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Dispersion Analysis of the ADI-FDTD and S-FDTD Methods

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Abstract

Numerical dispersion performances of ADI-FDTD and S-FDTD methods have been compared. It has been shown that for time steps below the stability limits of the S-FDTD method it has much better dispersion performance compared with the ADI-FDTD method and that the S-FDTD method can be usefully employed for space increments in the order of $\lambda/25$ *to* $\lambda/50$.

Key Words: *Symplectic finite-difference time-domain (S-FDTD) method, alternating-direction-implicit finite-difference time-domain (ADI-FDTD) method.*

1. Introduction

The finite-difference time-domain (FDTD) method has widely been used for the solutions of electromagnetic problems [1]. The stability condition for this method [1] imposes a limitation on the time step size. When the method is applied to electrically small problems this limitation necessitates unnecessarily small time steps, which considerably increases the computational time. Alternating-direction implicit finite-difference time-domain (ADI-FDTD) method [2] is unconditionally stable and theoretically there is no limitation on the time step size. But as the size of time steps is increased, numerical dispersion errors become large; so the time step size for the ADI-FDTD method is limited in use by the level of the numerical dispersion error that can be tolerated.

On the other hand, Symplectic FDTD (S-FDTD) method [3] is an explicit scheme which uses fourth-order finite differencing for space discretization and exponential differential operators for time discretization. The method reduces the numerical dispersion errors significantly. It has been shown [3] that the stability limit of this method is much higher than the Yee's FDTD method and that the stability limit depends linearly on the number of the exponential coefficients.

In this paper performances of the ADI-FDTD method, with second order and fourth order finite differencing in space, are compared with the performances of the S-FDTD method.

2. Numerical Dispersion Performance

When an electromagnetic problem is simulated in a discretized domain the phase velocity of the electromagnetic wave differs slightly from the phase velocity of the natural medium. The variation in the phase velocity is not constant but varies with the frequency, direction of propagation and the sizes of the time and spatial steps. There have been many publications dealing with the numerical dispersion of the ADI-FDTD method [4–7]. In this paper the dispersion relation formula used by Weiming Fu et al. [5] is used for calculating the three dimensional (3-D) dispersion error of the ADI-FDTD method, which is given by the relation

$$\sin^2(\omega\Delta t) = \frac{4s^2\left[\eta^2 + s^2\left(\eta_x^2\eta_y^2 + \eta_y^2\eta_z^2 + \eta_z^2\eta_x^2\right)\right]\left[1 + s^6\eta_x^2\eta_y^2\eta_z^2\right]}{\left[\left(1+s^2\eta_x^2\right)\left(1+s^2\eta_y^2\right)\left(1+s^2\eta_z^2\right)\right]^2}, \tag{1}$$

where $\eta = \sqrt{\eta_x^2 + \eta_y^2 + \eta_z^2}$, and for second order and fourth order space discretization:

$$\eta_\gamma = \sin\left(\tilde{k}A_\gamma\right); \quad \gamma \in \{x, y, z\} \tag{2}$$

$$\eta_\gamma = \frac{27}{24}\sin\left(\tilde{k}A_\gamma\right) - \frac{1}{24}\sin\left(3\tilde{k}A_\gamma\right), \tag{3}$$

respectively. $\tilde{k}$ represents the numerical wavenumber, Δ $(\Delta = \Delta x = \Delta y = \Delta z)$ is the cell size, Δt is the time increment, and $s = c\Delta t/\Delta$ is the stability factor. Parameters A_γ in these equations are defined as $A_x = \frac{\Delta}{2}\cos\varphi\sin\theta$, $A_y = \frac{\Delta}{2}\sin\varphi\sin\theta$ and $A_z = \frac{\Delta}{2}\cos\theta$.

The dispersion error of the Exponential Coefficient Optimized S-FDTD Method can be optimized for chosen parameters of the method and the numerical error relationship for the S-FDTD is given by [3]:

$$\cos(\omega\Delta t) = 1 + \frac{1}{2}\sum_{p=1}^{m} g_p\left\{4s^2\left(\eta_x^2 + \eta_y^2 + \eta_z^2\right)\right\}^p, \tag{4}$$

where

$$\begin{aligned} g_p = &\sum_{1\le i1\le j1<i2\le j2<\ldots<ip\le jp\le m} c_{i1}d_{j1}c_{i2}d_{j2}\ldots c_{ip}d_{jp} \\ &+ \sum_{1\le i1<j1\le i2<j2\le\ldots\le ip<jp\le m} d_{i1}c_{j1}d_{i2}c_{j2}\ldots d_{ip}c_{jp} \end{aligned} \tag{5}$$

and m is an integer and equals to (number of coefficients)/2. Parameters η_γ are defined in equations (2) and (3). c_i and d_i are time step coefficients to be determined.

Equations (1)–(5) have been used to obtain the three dimensional numerical dispersion performances of the two methods. As the S-FDTD Method has a stability limit it was only possible to compare the performances of the two methods for time steps corresponding to this limit or below.

The graphical results of the numerical dispersion study carried out for the S-FDTD (with 6 and 10 coefficients) and the ADI-FDTD methods (second and fourth order) are given by Figure 1 to Figure 5. The Figure 1 and Figure 2 show the dispersion errors against the angle θ for space increments $\lambda/25$ and $\lambda/50$ at stability factors close to the stability limits of the S- FDTD method. The stability limit of the 6 coefficient S-FDTD is 1.48 and the stability limit of the 10 coefficient S-FDTD is 2.47. The results are given for the 6

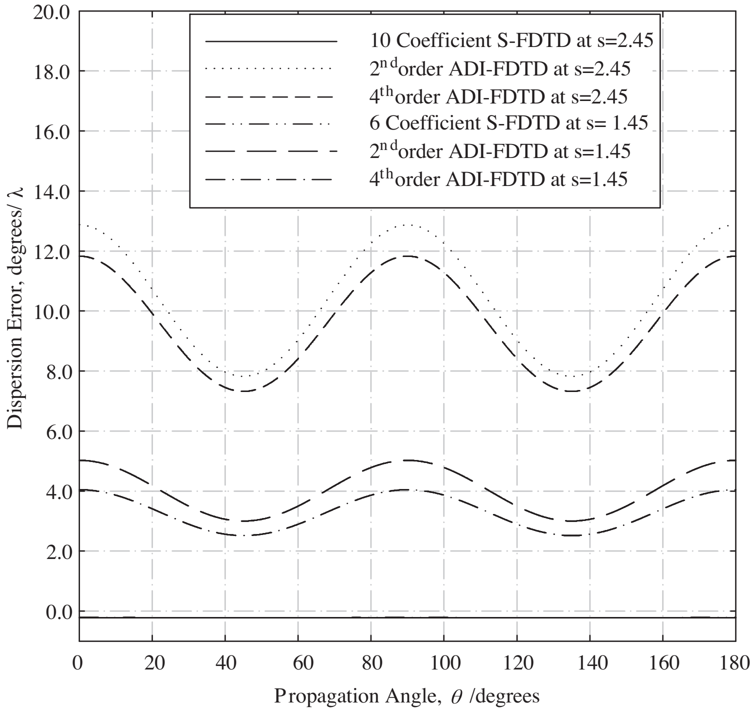

Figure 1. Dispersion error as a function of propagation angle θ, for $\varphi = 90^\circ$ and $\Delta = \lambda/25$.

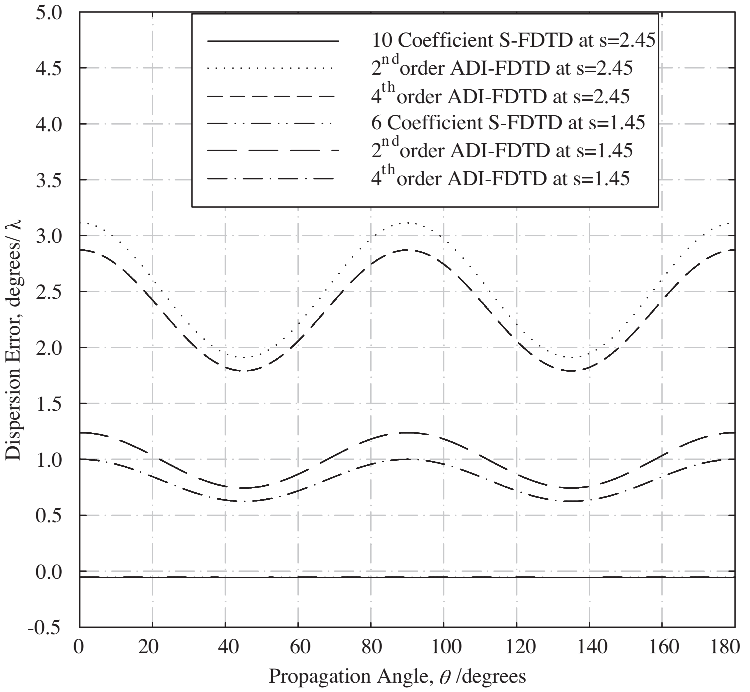

Figure 2. Dispersion error as a function of propagation angle θ, for $\varphi = 90^\circ$ and $\Delta = \lambda/50$.

and 10 coefficient S-FDTD (which are almost the same) as well as for the conventional second order ADI-FDTD and the fourth order ADI-FDTD methods. The results show that although there is not a significant difference in error performance of the two ADI-FDTD methods, the errors of the S-FDTD method are much smaller for both of the S-FDTD methods. The increasing the number of coefficients for S-FDTD does not have a significant improvement in dispersion performance but increases the stability limit. No results are presented against the φ angle as there is no significant variation with φ. Figure 3 shows the dispersion error against space increment Δ at the stability factors $s = 2.45$ and $s = 1.45$. For both of the ADI-FDTD

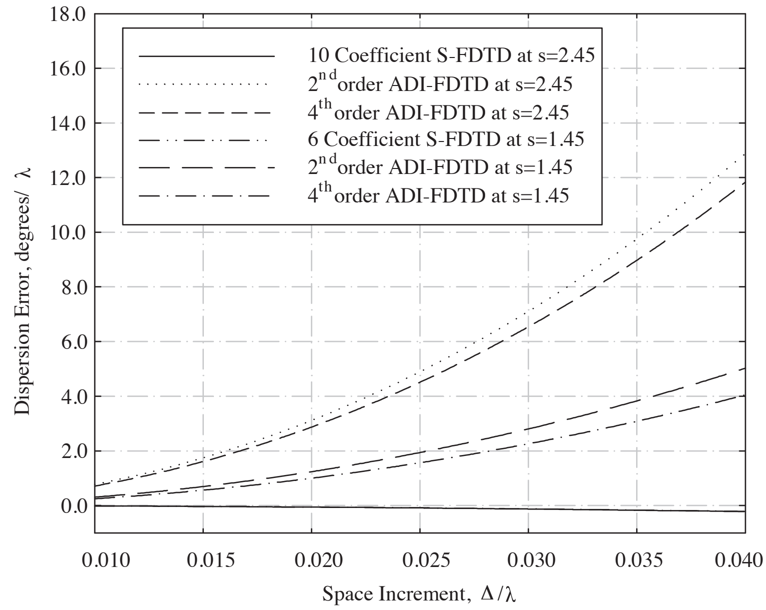

Figure 3. Dispersion error as a function of space increment per wavelength, for $\theta = 0^\circ$ and $\varphi = 90^\circ$.

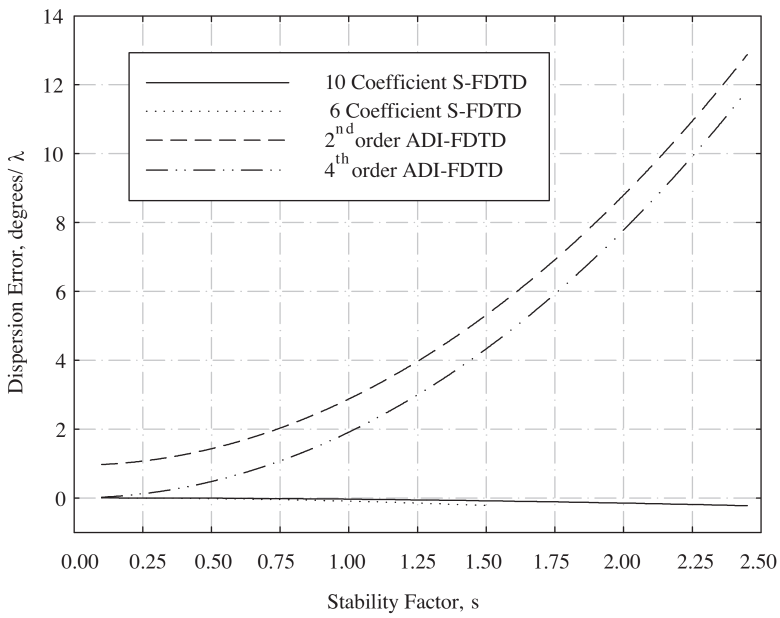

Figure 4. Dispersion error as a function of stability factor, for $\theta = 0^\circ$, $\varphi = 90^\circ$ and $\Delta = \lambda/25$.

methods the errors become unacceptably high for larger values of Δ. Figure 4 and Figure 5 show dispersion errors against the stability factor s up to the stability limits of the S-FDTD for space increments $\lambda/25$ and $\lambda/50$. As the s increases (in other words, as Δt increases) the errors for both of the ADI-FDTD methods increases while the errors for both of the S-FDTD method s remains low.

The stability limit of the conventional Yee's FDTD method is 0.577 [1] in 3-D, so it can not be used for stability factors above this limit. As the performances of the S-FDTD methods introduced in this paper are better than the ADI-FDTD method up to the stability limits of the S-FDTD methods for $\lambda/25$ and $\lambda/50$ space increments the new S-FDTD methods can be usefully employed in these regions.

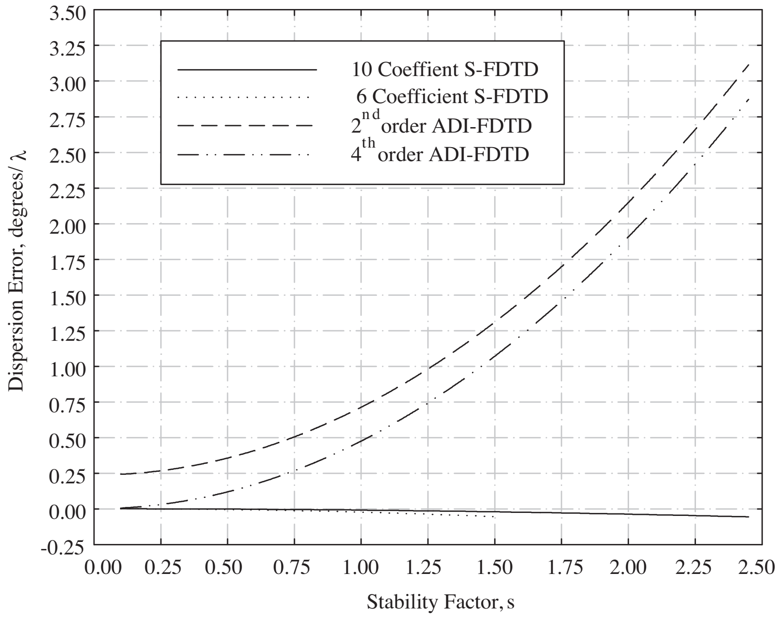

Figure 5. Dispersion error as a function of stability factor, for $\theta = 0^\circ$, $\varphi = 90^\circ$ and $\Delta = \lambda/50$.

3. Conclusion

It has been shown that ADI-FDTD method has large dispersion errors when the space increments are on the order of $\lambda/25$ to $\lambda/50$ for stability factors of larger than 1. It has also been shown that for time steps below the stability limits of the S-FDTD method it has much better dispersion performance compared with the ADI-FDTD method. Therefore the S-FDTD method can be usefully employed for space increments in the order of $\lambda/25$ to $\lambda/50$ for stability factors below the stability limit of the S-FDTD.

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