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An Exact Line Integral Representation of the PO Radiation Integral from a Flat Perfectly Conducting Surfaces Illuminated by Elementary Electric or Magnetic Dipoles

Matteo ALBANI, Stefano MACI
Information Engineering Dept., University of Siena-ITALY
e-mail: albanim@ing.unisi.it, macis@ing.unisi.it

Abstract

In this paper, a line integral representation for the PO radiation integral from a flat perfectly conducting surface, illuminated by an arbitrary oriented elementary either electric or magnetic dipole, is presented. No restriction is imposed on the position of the source and of the observation point. The main application of this result is the acceleration of the numerical PO integration for electrically large surfaces. The formulation is based on the application of the equivalence principle to a projecting surface which allows the analytical evaluation in closed form of one of the two-fold surface integral which define the radiated field at any space point. Although similar solutions has been suggested by other authors, our final outcome is simple, clearly interpretable, and easily applicable with respect to previous results.

Key Words: *PO radiation integral, Kirchhoff aperture field, equivalence principle*

1. Introduction

The possibility of reducing the radiation from a finite aperture to incremental contributions distributed on its rim was initially conjectured by Young nearly two centuries ago, and mathematically proved independently by Maggi [1] and Rubinowicz [2] for the scalar optical-acoustical case. These pioneering works and the subsequent stream of research literature had the objective to investigate and to demonstrate the existence of the *boundary waves* which produce the diffracted field (see [1] for a brief review and references). A first complete analytical treatment is due to Miyamoto and Wolf [4]-[5]. In these works a systematic algorithm is presented for the surface-to-line reduction of the radiation integrals by using a proper vector potential, whose definition is however based on a quite involved mathematics. The same result was demonstrated to be obtainable via a simpler physical derivation based on the Huyghens principle [6]-[7]. The same treatment was successfully applied to the problem of scattering from objects in the Physical Optics (PO) approximation [8] from perfectly conducting surfaces. Refinement of the solutions through the investigation on the physical properties of the individual boundary element connected with its uniqueness are conducted in [9]. The work in [10] represents a first attempt to extend the treatment to the electromagnetic case, but no explicit applicable formulas was presented there; while [11] contains the application of the procedure to the exact solution of infinite wedge diffraction. A complete and detailed review of these earlier results is given in [12].

In modern key, the PO reduction to line integration is intended to speed up the computation time in calculating the PO scattering from very large bodies described in terms of facet segmentation. Typical example is that of reflector antennas, shown in [13]. Facet PO description of large surfaces is also used in hybrid Method of Moment (MOM)-PO approach for antenna installation on large platforms. This method, although do not improve the matrix filling time, allows for a strong compression of the MoM matrix. In this framework, describing in terms of line integral the PO radiation from each of the myriad of flat faces of the contour leads to a drastic filling time improvement.

A modern application-oriented reformulation of the PO surface-to-line reduction technique and of the relevant numerical problems, is due to Asvestas [14]-[16]; his technique is applied in [17] to find explicit form for a PO contour line integration from a perfectly conducting flat face illuminated by an Hertzian dipole. The same procedure as that in [17] has been followed in [18], with extension to the case of surface impedance boundary conditions but with the assumption – significantly restricting in hybrid-MoM applications – of plane wave incidence. The work in [17], in our knowledge, contains the only explicit expression present in literature for an exact PO-line integration for dipole wave illumination, which include the reactive incident field of the dipole. We also note that a spherical wave illumination is also treated in [20] without including the treatment of the near field of the incident wave, i.e., by supposing the scatterer in the far region of the incident field, this latter still remaining with spherical wavefront. This leads a drastic simplification with respect to the expressions presented in [17]. These latter expressions are indeed constituted by a large number of involved contributions, thus rendering their applicability quite cumbersome. For this reason there is still a motivation for investigating on simpler and easily applicable expressions, which are not however affected by assumption of far zone illumination. To this purpose, we remark that the exact incremental diffraction PO contribution (i.e., the integrand of the final exact line integral representation) *is not unique*; indeed, an arbitrary irrotational field can be added to the integrand without affecting the final closed contour integration.

We also mention that the scalar procedure described in [7], has been rephrased in [19] for the electromagnetic near field radiation of a rectangular waveguide modal sheet distribution of currents, leading to incremental line contributions cast in a simple and nice form, which is however adaptable in the present framework only for the case of plane wave incidence.

The exact formulation used here for spherical source dipoles, takes inspiration from both the procedure presented in [7] and in [17] but is essentially different from both. From [17] our procedure maintains the clever idea of the projection surface from the observation point on which the geometrical construct is based; from the original Rubinowicz scalar work [7] (re-addressed in [20] for the electromagnetic case without incident reactive field components) the present formulation preserves the elegant and physical appealing application of the equivalence theorem. The final outcome is a simpler, but nevertheless “exact”, alternative to [17] of the incremental PO coefficients.

2. Electromagnetic Line Integral Representation of the Kirchhoff Aperture Field Integration

Consider an elementary electric dipole source $\mathbf{J}^d$ located at P' and illuminating the aperture region A, $\mathbf{J}^d$ denote the direction unit vector times the current amplitude of the elementary dipole. The purpose of this section is to describe the Kirchhoff radiation from A in terms of the integration along its contour C. Hereinafter, the observation point is at P, and the integration point (on every type of surface or line that will be defined), will be denoted by Q. The primary field radiated by the dipole at Q will be denoted by $\mathbf{E}^i(P',Q)$, $\mathbf{H}^i(P',Q)$. We construct a projection cone with the tip at P and whose lateral surface passes through the edge of the aperture A (Figure 1). We now apply the equivalence theorem to the surface comprising A and the lateral surface S of the frustum cone from A to infinity. Denoting by V the volume inside this surface and V^* the complementary volume, the equivalent currents on $A+S$ radiate in free space either the primary field or zero when the source lies either in V or V^*, respectively. By introducing the existence function U^i which is zero in V and unity in V^*, we have

$$\mathbf{E}^i(P',P)\,U^i = \iint_{A+S} \left\{ \tfrac{\zeta}{jk} \nabla_P \times \nabla_P \times \left[G(Q,P)\, \hat{n} \times \mathbf{H}^i(P',Q) \right] - \nabla_P \times \left[G(Q,P)\, \mathbf{E}^i(P',Q) \times \hat{n} \right] \right\} ds \quad (1)$$

and

$$\mathbf{H}^i(P',P)\,U^i = \iint_{A+S} \left\{ \nabla_P \times \left[G(Q,P)\, \hat{n} \times \mathbf{H}^i(P',Q) \right] + \tfrac{1}{jk\zeta} \nabla_P \times \nabla_P \times \left[G(Q,P)\, \mathbf{E}^i(P',Q) \times \hat{n} \right] \right\} ds \quad (2)$$

in which $\hat{n}$ is the outgoing normal to the surface $A+S$ at the integration point Q (Figure 1), ∇_P is the gradient operator with respect to the coordinates of the point P, and

$$G(Q,P) = \frac{e^{-jk|P-Q|}}{4\pi|P-Q|}, \quad (3)$$

with $|P-Q|$ denoting the distance between P and Q. Thus, the electric and magnetic fields radiated by the aperture A can be written as

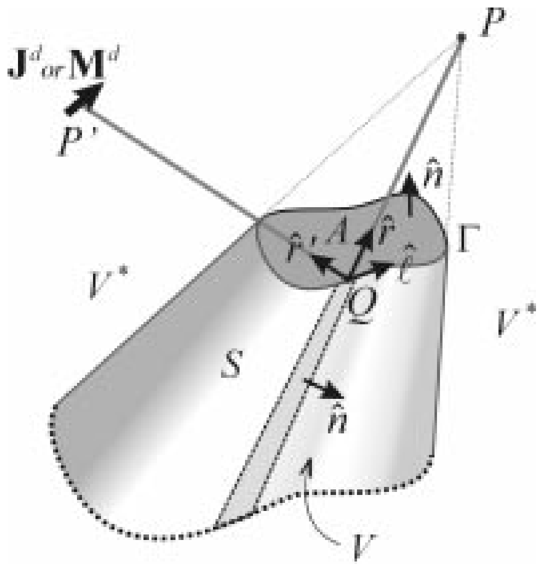

Figure 1. Projection cone from P of the aperture A and associated reference unit vectors at the integration point Q along the rim Γ. The radiation integral associated to the strip from Q to infinity may be calculated in a closed form using the Kottler formulation of the radiation integral.

$$
\begin{aligned}
&\mathbf{E}^{A}(P)=\iint\limits_{A}\left\{\tfrac{\zeta}{jk}\nabla_{P}\times\nabla_{P}\times\left[G(Q,P)\,\hat{n}\times\mathbf{H}^{i}(P',Q)\right]-\nabla_{P}\times\left[G(Q,P)\,\mathbf{E}^{i}(P',Q)\times\hat{n}\right]\right\}ds\\
&=\mathbf{E}^{i}(P',P)\,U^{i}-\iint\limits_{S}\left\{\tfrac{\zeta}{jk}\nabla_{P}\times\nabla_{P}\times\left[G(Q,P)\,\hat{n}\times\mathbf{H}^{i}(P',Q)\right]-\nabla_{P}\times\left[G(Q,P)\,\mathbf{E}^{i}(P',Q)\times\hat{n}\right]\right\}ds
\end{aligned},
\tag{4}
$$

and an analogous expression is valid for the magnetic field. The second terms in the r.h.s. of (4) can be cast in the form of the Kottler radiation integral [21], leading to

$$
\begin{aligned}
\mathbf{E}^{A}(P)=\mathbf{E}^{i}(P',P)\,U^{i}-\oint\limits_{\Gamma}G(Q,P)\,\mathbf{E}^{i}(P',Q)\times\hat{\ell}\,d\ell-\tfrac{\zeta}{jk}\oint\limits_{\Gamma}\nabla_{Q}G(Q,P)\,\mathbf{H}^{i}(P',Q)\cdot\hat{\ell}\,d\ell\\
+\iint\limits_{S}\left[G(Q,P)\,\hat{n}\cdot\nabla_{Q}\mathbf{E}^{i}(P',Q)-\hat{n}\cdot\nabla_{Q}G(P,Q)\,\mathbf{E}^{i}(P',Q)\right]ds
\end{aligned};
\tag{5}
$$

while the analogous for the magnetic field is

$$
\begin{aligned}
\mathbf{H}^{A}(P)=\mathbf{H}^{i}(P',P)\,U^{i}-\oint\limits_{\Gamma}G(Q,P)\,\mathbf{H}^{i}(P',Q)\times\hat{\ell}\,d\ell+\tfrac{1}{jk\zeta}\oint\limits_{\Gamma}\nabla_{Q}G(Q,P)\,\mathbf{E}^{i}(P',Q)\cdot\hat{\ell}\,d\ell\\
+\iint\limits_{S}\left[G(Q,P)\,\hat{n}\cdot\nabla_{Q}\mathbf{H}^{i}(P',Q)-\hat{n}\cdot\nabla_{Q}G(P,Q)\,\mathbf{H}^{i}(P',Q)\right]ds
\end{aligned}.
\tag{6}
$$

In (5) and (6), ∇_Q is the gradient operator with respect to the coordinates of the point Q. The first and the second terms in the r.h.s. of (5) and (6) are line integrals along the edge Γ of the aperture, while the last terms are still a surface integral on the cone lateral surface S. Following the way traced by Rubinowicz for the scalar case [7], also this latter integral can be reduced into a line integral on Γ . The main difficulty is to reduce the integrand to quantities which can be assimilated to those for the scalar case. To this end, we notice first that thanks to the particular construction of the projection cone, $\nabla_Q G(P,Q)$ is orthogonal to the normal unit vector $\hat{n}$, hence the second term inside the surface integral vanishes. Furthermore, since the incident electric and magnetic fields are that produced by an elementary electric dipole $\mathbf{J}^d$ at P', we have

$$
\mathbf{E}^{i}(P',Q)=\tfrac{\zeta}{jk}\nabla_{Q}\times\nabla_{Q}\times\left[G(P',Q)\,\mathbf{J}^{d}\right]=\tfrac{\zeta}{jk}\nabla_{P'}\times\nabla_{P'}\times\left[G(P',Q)\,\mathbf{J}^{d}\right]
\tag{7}
$$

and

$$
\mathbf{H}^{i}(P',Q)=\nabla_{Q}\times\left[G(P',Q)\,\mathbf{J}^{d}\right]=-\nabla_{P'}\times\left[G(P',Q)\,\mathbf{J}^{d}\right];
\tag{8}
$$

where $G(P',Q)$ is the free-space scalar Green's function in (3) and $\nabla_{P'}$ is the gradient operator with respect to P'. The above relations holds because $\mathbf{J}^d$ is independent of both P' and Q. Equations (7) and (8) leads to the following identities for the residual surface integrals to be evaluated in (5) and (6)

$$
\iint\limits_{S}G(Q,P)\,\hat{n}\cdot\nabla_{Q}\mathbf{E}^{i}(P',Q)\,ds=\tfrac{\zeta}{jk}\nabla_{P'}\times\nabla_{P'}\times\left[\iint\limits_{S}G(Q,P)\,\hat{n}\cdot\nabla_{Q}G(P',Q)\,ds\,\mathbf{J}^{d}\right]
\tag{9}
$$

$$\iint_S G(Q,P)\ \hat{n}\cdot\nabla_Q \mathbf{H}^i(P',Q)\ ds = -\nabla_{P'}\times\left[\iint_S G(Q,P)\ \hat{n}\cdot\nabla_Q G(P',Q)\,ds\,\mathbf{J}^d\right] \tag{10}$$

where the order of surface integral and spatial gradients with respect to P' have been legitimately interchanged. The surface integral between bracket in the r.h.s. of (9) and (10) is *exactly* the same of the scalar case, which, as shown in [7], can be reduced to a line integral along the aperture rim Γ. This is obtained by first parameterizing the lateral surface S into elementary strips from Q to infinity (Figure 1) and next by evaluating in an exact closed form the radiation integral for each semi-infinite strip; thus, obtaining [3, page 449, eq. 13]

$$\iint_S G(Q,P)\ \hat{n}\cdot\nabla_Q G(P',Q)\ ds = \oint_\Gamma G(Q,P)\,G(P',Q)\,\frac{(\hat{n}\cdot\hat{r}')\left|\hat{r}\times\hat{\ell}\right|}{1+\hat{r}'\cdot\hat{r}}\,d\ell \tag{11}$$

where $\hat{r}$ and $\hat{r}'$ are the unit vectors associated to $P-Q$ and $P'-Q$, respectively, and $\hat{\ell}$ is the unit vector tangent to the rim (see Figure 1). Since $\hat{n}=\frac{\hat{\ell}\times\hat{r}}{\left|\hat{\ell}\times\hat{r}\right|}$, (11) becomes

$$\iint_S G(Q,P)\ \hat{n}\cdot\nabla_Q G(P',Q)\ ds = \oint_\Gamma G(Q,P)\,G(P',Q)\,\frac{\hat{r}\times\hat{r}'}{1+\hat{r}'\cdot\hat{r}}\cdot\hat{\ell}\,d\ell \tag{12}$$

that yields, via (9) and (10)

$$\iint_S G(Q,P)\ \hat{n}\cdot\nabla_Q \mathbf{E}^i(P',Q)\ ds = \oint_\Gamma G(Q,P)\,\tfrac{\zeta}{jk}\nabla_{P'}\times\nabla_{P'}\times\left[G(P',Q)\,\mathbf{J}^d\frac{\hat{r}\times\hat{r}'\cdot\hat{\ell}}{1+\hat{r}'\cdot\hat{r}}\right]d\ell \tag{13}$$

and

$$\iint_S G(Q,P)\ \hat{n}\cdot\nabla_Q \mathbf{H}^i(P',Q)\ ds = -\oint_\Gamma G(Q,P)\,\nabla_{P'}\times\left[G(P',Q)\,\mathbf{J}^d\frac{\hat{r}\times\hat{r}'\cdot\hat{\ell}}{1+\hat{r}'\cdot\hat{r}}\right]d\ell. \tag{14}$$

Finally, using (13) in (5) and (14) in (6), we obtain

$$\mathbf{E}^A(P) = \mathbf{E}^i(P',P)\,U^i + \oint_\Gamma \mathbf{e}(P',Q,P)\ d\ell,\quad \mathbf{H}^A(P) = \mathbf{H}^i(P',P)\,U^i + \oint_\Gamma \mathbf{h}(P',Q,P)\ d\ell \tag{15}$$

in which the "incremental Kirchhoff diffraction" electric and magnetic fields are given by

$$\begin{gathered}\mathbf{e}(P',Q,P) = -G(Q,P)\,\mathbf{E}^i(P',Q)\times\hat{\ell} + \tfrac{\zeta}{jk}\nabla_P G(Q,P)\,\mathbf{H}^i(P',Q)\cdot\hat{\ell}\\ +G(Q,P)\,\tfrac{\zeta}{jk}\nabla_{P'}\times\nabla_{P'}\times\left[G(P',Q)\,\mathbf{J}^d\tfrac{\hat{r}\times\hat{r}'\cdot\hat{\ell}}{1+\hat{r}'\cdot\hat{r}}\right]\end{gathered} \tag{16}$$

and

$$\begin{array}{c}\mathbf{h}\left(P',Q,P\right)=-G\left(Q,P\right)\mathbf{H}^{i}\left(P',Q\right)\times\hat{\ell}-\frac{1}{jk\zeta}\nabla_{P}G\left(Q,P\right)\mathbf{E}^{i}\left(P',Q\right)\cdot\hat{\ell}\\ -G\left(Q,P\right)\nabla_{P'}\times\left[G\left(P',Q\right)\mathbf{J}^{d}\frac{\hat{r}\times\hat{r}'\cdot\hat{\ell}}{1+\hat{r}'\cdot\hat{r}}\right]\end{array},\tag{17}$$

respectively. The latter relationships constitute the final expressions we were looking for. In order to allow the practical implementation, we explicitly perform the differential operations in (16) and (17), thus obtaining the closed-form expressions

$$\begin{array}{c}\mathbf{e}\left(P',Q,P\right)=-G\left(Q,P\right)\mathbf{E}^{i}\left(P',Q\right)\times\hat{\ell}-\zeta\left(1+\frac{1}{jkr}\right)G\left(Q,P\right)\mathbf{H}^{i}\left(P',Q\right)\cdot\hat{\ell}\,\hat{r}\\ -jk\zeta G\left(Q,P\right)G\left(P',Q\right)\frac{\left|\hat{r}\times\hat{\ell}\right|}{1+\hat{r}'\cdot\hat{r}}\underline{\underline{\mathbf{D}}}\cdot\mathbf{J}^{d}\end{array}\tag{18}$$

and

$$\begin{array}{c}\mathbf{h}\left(P',Q,P\right)=-G\left(Q,P\right)\mathbf{H}^{i}\left(P',Q\right)\times\hat{\ell}+\frac{1}{\zeta}\left(1+\frac{1}{jkr}\right)G\left(Q,P\right)\mathbf{E}^{i}\left(P',Q\right)\cdot\hat{\ell}\,\hat{r}\\ +jkG\left(Q,P\right)G\left(P',Q\right)\frac{\left|\hat{r}\times\hat{\ell}\right|}{1+\hat{r}'\cdot\hat{r}}\mathbf{V}\times\mathbf{J}^{d}\end{array}.\tag{19}$$

In (18) and (19), $r=|P-Q|$, the dyad $\underline{\underline{\mathbf{D}}}$ and the vector $\mathbf{V}$ can be expressed in a simple form in the ray fixed spherical coordinate system, which is defined by the unit vectors $\left(\hat{r}',\hat{\theta}',\hat{\phi}'\right)$ with $\hat{\phi}'=\frac{\hat{r}\times\hat{r}'}{|\hat{r}\times\hat{r}'|}$ and $\hat{\theta}'=\hat{\phi}'\times\hat{r}'$, as

$$\underline{\underline{\mathbf{D}}}=\sum_{u,v=r',\theta',\phi'}D_{uv}\hat{u}\hat{v},\tag{20}$$

where

$$D_{r'r'}=-2\left(\frac{1}{jkr'}+\frac{1}{(jkr')^{2}}\right)\hat{n}\cdot\hat{r}',\tag{21}$$

$$D_{r'\theta'}=D_{\theta'r'}=\left(\frac{1}{jkr'}+\frac{2}{(jkr')^{2}}\right)\hat{r}\times\hat{n}\cdot\hat{\phi}',\tag{22}$$

$$D_{r'\phi'}=D_{\phi'r'}=\left(\frac{1}{jkr'}+\frac{2}{(jkr')^{2}}\right)\hat{n}\cdot\hat{\phi}',\tag{23}$$

$$D_{\theta'\theta'}=\left(1+\frac{1}{jkr'}+\frac{\hat{r}\cdot\hat{r}'}{1+\hat{r}\cdot\hat{r}'}\frac{1}{(jkr')^{2}}\right)\hat{n}\cdot\hat{r}',\tag{24}$$

$$D_{\theta'\phi'}=-\frac{1-\hat{r}\cdot\hat{r}'}{|\hat{r}\times\hat{r}'|}\frac{1}{(jkr')^{2}}\hat{n}\cdot\hat{\phi}',\tag{25}$$

$$D_{\phi'\theta'}=\frac{1}{1+\hat{r}\cdot\hat{r}'}\frac{1}{(jkr')^{2}}\hat{r}\times\hat{n}\cdot\hat{r}',\tag{26}$$

$$D_{\phi'\phi'}=\left\{1+\frac{1}{jkr'}+\left(1+\frac{1-\hat{r}\cdot\hat{r}'}{|\hat{r}\times\hat{r}'|^{2}}\right)\frac{1}{(jkr')^{2}}\right\}\hat{n}\cdot\hat{r}';\tag{27}$$

and

$$\mathbf{V} = \left(1 + \tfrac{1}{jkr'}\right) \hat{n} \cdot \hat{r}' \hat{r}' - \tfrac{1}{jkr'} \hat{n} \times \hat{r} \cdot \hat{\phi}' \hat{\theta}' - \tfrac{1}{jkr'} \hat{n} \cdot \hat{\phi}' \hat{\phi}'. \tag{28}$$

In the above expressions $r' = |P' - Q|$ accordingly with the introduced reference system.

When, in place of an electric dipole, we consider an elementary magnetic dipole $\mathbf{M}^d$ located at P', the same procedure could be followed; however, the final result can be straightforwardly obtained from (18) and (19) by invoking the duality principle. We eventually obtain the integral representations (15), in which the incremental diffraction electric and magnetic fields are now given by

$$\begin{aligned} \mathbf{e}\left(P', Q, P\right) = -G\left(Q, P\right) \mathbf{E}^i\left(P', Q\right) \times \hat{\ell} - \zeta \left(1 + \tfrac{1}{jkr}\right) G\left(Q, P\right) \mathbf{H}^i\left(P', Q\right) \cdot \hat{\ell}\, \hat{r} \\ + jkG\left(Q, P\right) G\left(P', Q\right) \tfrac{\left|\hat{r} \times \hat{\ell}\right|}{1 + \hat{r}' \cdot \hat{r}} \mathbf{V} \times \mathbf{M}^d \end{aligned} \tag{29}$$

and

$$\begin{aligned} \mathbf{h}\left(P', Q, P\right) = -G\left(Q, P\right) \mathbf{H}^i\left(P', Q\right) \times \hat{\ell} + \tfrac{1}{\zeta} \left(1 + \tfrac{1}{jkr}\right) G\left(Q, P\right) \mathbf{E}^i\left(P', Q\right) \cdot \hat{\ell}\, \hat{r} \\ - \tfrac{jk}{\zeta} G\left(Q, P\right) G\left(P', Q\right) \tfrac{\left|\hat{r} \times \hat{\ell}\right|}{1 + \hat{r}' \cdot \hat{r}} \underline{\underline{\mathbf{D}}} \cdot \mathbf{M}^d \end{aligned} \tag{30}$$

in which the vector $\mathbf{V}$ and the dyad $\underline{\underline{\mathbf{D}}}$ are the same defined in (28) and (20)-(27), respectively.

Asymptotic limit for locally plane spherical wavefront

When the source location moves far from the aperture in such a way that all the aperture is in the far field associated to the dipole, the reactive field component of the source can be neglected thus obtaining $\mathbf{E}^i\left(P', Q\right) \approx jk\zeta G\left(P', Q\right) \hat{r}' \times \hat{r}' \times \mathbf{J}^d$ and $\mathbf{H}^i\left(P', Q\right) \approx jkG\left(P', Q\right) \hat{r}' \times \mathbf{J}^d$. The dyad $\underline{\underline{\mathbf{D}}}$ and the vector $\mathbf{V}$ simply reduce to

$$\underline{\underline{\mathbf{D}}} \approx \left(\hat{n} \cdot \hat{r}'\right) \left[\hat{\theta}' \hat{\theta}' + \hat{\phi}' \hat{\phi}'\right] \text{ and } \mathbf{V} \approx \left(\hat{n} \cdot \hat{r}'\right) \hat{r}', \tag{31}$$

respectively, and (18) and (19) are rearranged in such a way to express each incremental diffraction contribution as a function of the homologous incident field only

$$\begin{bmatrix} \mathbf{e}\left(P', Q, P\right) \\ \mathbf{h}\left(P', Q, P\right) \end{bmatrix} \approx \underline{\underline{\mathbf{D}}}_0 \cdot \begin{bmatrix} \mathbf{E}^i\left(P', Q\right) \\ \mathbf{H}^i\left(P', Q\right) \end{bmatrix} G\left(Q, P\right) \tag{32}$$

where

$$\underline{\underline{\mathbf{D}}}_0 = \left(\hat{\ell} \times \underline{\underline{I}} - \left(1 + \tfrac{1}{jkr}\right) \hat{r} \left(\hat{r}' \times \hat{\ell}\right) + \tfrac{\left(\hat{r} \times \hat{r}'\right) \cdot \hat{\ell}}{1 + \hat{r}' \cdot \hat{r}} \underline{\underline{I}}\right) \tag{33}$$

where $\underline{\underline{I}}$ is the unit dyad so that the notation $\left(\hat{\ell} \times \underline{\underline{I}}\right) \cdot \mathbf{H}^i$ denotes $\hat{\ell} \times \mathbf{H}^i$. The same formal expressions as a function of the incident field are also obtained when in place of the electric dipole we consider a magnetic dipole by using (29) and (30) as starting point. Since the above relationships depend only on incident fields, one can apply these expressions using the far field of an arbitrary antenna with a given phase center.

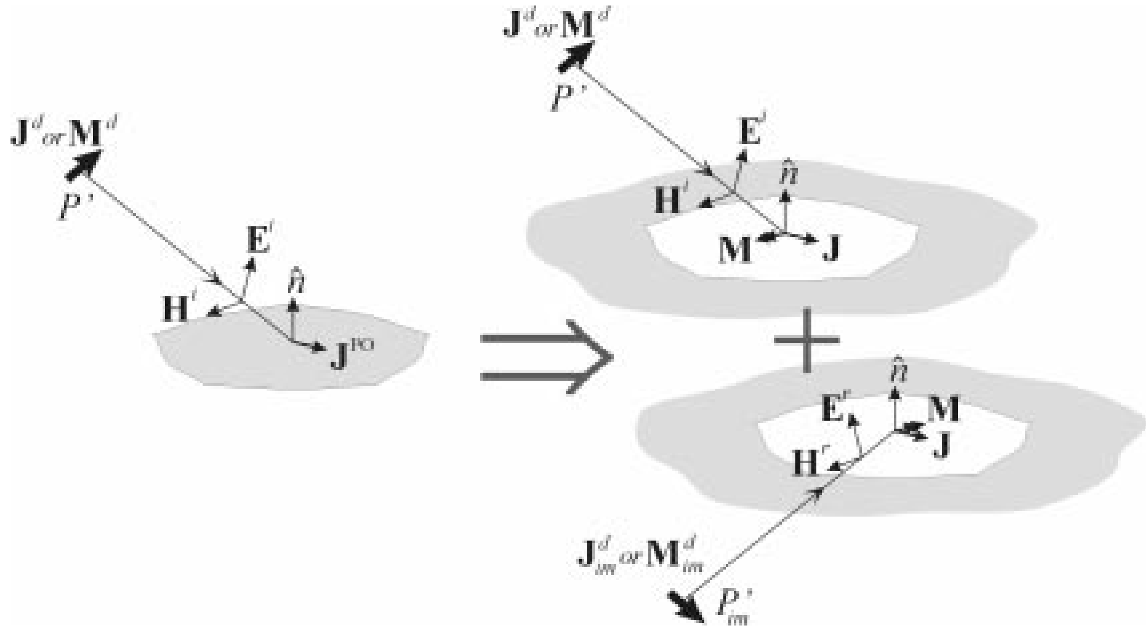

Figure 2. PO scattering from a flat impedance plate as superposition of aperture scattering illuminated by actual and image sources

3. Electromagnetic Line Integral Representation of the PO Radiation Integral

In the PO approximation, the field radiated by a arbitrary contoured flat perfectly conducting plate is

$$\mathbf{E}^{PO}\left(P',P\right)=\iint\limits_{A}\left\{\tfrac{\zeta}{jk}\nabla_P\times\nabla_P\times\left[G\left(Q,P\right)\,2\hat{n}\times\mathbf{H}^i\left(P',Q\right)\right]\right\}ds \tag{34}$$

$$\mathbf{H}^{PO}\left(P',P\right)=\iint\limits_{A}\left\{\nabla_P\times\left[G\left(Q,P\right)\,2\hat{n}\times\mathbf{H}^i\left(P',Q\right)\right]\right\}ds \tag{35}$$

where $\hat{n}$ the normal unit vector outgoing toward the lit region. As shown in Figure 2, the radiation integrals (34) and (35) can be thought of as the superposition of two aperture radiation integrals. The first aperture is that directly illuminated by the dipole as described in the previous section and the second one is that illuminated by its image through the perfectly electric infinite plane containing the plate (Figure 3). This image source is given by either $\mathbf{J}^d_{im}=-\mathbf{J}^d+2\left(\hat{n}\cdot\mathbf{J}^d\right)\,\hat{n}$ or $\mathbf{M}^d_{im}=\mathbf{M}^d-2\left(\hat{n}\cdot\mathbf{M}^d\right)\,\hat{n}$ for the electric or magnetic dipole, respectively, and is located at $P'_{im}=P'-2\left(\hat{n}\cdot P'\right)\hat{n}$. By superposing the two aperture surface current, the electric currents double and the magnetic currents cancel, thus reconstructing the PO currents. The aperture field radiated by both the actual and the image source can be reduced, as shown in the previous sections, to obtain a line integral representation. Applying (15), (18) and (19) to both the actual and to the image electric dipoles $\mathbf{J}^d$ and $\mathbf{J}^d_{im}$, leads to

$$\mathbf{E}^{PO}(P) = -\mathbf{E}^{i}(P', P)\, U^{i} + \mathbf{E}^{r}(P'_{im}, P)\, U^{r} + \oint_{\Gamma} \mathbf{e}^{PO}(P', Q, P)\, d\ell \tag{36}$$

and

$$\mathbf{H}^{PO}(P) = -\mathbf{H}^{i}(P', P)\, U^{i} + \mathbf{H}^{r}(P', P)\, U^{r} + \oint_{\Gamma} \mathbf{h}^{PO}(P', Q, P)\, d\ell \tag{37}$$

in which $\mathbf{E}^{r}(P'_{im}, P)$ and $\mathbf{H}^{r}(P'_{im}, P)$ denote the reflected fields observed at P (which constitutes the primary fields radiated by the image dipole at P'_{im}). These field are bounded by the existence function U^{r} that equals unity or vanishes if P'_{im} lies inside or outside the projection cone V, respectively (Figure 1). In (36)

$$\begin{aligned} \mathbf{e}^{PO}(P', Q, P) = {} & -2G(Q, P)\left[\mathbf{E}^{i}(P', Q) \cdot \hat{n}'\right]\hat{n}' \times \hat{\ell} - 2\zeta\left(1 + \tfrac{1}{jkr}\right) G(Q, P)\, \mathbf{H}^{i}(P', Q) \cdot \hat{\ell}\, \hat{r} \\ & -jk\zeta G(Q, P)\, G(P', Q)\left[\tfrac{|\hat{r} \times \hat{\ell}|}{1 + \hat{r'} \cdot \hat{r}} \underline{\underline{\mathbf{D}}} \cdot \mathbf{J}^{d} + \tfrac{|\hat{r} \times \hat{\ell}|}{1 + \hat{r'}_{im} \cdot \hat{r}} \underline{\underline{\mathbf{D}}}_{im} \cdot \mathbf{J}^{d}_{im}\right] \end{aligned} \tag{38}$$

while in (37)

$$\begin{aligned} \mathbf{h}^{PO}(P', Q, P) = {} & -2G(Q, P)\left[\mathbf{H}^{i}(P', Q) - \hat{n} \cdot \mathbf{H}^{i}(P', Q)\, \hat{n}\right] \times \hat{\ell} \\ & +jkG(Q, P)\, G(P', Q)\left[\tfrac{|\hat{r} \times \hat{\ell}|}{1 + \hat{r'} \cdot \hat{r}} \mathbf{V} \times \mathbf{J}^{d} + \tfrac{|\hat{r} \times \hat{\ell}|}{1 + \hat{r'}_{im} \cdot \hat{r}} \mathbf{V}_{im} \times \mathbf{J}^{d}_{im}\right] \end{aligned} \tag{39}$$

where the dyad $\underline{\underline{\mathbf{D}}}_{im}$ and the vector $\mathbf{V}_{im}$ are obtained from the respective quantities $\underline{\underline{\mathbf{D}}}$ and $\mathbf{V}$ by replacing $\hat{r'}$ in (20)-(27) and in (28), respectively, with the unit vector $\hat{r'}_{im} = \hat{r}' - 2(\hat{n}' \cdot \hat{r}')\, \hat{n}'$ associated to the reflected wave.

In applying the procedure described in Section 2 to the incident field, we changed sign in front of the incident field, due to the change of sign of the normal to the PO surface with respect to the normal to the aperture surface. Note that in the asymptotic regime, the field on the shadow region should be composed by only the diffracted PO field contribution; as a consequence, the incident field at the first term of (36)-(37) should be cancelled in the shadow region there by the line integral field contribution.

For magnetic dipole illumination, following the same scheme, we apply (29) and (30) to the actual and image magnetic dipole $\mathbf{M}^{d}$ and $\mathbf{M}^{d}_{im}$, thus obtaining

$$\begin{aligned} \mathbf{e}^{PO}(P', Q, P) = {} & -2G(Q, P)\left[\mathbf{E}^{i}(P', Q) \cdot \hat{n}\right]\hat{n} \times \hat{\ell} - 2\zeta\left(1 + \tfrac{1}{jkr}\right) G(Q, P)\, \mathbf{H}^{i}(P', Q) \cdot \hat{\ell}\, \hat{r} \\ & +jkG(Q, P)\, G(P', Q)\left[\tfrac{|\hat{r} \times \hat{\ell}|}{1 + \hat{r'} \cdot \hat{r}} \mathbf{V} \times \mathbf{M}^{d} + \tfrac{|\hat{r} \times \hat{\ell}|}{1 + \hat{r'}_{im} \cdot \hat{r}} \mathbf{V}_{im} \times \mathbf{M}^{d}_{im}\right] \end{aligned} \tag{40}$$

and

$$\begin{aligned} \mathbf{h}^{PO}(P', Q, P) = {} & -2G(Q, P)\left[\mathbf{H}^{i}(P', Q) - \mathbf{H}^{i}(P', Q) \cdot \hat{n}\right]\hat{n} \times \hat{\ell} \\ & -jk\zeta G(Q, P)\, G(P', Q)\left[\tfrac{|\hat{r} \times \hat{\ell}|}{1 + \hat{r'} \cdot \hat{r}} \underline{\underline{\mathbf{D}}} \cdot \mathbf{M}^{d} + \tfrac{|\hat{r} \times \hat{\ell}|}{1 + \hat{r'}_{im} \cdot \hat{r}} \underline{\underline{\mathbf{D}}}_{im} \cdot \mathbf{M}^{d}_{im}\right] \end{aligned} . \tag{41}$$

Asymptotic limit for locally plane spherical wavefront

As for the aperture case, in the far field regime for the incident field, the above expression becomes formally the same for both electric and magnetic field provided that the incident and the reflected field can be used in the incremental PO contribution:

$$\left[\begin{array}{c}\mathbf{e}(P',Q,P)\\ \mathbf{h}(P',Q,P)\end{array}\right] \approx \left(-\underline{\underline{\mathbf{D}}}_0 \cdot \left[\begin{array}{c}\mathbf{E}^i(P',Q)\\ \mathbf{H}^i(P',Q)\end{array}\right] + \underline{\underline{\mathbf{D}}}_{0ref} \cdot \left[\begin{array}{c}\mathbf{E}^{ref}(P',Q)\\ \mathbf{H}^{ref}(P',Q)\end{array}\right]\right) G(Q,P) \tag{42}$$

where $\mathbf{E}^{ref}(P',Q)\, and H^{ref}(P',Q)$ are the reflected field, $\underline{\underline{\mathbf{D}}}_0$ is defined in (33) and $\underline{\underline{\mathbf{D}}}_{0ref}$ is obtained from $\underline{\underline{\mathbf{D}}}_0$ by replacing $\hat{r'}$ with $\hat{r'}_{im} = \hat{r}' - 2(\hat{n}' \cdot \hat{r}')\,\hat{n}'$.As mentioned for the case of the aperture field, the above relationships can be applied heuristically to a general source place placed in the far zone.

4. Illustrative Examples

Many numerical results have been calculated comparing the surface PO numerical integration and its line reduction formulated here. These tests have not only permitted to verify the exactness of the present formulation but also to test the CPU time acceleration rate with respect to a standard surface integration. Among them, we choose to show the results relevant to the geometry presented in Figure 3. We consider the PO scattered field from a perfectly conducting $3\lambda \times 2\lambda$ rectangular flat plate with vertices $V_1 = (0,0,0)$, $V_2 = (2\lambda,0,0)$, $V_3 = (2\lambda,3\lambda,0)$, $V_4 = (0,3\lambda,0)$, illuminated by either a vertical electric or a vertical magnetic dipole with a unit moment placed at $P' = (\lambda, 1.5\lambda, 3\lambda)$ ($\mathbf{J}^d = \hat{u}_d \times 1\, A \cdot m$, $\mathbf{M}^d = \hat{u}_d \times 1\, V \cdot m$, with $\hat{u}_d = \hat{z} = (0,0,1)$). In the origin of the reference system a spherical coordinates system $P = (r,\theta,\phi)$ is defined, and the observation scan is taken with $r = 5\lambda$, $\phi = 45°$ and θ ranging from $0°$ to $90°$. Figure 4 shows the amplitude of the spherical components of the PO electric field, when the plate is illuminated by the electric dipole. The continuous thick line refers to the PO scattered field calculated via the exact line integral (36) and (38), while the triangles mark the standard PO surface numerical integration (34). The two results are in perfect agreement, as expected. The third group of results, continuous thin lines with dots, are calculated via an asymptotic line integral reduction of PO using the simplified formulas (42). Despite the rim of the plate is sufficiently far from the dipole source (more than 3λ) to assume the incident field as a local plane wave, the asymptotic results are in a very poor accordance with the exact one. Indeed, the source illuminates the center of the plate with a far-field pattern null, therefore the boundary wave diffraction mechanism is also strongly affected by higher order slope diffraction contributions which are neglected in the asymptotic approximation. This results in a significant discrepancy, difficult to predict *a priori*, between the exact and the asymptotic results. This is particularly evident near the shadow boundary of the reflected geometrical optics fields, where the asymptotic-based solution exhibit a discontinuity in the first derivative. This comparison is intended to highlight the importance of the exact line-integral derivation, where the higher order contribution of the incident field are treated exactly, with respect to solution based on approximation which do not treat higher order incident field in a proper way.

Analogously, Figure 5 presents the amplitude of the spherical components of the PO scattered magnetic field. The exact line integral (37) and (39) (continuous thick line) superimpose the standard PO surface numerical integration (35) (triangles), while the asymptotic line integral result is again in scarce accordance. The same quantities are plotted again in Figures 6 and 7 when the plate is excited by a magnetic dipole.

The same remarks previously done still applies.

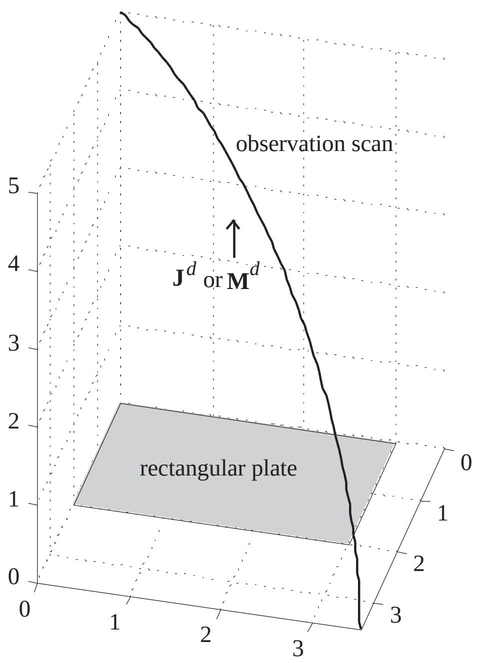

Figure 3. Geometry for the numerical results presented in Figures 4-7.

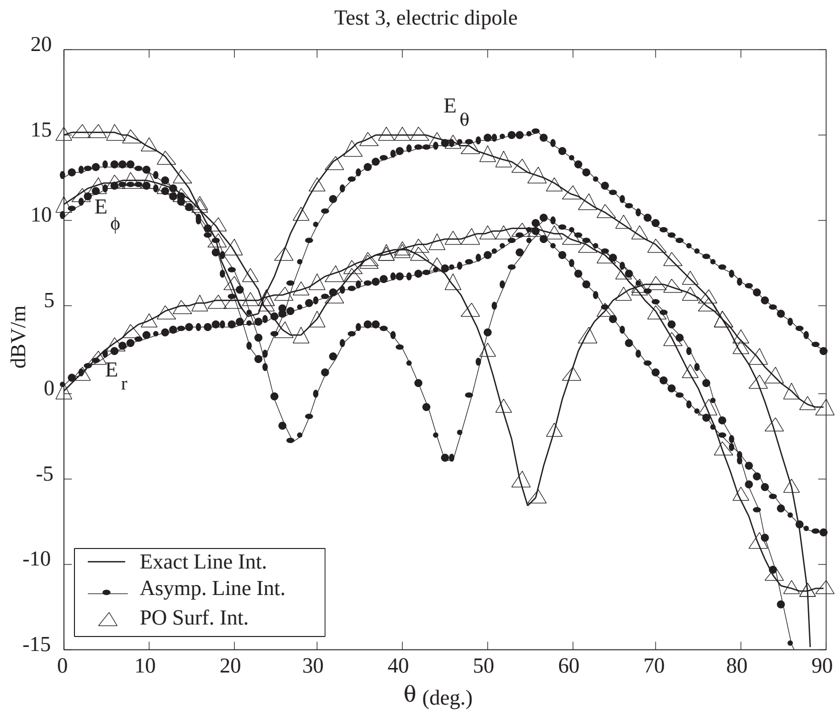

Figure 4. Scattered electric fields components for the case shown in Figure 3 with illumination by electric dipole. Exact line integration (—-), asymptotic line integration (•) and PO surface integration (Δ).

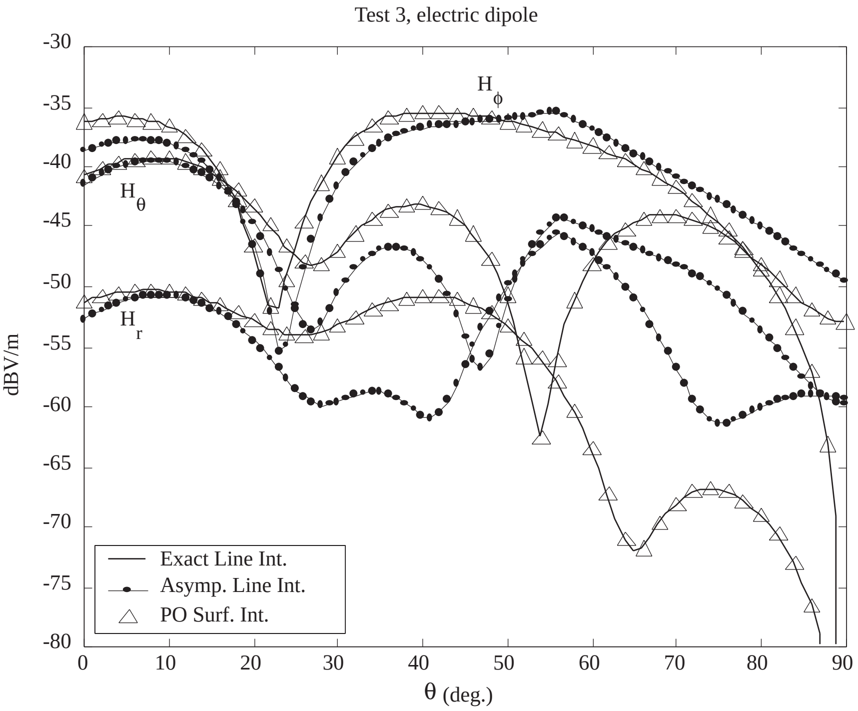

Figure 5. Scattered magnetic fields components for case shown in Figure 3 with illumination by an electric dipole. Exact line integration (—-), Asymptotic line integration (•) and Surface integration (Δ).

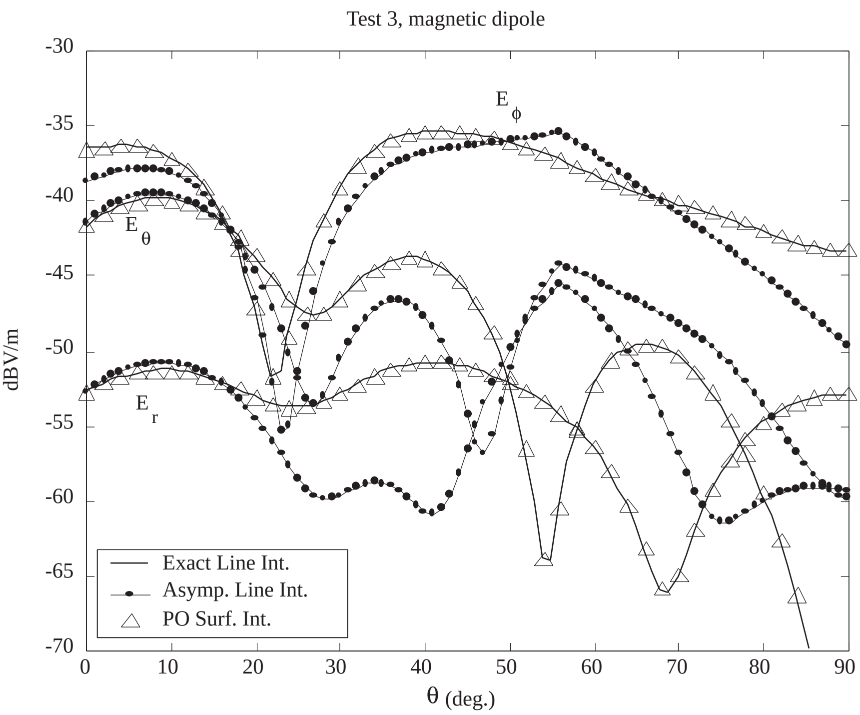

Figure 6. Scattered electric fields components for the case shown in Figure 3 with illumination by a magnetic dipole. Exact line integration (—-), asymptotic line integration (•) and PO surface integration (Δ).

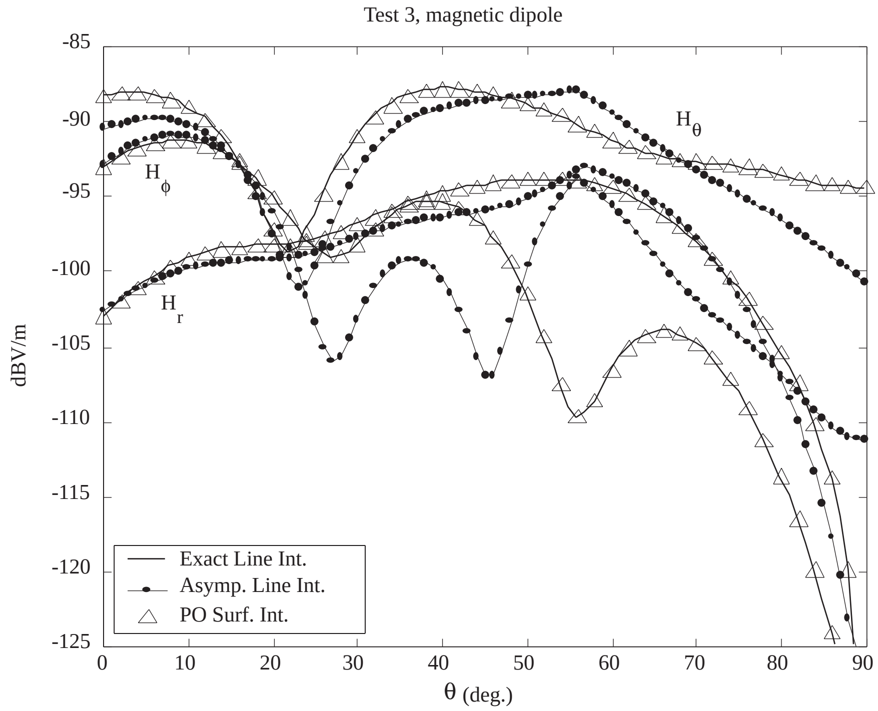

Figure 7. Scattered magnetic fields components for the case shown in Figure 3 with illumination by a magnetic dipole. Exact line integration (—-), Asymptotic line integration (•) and Surface integration (Δ).

5. Conclusions

An exact line integral representation has been presented for the PO field radiated by a planar, arbitrarily contoured perfectly conducting surface illuminated by both an electric or a magnetic arbitrarily located dipole. The formulation is based on the equivalence theorem and on the Kottler representation of the Kirchoff radiation integral. Although a line PO integral representation has been obtained by other authors with a similar approach [17], the final representation differs from that in [17] and appears quite simpler. The fact that two integrand of the closed line spatial representation of the same field can be different one to the other is not surprising, since an arbitrary irrotational vector potential gives zero when circuited on a closed contour. The present formulation, allows for a drastic reduction of computer time when the PO field from large bodies is obtained by facet description of the body surface, especially in the framework of a hybrid MoM-PO method. Furthermore the above formulation is useful when PO field is augmented by fringe contributions calculated in the framework of the Physical Theory of Diffraction [22]or of the Incremental Theory of Diffraction [23]-[24].

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