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Turbo Codes: The Issue of Average Union Upper Bound under Imperfect Channel State Information in Rayleigh Fading Channels

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Abstract

The potential of turbo codes to demonstrate excellent performance in the region of low signal to noise ratio is now a familiar fact. The idea of turbo codes was first introduced by Berrou and was based on the clever utilization of the existing decoding mechanism of BCJR with a change in the encoder by introducing feedback and an interleaver in the overall scheme. Since then, these powerful forward error correcting codes have attained a significant attention and have either replaced or have become a sturdy candidate for many applications in this era of modern communication systems. So far, a lot of work has been published regarding the performance of turbo codes by using both simulation and average upper bounds, but unfortunately, no author has yet obtained a mechanism to resolve the issue of obtaining an average upper bound under imperfect channel state information. In this article we will address the issue of obtaining an average union upper bound for the bit error rate using the transfer function approach. In this work, the bound for coherent BPSK over independent Rayleigh fading channels will be discussed and it will be showed that this way of computing the average union upper bound is a generalized approach for obtaining upper bounds. In order to incorporate imperfect estimation, it is assumed that the estimation errors are Gaussian distributed having a variance equal to a fraction of the variance of normalized Rayleigh distribution. The work also shows that the pairwise error probability expression with no estimation error is a particular instance of an imperfect estimation case.

Key Words: *Turbo code, imperfect estimate, channel state information, pairwise error probability, average union upper bound.*

1. Introduction

Turbo codes have received considerable attention since their introduction in 1993 by Berrou [1]. These powerful codes have diminished the gap between the theoretical limit as proposed by Shannon and the practical attainable performance by a coding scheme. The decoding mechanism for turbo codes is derived from the existing algorithm known as the BCJR algorithm [2]. The performance of this powerful coding scheme is studied either by computer simulation or by utilizing average union upper bound. The performance of turbo codes in the region of low signal to noise ratio is dominated by simulation analysis as the average union bound in this area has a diverging behavior. On the other hand, performance analysis of turbo codes through simulation in the region of high signal to noise ratio is not a simple task, as it involves an immense amount of computing power as well as impractical simulation time. Moreover, attention must be paid to the period of the random number generators for the simulation setup, as it is most likely that most of the standard and familiar random number generators will produce erroneous results due to the exhaustion of their period. Benedetto and Divsalar proposed different union upper bound schemes in [3, 4] for exploring the code behavior in the region of strong signal strength.

The issue of the performance obtained by simulation of turbo codes under imperfect channel state information (ICSI) is studied recently in papers [5, 6] and a mechanism is introduced to obtain channel state information (CSI) from the received sequence as the addition of this factor in the turbo decoder metric improves the overall performance of the system. There is still a lack of work to be done for the theoretical bound to represent the case of ICSI as it is common to observe the performance of turbo codes in the region of high signal strength using the bounds. It is also indispensable to disclose the issue of ICSI as most of the practical channel estimators suffers from overestimation or underestimation errors of the fading channel coefficients.

Here a rate 1/3 turbo code with a memory of two and an input block size of K bits and an output stream of $N = 3(K+2)$ bits will be used to study a (1,5/7,5/7) encoder. Although, the discussion of obtaining the bound is centered at this specific encoder due to its better performance as compared to other memory two encoders, the idea is in general can be applied to obtain theoretical bounds for any other encoding scheme.

The paper is organized as follows. In Section 2 the encoding and the received signal model for turbo codes is presented. Section 3 explains the concept of imperfect estimate. Section 4 provides the details about the pairwise error probability. Section 5 discusses the approach behind transfer function based union upper bounds and provides the results. Finally, Section 6 concludes with the findings of the work.

2. System Model

Turbo codes, a parallel concatenated error correcting scheme is obtained by transmitting together, the message sequence $\mathbf{x}$ called the systematic bit c_0 and the two parity sequences c_1 and c_2, which are obtained by encoding the message sequence and the interleaved message sequence. Figure 1 shows the general encoding structure for a turbo encoding scheme where U represents the interleaving, which is obtained by permuting the message bits in a random order.

In this article, we consider a turbo encoded sequence through an independent Rayleigh fading channel with imperfect channel estimate using antipodal binary phase-shift keying (BPSK) modulation. The discrete time system model with $c_k \in \pm\sqrt{E_s}$, and $1 \leq k \leq (N+2)/R$, is the BPSK transmitted symbol, where N and R represent the information block size and the code rate respectively. For this case, the relationship between the symbol energy and bit energy can be stated as $E_s = RE_b$. The discrete output at the receiving end obtained from the matched filter assuming perfect synchronization at any time instant k is given by

$$r_k = h_k c_k + n_k \tag{1}$$

where

$$h_k = a_k + m_k \tag{2}$$

In the above expressions r_k represents the received signal, h_k is the Rayleigh distributed noisy fading channel coefficient, a_k is the ideal Rayleigh fading coefficient, m_k models the error in the fading coefficient and n_k represents zero-mean additive white Gaussian noise(AWGN).

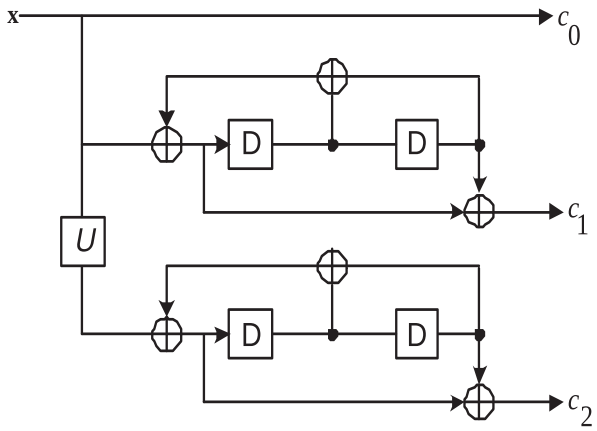

Figure 1. Encoding structure for (1,5/7,5/7) turbo code.

3. Imperfect Estimate

In order to incorporate the effect of channel estimation errors one needs to study a specific channel estimator; different estimators behave differently according to the underlying algorithm. Since it is not the purpose of this article to discuss the behavior of different channel estimators, we will resolve this issue by assuming that the estimation errors correspond to a fraction of the variance of normalized Rayleigh distribution and are Gaussian distributed. The normalized Rayleigh distribution is given as

$$f(a) = 2ae^{-a^2}, \qquad a \geq 0. \tag{3}$$

The normalization condition implies that $E[a^2] = 1$, the mean and variance for the normalized Rayleigh fading model are $m_a = 0.8862$ and $\sigma_a^2 = 0.2146$, respectively. From equation (2) one can observe that m_k is Gaussian distributed with $E[m_k^2] = \sigma_m^2$ and $E[m_k] = 0$. From this, it follows that $E[h_k] = 0.8862$, and since both a_k and m_k are considered to be independent, we obtain $\sigma_h^2 = 1 + \sigma_m^2$. The error in estimates can be written as $\sigma_m^2 = \alpha\sigma_a^2$, with α chosen as $0.0, 0.1, 0.2$ and 0.4 in this work.This described model, though simple is in fact valid for most of the practical channel estimators and will serve well to demonstrate the behavior of upper bounds.

4. Pairwise Error Probability

In the case of coherent BPSK the imperfect channel coefficients can be represented as

$$h_k = a_k + m_k \tag{4}$$

where h_k represents the erroneous channel estimate, a_k models the perfect Rayleigh distributed channel state information and m_k is the estimation error. The pairwise error probability $p(\mathbf{x_n}, \hat{\mathbf{x}}_\mathbf{n})$ is a measure of the probability that the decoder chooses as its estimate a length n sequence $\hat{\mathbf{x}}_{\mathbf{n}} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ when the transmitted sequence was $\mathbf{x}_{\mathbf{n}} = (x_1, x_2, ..., x_n)$. This can be obtained by evaluating the Euclidean distance between the two codewords and can be expressed as

$$\sum_{i=1}^{n} |r_i - h_i x_i|^2 \geq \sum_{i=1}^{n} |r_i - h_i \hat{x}_i|^2 \tag{5}$$

after substituting (4) into (5) and applying simplifications one can obtain the conditional pairwise error probability(CPEP) and the pairwise error probability (PEP) as given by (6) and (7) (see Appendix I for the derivation).

$$p(\mathbf{x_n}, \hat{\mathbf{x}}_\mathbf{n} | \mathbf{h_n}) = Q\left(\sqrt{\frac{2RE_b \sum_{k=1}^{d} h_k^2}{(1+\sigma_m^2)N_0}}\right) \tag{6}$$

$$p(\mathbf{x_n}, \hat{\mathbf{x}}_\mathbf{n}) = E\left[p\left(\mathbf{x_n}, \hat{\mathbf{x}}_\mathbf{n} | \mathbf{h_n}\right)\right] \tag{7}$$

where $E[\cdot]$ is the expected value taken over the noisy estimate vector $\mathbf{h}$ and d is the Hamming distance between $\mathbf{x_n}$ and $\hat{\mathbf{x}}_\mathbf{n}$. Obtaining a closed form expression for PEP is not an easy task and even approximating the error function by an exponential function will not lead to a closed form solution due to the presence of noisy estimation factor. Since, the expression for PEP is obtained as the expected value, rather than trying to integrate the term we can calculate the average statistical mean value of (7) from a_k and m_k by generating them from their distributions. Also it is worth nothing that it is a simple exercise to show that when the estimation errors m_k's are equal to zero equation (6) becomes the standard expression for the case with perfect channel state information (PCSI) which is shown in [7] and is

$$p(\mathbf{x_n}, \hat{\mathbf{x}}_\mathbf{n}) = E\left[Q\left(\sqrt{2\frac{E_s}{N_0}\sum_{i=1}^{d} a_i^2}\right)\right] \tag{8}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt. \tag{9}$$

An exact expression for the pairwise error probability under perfect channel state information is provided in [8] as

$$p_2(d) = G(d, E_s/N_0)\left[1 + E_s/N_0\right]^{-d}, \tag{10}$$

where

$$G(d, E_s/N_0) = \frac{1}{2^{2d}}\left[\sum_{i=1}^{d} \binom{2d-i-1}{d-1} \left(\frac{2}{1+\sqrt{\frac{E_s/N_0}{1+E_s/N_0}}}\right)^i\right] \tag{11}$$

The above expression is computationally more efficient than finding the result through the expected value of the generated random numbers, but is only valid for the PCSI case. However, it will serve to verify our result when the estimation error is zero.

5. BER Bound

The classical rate 1/3 turbo encoder produces a codeword for an input stream by concatenating the three output sequences. Since the codeword is obtained by concatenation of the three sequences, the Hamming weight of the codeword becomes the sum of the individual weights of three sequences (i.e., $d = i + d_1 + d_2$). Following [3] the union upper bound on the average bit error probability becomes

$$\bar{P}_b \leq \sum_{d=d_{min}}^{N} \sum_{i=1}^{K} \sum_{d_1} \sum_{d_2} \frac{i}{K} \binom{K}{i} p(d_1|i) p(d_2|i) p(\mathbf{x_n}, \hat{\mathbf{x}}_\mathbf{n}) \tag{12}$$

where d_{min} is the minimum Hamming distance of the code, $p(\mathbf{x_n}, \hat{\mathbf{x}}_\mathbf{n})$ is the pairwise error probability and $p(d|i)$ denotes the conditional probability of producing a codeword fragment of weight d for a randomly selected input sequence of weight i. In order to evaluate the expression in (12) the distribution of the weight of parity sequences, d_1 and d_2 needs to be determined. The expression for their distribution is obtained as shown by [3] and equals

$$p(d|i) = \frac{t(l,i,d)}{\binom{K}{i}} \tag{13}$$

where d represents the weight of the parity sequence and i is the weight of the message block. Moreover, $t(l,i,d)$ is obtained from the code's transfer function and represents the total number of paths of length l, input weight i, and output weight d, emerging from and terminating in the zero state (see Appendix II for the derivation of $t(l,i,d)$).

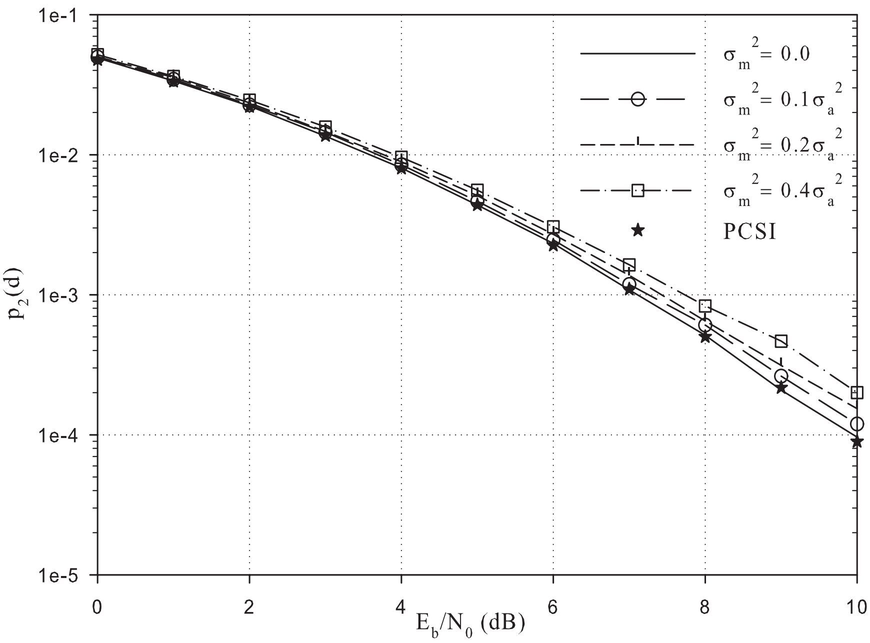

Figure 2. Pairwise error probability plot for $d = 5$.

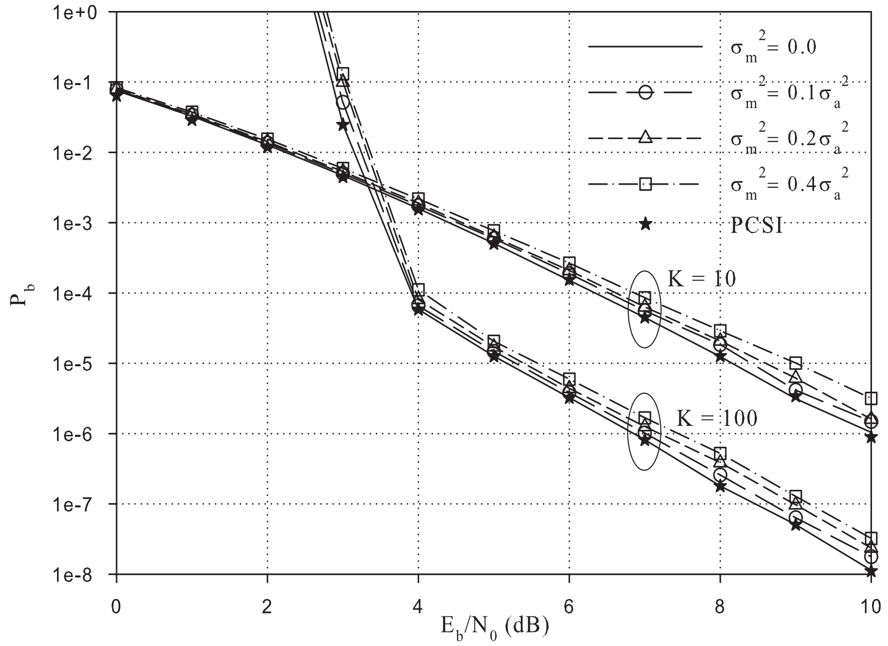

Figure 3. Bit error probability bounds for $(1, 5/7, 5/7)$ scheme.

Figure 2 shows the pairwise error probability plot when $d = 5$. From the plot it is clear that as the estimation error is increased the gap between the perfect and imperfect case also increases. Figure 3 show the BER versus E_b/N_0 bounds for input blocks of size 10 and 100 having $(1, 5/7, 5/7)$ generator polynomial. It is obvious from the plots that as the error in the estimation is increased the bound also shifts showing a trend of performance degradation. It can also be examined that the case with no estimation error case $\alpha = 0$ matches with the PCSI case as provided by [8].

6. Conclusions

In this paper the idea of obtaining BER bounds under ICSI is explained. From the results it is observed that the new pairwise error probability expression serve as a generalized representation and the case of PCSI is a particular instance of it. Additionally, the simple estimation error model helped to avoid the complexity to model channel estimation errors and shows an expected degradation in the performance as the estimation error is increased.

APPENDIX I

The conditional pairwise error probability for a given realization of noisy fading variable $\mathbf{h_n}$ is given by

$$p(\mathbf{x_n}, \hat{\mathbf{x}}_\mathbf{n}|\mathbf{h_n}) = Pr\left(\sum_{i=1}^{n}|r_i - h_i x_i|^2 \geq \sum_{i=1}^{n}|r_i - h_i\hat{x}_i|^2\right) \tag{14}$$

Opening the square and rearranging the terms we get

$$= Pr\left(\sum_{i=1}^{n} 2n_i h_i(\hat{x_i} - x_i) \geq \sum_{i=1}^{n}[h_i^2(\hat{x_i}^2 - x_i^2) - 2h_i(x_i\hat{x_i} - x_i^2)]\right) \tag{15}$$

If we look for differing bits in the transmitted frame, we get

$$= Pr\left(\sum_{i=1}^{n} 2n_i h_i(\hat{x_i} - x_i) \geq \sum_{i=1}^{d}[h_i^2(1-1) - 2h_i(-1-1)]\right) \tag{16}$$

$$= Pr\left(\sum_{i=1}^{n} 2n_i h_i(\hat{x_i} - x_i) \geq 4\sum_{i=1}^{d} h_i^2\right) \tag{17}$$

Let

$$Z = \sum_{i=1}^{n} 2n_i h_i(\hat{x_i} - x_i) \tag{18}$$

then

$$Var(Z) = \sigma_n^2 = 16\sigma^2 \sum_{i=1}^{d} h_i^2 \tag{19}$$

$$p(\mathbf{x_n}, \hat{\mathbf{x}}_\mathbf{n}|\mathbf{h_n}) = Q\left(\frac{4\sum_{k=1}^{d} h_k^2}{\sigma_n}\right) \tag{20}$$

For BPSK

$$\frac{1}{\sigma} = \sqrt{\frac{2E_s}{(1+\sigma_m^2)N_0}} \tag{21}$$

Hence

$$p(\mathbf{x_n}, \hat{\mathbf{x}}_\mathbf{n}|\mathbf{h_n}) = Q\left(\sqrt{\frac{2RE_b\sum_{k=1}^{d} h_k^2}{(1+\sigma_m^2)N_0}}\right) \tag{22}$$

APPENDIX II

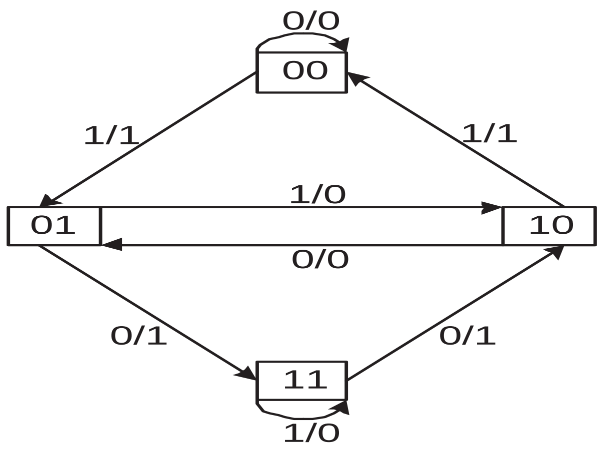

Figure 4. State diagram for (1,5/7,5/7) turbo code.

The information in the state diagram as shown by figure 4 can be summarized by state transition matrix $\mathbf{A}(L, I, D)$

$$\mathbf{A}(L, I, D) = \begin{pmatrix} L & LID & 0 & 0 \\ 0 & 0 & LD & LI \\ LID & L & 0 & 0 \\ 0 & 0 & LI & LD \end{pmatrix} \tag{23}$$

where each edge in the state diagram is replaced by a monomial $L^l I^i D^d$. Here, l is always unity, and i and d are either 0 or 1, which is determined by the corresponding input and output bits. Following [3], the transfer function (generating function) can be defined as

$$T(L, I, D) \approx \frac{det(\mathbf{I} - \mathbf{A})}{det(\mathbf{I} - \mathbf{X})} \tag{24}$$

where $det(\cdot)$ is the determinant operator, $\mathbf{I}$ is the identity matrix and $\mathbf{X}$ is the reduced matrix obtained by eliminating the first row and first column from $\mathbf{A}$. Hence, we have

$$T(L, I, D) \approx \frac{1 - LI - L^2 I - L^3(D^2 - I^2)}{1 - L(1 + I) - L^3(D^2 - I - I^2 + I^3 D^2) + L^4(D^2 - I^2 - I^2 D^4 + I^4 D^2)}. \tag{25}$$

In order to obtain $t(l, i, d)$ we multiply both sides of (25) by the denominator of the right hand side and replace each $T(L, I, D)$ term and its coefficient with $t(l - a, i - b, d - c)$, where a, b and c are the powers of L, I and D respectively. Similarly, each term on the right hand side (the numerator) is replaced with the delta function $\delta(l - a, i - b, d - c)$, where a, b and c represents the power of L, I and D for every coefficient. The recursive equation for $t(l,i,d)$ becomes

$$\begin{aligned} t(l,i,d) = {} & t(l-1,i-1,d) + t(l-1,i,d) + t(l-3,i-3,d-2) - \\ & t(l-3,i-2,d) - t(l-3,i-1,d) + t(l-3,i,d-2) - \\ & t(l-4,i-4,d-2) + t(l-4,i-2,d-4) + t(l-4,i-2,d) - \\ & t(l-4,i,d-2) + \delta(l,i,d) - \delta(l-1,i-1,d) - \\ & \delta(l-2,i-1,d) - \delta(l-3,i,d-2) + \delta(l-3,i-2,d) \end{aligned} \tag{26}$$

where $t(l,i,d) = 0$ if any index is negative and $\delta(l,i,d) = 1$ if $l = i = d = 0$ else $\delta(l,i,d) = 0$.

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