2$) influenced by an adversary $\mathcal{A}$ exists if and only if, for each of the 2-sized topology knowledge ($\mathcal{Z}$) that can be derived from $\mathcal{TK}$, there exists a protocol solving π with topology knowledge $\mathcal{Z}$.*

Proof: Necessity: We need to prove that if π cannot be solved with one of the 2-sized topology knowledge $\mathcal{Z}$ derived from $\mathcal{TK}$ then π cannot be solved with $\mathcal{TK}$. The proof is trivial following Note 2. If a protocol does not exist to solve a problem using a greater topology knowledge $\mathcal{Z}$, then it certainly cannot solve using lesser $\mathcal{TK}$.

Sufficiency: We need to prove the statement: If there exists a protocol Π_j that can solve the problem π for the j^{th} 2-sized topology knowledge $\mathcal{Z}_j$ derived from $\mathcal{TK}$, then there exists a protocol Π that can solve π with topology knowledge $\mathcal{TK}$. We give a proof by induction on the size of $\mathcal{TK}_i$.

Base Case: In this case, $N = 3$. Let $\mathcal{TK}_i = \{\mathcal{E}_1^i, \mathcal{E}_2^i, \mathcal{E}_3^i\}$, $1 \leq i \leq |\mathbb{P}|$. Let Π_1 be the protocol that solves π using the topology knowledge $\mathcal{Z}_1$ where $\mathcal{Z}_1$ differs from $\mathcal{TK}$ only with respect to player 1, namely, in $\mathcal{Z}_1$ the topology knowledge of player 1 is set as $\mathcal{Z}_{11} = \{\mathcal{E}_1^1, \mathcal{E}_2^1\}$ and let Π_2 be the protocol that solves π using the topology knowledge $\mathcal{Z}_2$ where $\mathcal{Z}_2$ differs from $\mathcal{TK}$ only with respect to player 1, namely, in $\mathcal{Z}_2$ the topology knowledge of player 1 is set as $\mathcal{Z}_{12} = \{\mathcal{E}_1^1, \mathcal{E}_3^1\}$ and let Π_3 be the protocol that solves π using the topology knowledge $\mathcal{Z}_3$ where $\mathcal{Z}_3$ differs from $\mathcal{TK}$ only with respect to player 1, namely, in $\mathcal{Z}_3$ the topology knowledge of player 1 is set as $\mathcal{Z}_{13} = \{\mathcal{E}_2^1, \mathcal{E}_3^1\}$. Notice that using above three protocols Π_1, Π_2 and Π_3, the problem π can be solved even if the player 1 is not aware of the actual topology. Specifically, all the players run all the three protocols Π_1, Π_2 and Π_3 and obtain three outputs O_1, O_2 and O_3 respectively. Note that the actual graph is part of all the individual player's topology knowledge in at least two of the three cases. Thus, at least two of the three outputs must be same and equal to the output that a protocol solving π would produce. Note that this follows from IO Condition and Input Honesty Condition or IO Condition and Output Agreement Condition in Note 3. IO Condition and Input Honesty Condition together imply that there is a unique output for a given input for the problem π. IO Condition and Output Agreement Condition force the inputs to be consistent so as to make the outputs same. Thus, every player can take the majority of the three outputs and thereby solve π. Thus, a protocol for π with topology knowledge $\mathcal{TK}$ exists if and only if a protocol for solving π exists with each of the two valid topology knowledge sets (i.e. those that contain the actual edge-set) among $\mathcal{Z}_{11}, \mathcal{Z}_{12}$ and $\mathcal{Z}_{13}$.

Note that in a valid $\mathcal{Z}_{jk}$, the number of elements in the first player's topology knowledge is two (and not three). Now, we repeat the idea for *player 2*; that is, we create three more topology knowledge sets from each of the three $\mathcal{Z}_{jk}$'s with the second player's topology knowledge split into three parts such that each edge-set occurs in two of them. This would imply that a protocol for π with topology knowledge $\mathcal{TK}$ exists if and only if a protocol for solving π exists with (several) valid

topology knowledge sets with the size of the topology knowledge sets of the players 1 and 2 being of size two. Repeating this process of all the $|\mathbb{P}|$ players, we can say that if some protocol solves a problem for each of its 2-sized $\mathcal{TK}$s derived from 3-sized $\mathcal{TK}$, then Π solves the problem with 3-sized $\mathcal{TK}$.

Induction Hypothesis: Let us suppose that the statement is true for up to any m-sized $\mathcal{TK}$. That is to say, If π is solvable for each of its 2-sized $\mathcal{TK}$s derived from the m-sized $\mathcal{TK}$, then π can be solved with topology knowledge $\mathcal{TK}$.

Induction: Let $\mathcal{TK}_i$ be $m+1$-sized $\mathcal{TK}_i$. Since $m+1 > 3$, we can partition $\mathcal{TK}_i$ into 3 sets, say A, B, C each of which is of size less than $m+1$ such that every element in $\mathcal{TK}_i$ occurs in two among A, B and C. From our induction hypothesis, we know that π can be solved with topology knowledge X_A (which differs from at player i's topology knowledge, namely $\mathcal{TK}_i$ is replaced with A) if it can be solved for each of the 2-sized topology knowledge sets derived from X_A. Similarly for X_B and X_C. Also, if π is solved over two of the valid topology knowledge sets among X_A, X_B and X_C, then it can be solved with $\mathcal{TK}$ too (by majority voting). Note that in each of the two valid topology knowledge sets among X_A, X_B and X_C, the size of topology knowledge of player i is less than m. Repeating the above process across all players, we can bring down the size to the topology knowledge to less than m for all players.

Thus, the statement is true for an $m+1$-sized $\mathcal{TK}$.Therefore, by induction, it is true $\forall m \in \mathbb{N}$. □

C Sufficiency Proof of Theorem 3

Proof. Sufficiency: For Sufficiency, Network $\mathcal{N}$ that is not in Critical Combination guarantees the existence of a PRC protocol from S to R in $\mathcal{N}$. So, we prove by giving a protocol and its proof of correctness.

Note 4 gives us the glimpse of the state of the network $\mathcal{N}$ following Definition 13.

Since there are n players, and t can be corrupted, there are $\binom{|\mathbb{P}|}{t}$ options in front of the adversary,that is there are exactly $\binom{|\mathbb{P}|}{t}$ distinct ways of corrupting exactly t players. Let each of the $\binom{|\mathbb{P}|}{t}$ distinct subsets of size t be represented as $\{B_1, B_2, B_3, \ldots, B_{\binom{|\mathbb{P}|}{t}}\}$ where $B_i \subset \mathbb{P}$ and $|B_i| = t$. First, we show how to design a "PRC" sub-protocol assuming that the adversary is allowed to choose only from *two* of the $\binom{|\mathbb{P}|}{t}$ options that originally existed. In other words, we are only concerned about an adversary that may corrupt the players in the set B_α or the set B_β, where $1 \leq \alpha, \beta \leq \binom{|\mathbb{P}|}{t}$ and $\alpha \neq \beta$. Let us denote the resulting sub-protocol as $\Pi_{\alpha\beta}$. In the sequel, we show how to use all the sub-protocols $\Pi_{\alpha\beta}$ (there are clearly $\binom{\binom{|\mathbb{P}|}{t}}{2}$ of them) to design a grand protocol Π that can be proved to be the required PRC protocol. In the Definition 13, the two t-sized sets B_1 and B_2 can be understood as the sets of players that an adversary may corrupt in one of the instances of B_α and B_β.

An honest player is the player not corrupted by the adversary. An honest path is understood as the path that avoids the sets B_α and B_β. An honest player in the network in possession of the actual edge-set behaves differently from an honest player which has a doubleton set. When the player knows $\mathcal{E}$, all their communication, that is, sending and receiving messages is along the lines of edges in $\mathcal{E}$ only. When the player has a doubleton set as its $\mathcal{TK}_i$, it sends messages along both, the edges in actual edge-set $\mathcal{E}$ and the other edge-set in its $\mathcal{TK}_i$. It is never sure which of its message is valid, because all communication along the false edges is lost, and the identity of the false

edges is not with the player. Now, an honest player with a doubleton set as its $\mathcal{TK}_i$ accepts all messages that it receives which it identifies as valid, that is, those belonging to the edges in any of the two edge-sets in its $\mathcal{TK}_i$. All players corrupted by the adversary, w.l.o.g.,can be assumed to know $\mathcal{TK}$ and the actual edge-set $\mathcal{E}$. This is a modest assumption when dealing with an adversary. An honest player drops all messages from a player if it identifies that it is corrupted by the adversary.

Designing the sub-protocol $\Pi_{\alpha\beta}$: Critical Combination does not occur in network $\mathcal{N}$ and this implies all that all the conditions that cause critical combination are falsified. We see the consequences of the same here, and use this to design our sub-protocol.

Neither of B_α or B_β cut across all strong paths between **S** and **R**. Since B_α and B_β are the sets chosen by adversary one of which it can corrupt, there must be at least one honest strong path from **S** and **R** that does not pass through either B_α or B_β in $\mathcal{N}$.

The deletion of both the sets B_α and B_β from the network $\mathcal{N}$ does not cut across all weak paths between **S** and **R**. There must exist at least one honest weak path from **S** to **R** in $\mathcal{N}$ that avoids both the sets B_α and B_β.

We start with this honest weak path, say p. We consider the following two cases in the design of the sub-protocol $\Pi_{\alpha\beta}$:

Case (1) :*The path p is such that $w =$ **S**:* In this case, the path p contains a player y (which may be **S** or **R** too) such that p is the combination of the strong path from y to **S** and the strong path from y to **R**. In other words, $y \in$(**S**-group $\cap$**R**-group). We know that **R**-group has the edge-sets $\{E_0, E_1\}$ as its topology knowledge. If **S**-group$\cap X \neq \emptyset$, then **S** would know the actual graph, else, it would have the same topology knowledge as **R**-group, $\{E_0, E_1\}$. We give the protocol for the case where both **S** and **R** do not have the actual graph $\mathcal{N}$ and are in possession of $\{E_0, E_1\}$, one of which is known to be $\mathcal{N}$, as per the Definition 5. The other case is similar and follows the same approach. *case (i):* Notice that $y \in$ **R**-group, so even y has $\{E_0, E_1\}$ as its topology knowledge. Each of these edge-sets is such that each has a weak path that does not pass through the two sets B_α and B_β. Let the path along the actual edge-set (w.l.o.g, say E_0) be p and along E_1 be p'. In p', we have a y' which has a strong path from it to **S** and **R**. The state as defined in Note 4 is the same in both the edge-sets. Note that all players in p and p' are honest. The protocol that is run on one path p is correspondingly replicated on p'. We give the protocol for p: First, y sends to both **S** and **R**,along its both edge-sets, a message with two parts: one, a set of random keys, two, an array of signatures. Each player appends its signature to the second part of the message as it forwards the message to the next player in the path p. The random keys K_1, K_2 and K_3, along with the list of signatures of the players that the messages have seen, is sent to both **S** and **R** along the path p. **S, R** receive two sets of the same three keys along the actual edge-set, and E_1 from y in p. Along p', suppose the random keys sent by y' be K_1', K_2' and K_3'. If **S, R** receive these three keys along both the edge-sets E_0 and E_1, then they accept them as they cannot distinguish between the two edge-sets as to which is the correct one. **S,R** end up with two distinct sets of keys - (K_1', K_2' and K_3') and (K_1, K_2 and K_3). Next,**S** computes two values: ψ, ψ_1; two signatures: χ,χ_1; where $\psi = (M + K_1)$, $\chi = (K_2(M + K_1) + K_3)$ and $\psi_1 = (M + K_1')$, $\chi_1 = (K_2'(M + K_1') + K_3')$, and M is the message that needs to be reliably transmitted. **S** sends two messages in each edge-set E_0 and E_1 [4] to **R** along *all* the vertex-disjoint strong paths each containing: a value (ψ/ψ_1), a signature(χ/χ_1), and

[4] Note that **S** can verify if it has at least one honest strong path from it to **R** or not, and distinguish it with E_1

an array of signatures. Each player appends its signature to the array of signatures as it forwards the message to the next player in the path p or correspondingly in p'. Now, $\mathbf{R}$ receives two values - two each of ψ' and ψ'_1; two signatures - two each of χ' and χ'_1 along two different paths as are given in each of the edge-sets E_0 and E_1. Notice that, $\mathbf{R}$ has knowledge of (K_1, K_2 and K_3) and (K'_1, K'_2 and K'_3). Hence it can easily verify if $\chi' \stackrel{?}{=} K_2 * \psi' + K_3$ (correspondingly it verifies for χ'_1). $\mathbf{R}$ reacts as follows: If the received value ψ' has a valid signature ($\chi' = K_2 * \psi' + K_3$), then $\mathbf{R}$ outputs $(\psi' - K_1)$ (correspondingly it outputs $(\psi'_1 - K'_1)$ in the other edge-set); furthermore, among all the received values, at least one of them is guaranteed to be valid (because at least one honest strong path exists!). The probability that $\mathbf{R}$ outputs the same message in both the edge-sets is high,namely $1 - \frac{1}{|\mathbb{F}|}$,which can be made $(1 - \delta)$ by suitably choosing $\mathbb{F}$.

Case (2): *The path p is such that there are $k > 0$ players like w ($w \neq \mathbf{S}$), say $w_1, \ldots, w_k$ along p:* We will first consider the case when $k = 1$. For each of the subsequent cases ($k > 1$), we repeat the appropriate protocols given below on all w_i's ($1 \leq i \leq k$) and in the sequel succeed in establishing reliable communication between $\mathbf{S}$ and $\mathbf{R}$ with a high probability. Owing to space constraints rest of the proof is given in Appendix C.

Since we start with the assumption that conditions on Definition 13 are falsified, note that every strong path from w to $\mathbf{R}$ does not pass through both B_α and B_β for a $w \in W$ that is on the honest weak path p from w_1 to $\mathbf{R}$. That is, there must exist a strong path Q from w_1 to $\mathbf{R}$ that does not pass through nodes in either the set B_α or the set B_β.

Recall that p must contain a node y (which may be $\mathbf{R}$) such that there is strong path from y to w_1 (along p) and there is a strong path from y to $\mathbf{R}$ (also along p). In other words, $y \in (w_1 - group \cap \mathbf{R} - group)$. We know that $\mathbf{R}$-group has the edge-sets $\{E_0, E_1\}$ as its topology knowledge. If w_1-group $\cap$ $X \neq \emptyset$, then w_1 would know the actual graph, else, it would have the same topology knowledge as $\mathbf{R}$-group, $\{E_0, E_1\}$. The protocol we give below is in two parts. Part I deals with the protocols to for communication between w_1 and $\mathbf{R}$ for each of the four cases given above. Part II deal with how, from Part I, we move on to ensure reliable communication between $\mathbf{S}$ and $\mathbf{R}$ with a very high probability.

Part I: *Protocols for communication between w_1 and $\mathbf{R}$: case (i):* We give the protocol for the case where both w_1 and $\mathbf{R}$ do not have the actual graph $\mathcal{N}$ and are in possession of $\{E_0, E_1\}$, one of which is known to be $\mathcal{N}$, as per the Definition 5. The other case is similar and follows the same approach. The protocol for w_1, y,$\mathbf{R}$ is similar to the case (i) in Case 1 above till each of w_1, $\mathbf{R}$ end up with two distinct sets of keys - (K'_1, K'_2 and K'_3) and (K_1, K_2 and K_3) just as **S, R** do.

Next, w_1 computes two values: ψ, ψ_1; two signatures: χ,χ_1; where $\psi = (M_{w_1} + K_1)$, $\chi = (K_2(M_{w_1} + K_1) + K_3)$ and $\psi_1 = (M_{w_1} + K'_1)$, $\chi_1 = (K'_2(M_{w_1} + K'_1) + K'_3)$, and M_{w_1} is the message from w_1 that is to be reliably transmitted to $\mathbf{R}$. Let the path along the E_0 be Q and along E_1 be Q'. w_1 sends two messages in each edge-set (E_0 and E_1) to $\mathbf{R}$ along Q and Q' each containing: a value (ψ/ψ_1), a signature(χ/χ_1), and an array of signatures. Each player appends its signature to the array of signatures as it forwards the message to the next player in the path Q or correspondingly in Q'. Now, $\mathbf{R}$ receives two values - two each of ψ' and ψ'_1; two signatures - two each of χ' and χ'_1 along two different paths as are given in each of the edge-sets E_0 and E_1. $\mathbf{R}$ verifies using both the set of keys in its possession. Using these, it can easily verify if $\chi' \stackrel{?}{=} K_2 * \psi' + K_3$. It verifies on all combinations of ψ',ψ'_1,χ' and χ'_1. $\mathbf{R}$ reacts as follows: If the received value ψ' has a valid signature

on at least one of the combination (say $\chi' = K_2 * \psi' + K_3$), then **R** outputs $(\psi' - K_1)$; else (that is if either the signature is invalid($\chi' \neq K_2 * \psi' + K_3$) or the original message is not received), **R** *knows* the identity (among the two possibilities of α or β) of the set that is the corrupt set.

Since we start with the assumption that conditions on Definition 13 are falsified, we note that $\forall$ nodes in $(W \cup (Z \cap X))$, for $i \in \{0, 1\}$, B_1 is not a vertex cut-set to **R** in the edge-set $E_i \in \mathcal{TK}_G$, and B_2 is not a vertex cut-set to **R** in the edge-set $E_{\bar{i}} \in \mathcal{TK}_G$, the path Q completely avoids the players from one of these sets say B_j, $j \in \{1, 2\}$; This clearly means that a faulty path Q (since a wrong message was delivered) entails that set $B_{\bar{j}}$ is corrupt (where $\bar{j} = \{1, 2\} - \{j\}$).

In Case (2) and its sub-cases above, if the set B_j, $j \in \{1, 2\}$, is not corrupt (which means that the other set may be corrupt), then **R** receives the correct message with certainty while the adversary has no information about the message.On the other hand, if the set B_j is corrupt, then though the adversary still has no information about the transmitted message, he has complete control over **R**'s output. **R**'s output could therefore either be a valid message or a null message with the knowledge that (any subset of) B_j is corrupt. But, if **R** receives a valid message, it is the correct message with a very high probability.

Protocols for the four cases in Part I aim at one of the following: (a) Simulating a direct edge (in other sense, having a strong path that passes only through honest players) between w_1 and **R** so that message from w_1 can be successfully communicated to **R** (or) (b) Simulation of the direct edge fails, and **R** identifies which of the two sets in B_α, B_β is corrupt.

Part II: If a protocol in Part I succeeds in (a) above, then depending on whether w_1 (and thereby all such w_i's) belongs to **S**-group or vice-versa, that is, $w_1 \in$ **S**-group or **S** $\in w_1$-group we have two cases. Note that, there will exist w_i's where neither $w_i \in$ **S**-group nor **S** $\in w_i$-group. It is precisely for this reason that we repeat the appropriate protocols given below on all w_i's $(1 \leq i \leq k)$ so as to arrive at a case where one of these two cases occurs.

If $w_1 \in$ **S**-group, then, there exists a direct path from w_1 to **S**. From the protocol in Part I, there is a direct path from w_1 to **R**. That is, $w_1 \in$ **R**-group, which implies that $w_1 \in$ (**S**-group $\cap$ **R**-group). Notice that w_1 is similar to the player y in path p in Case (1) of the sub-protocol $\Pi_{\alpha\beta}$. So, in this case, we follow the protocol in Case (1) to establish reliable communication between **S** and **R**. If **S** $\in w_1$-group, then **S** sends the message that it wants to reliably communicate to **R** through the path between w_1 and **R**. Since the path is secure, and passes only through honest players, reliable communication takes place.

If a protocol in Part I succeeds in (b) above, then we do the following: Since there exists at least one honest strong path from **S** to **R**, it must avoid B_j. **S** sends the message along its both edge-sets which has two parts: the message M and an array of player indices. Each player appends its index to the array of player indices as it forwards the message to the next player along all these paths to **R**. The knowledge that B_j is corrupt is sufficient for **R** to recover the correct message passing through the honest path.

Note that this gives us the protocol for our uniform topology knowledge built with the help of $\mathcal{TK}_G$s. To give the protocol for a given 2-sized $\mathcal{TK}$, we use the means shown in the *If part* proof of the Theorem 2 given in Section 4. This completes our exercise of constructing the sub-protocol

$\Pi_{\alpha\beta}$ that is guaranteed to work correctly only if one of B_α or B_β is chosen by the adversary.

Note that **R** can simulate the sub-protocol $\Pi_{\alpha\beta\gamma}$ which assumes that one among the *three* sets B_α or B_β or B_γ is chosen by the adversary. The simulation is done as follows: **R** takes the majority among the outputs of the three protocols $\Pi_{\alpha\beta}$, $\Pi_{\beta\gamma}$ and $\Pi_{\alpha\gamma}$. A majority is bound to exist since any set chosen by the adversary is tolerated in *two* of the three protocols. Next, R can simulate the sub-protocol which behaves like a PRC protocol as long as any one among a collection of *four* sets is chosen by the adversary. Continuing further, **R** will be able to simulate the protocol that behaves correctly if one among the collection of $\binom{n}{t}$ sets is chosen by the adversary. This protocol by definition is the PRC protocol from **S** to **R**! We conclude the sufficiency part of the proof. □