Chapter 2

Signals and Spectra

(Review)

2.1 Signals and Spectra

signal

- **Time domain description**

  Waveform, Power, ……

- **frequency domain description**

  Frequency, bandwidth, ……

2.1 Signals and Spectra

Waveform

Rectangular pulse

$$\Pi\left(\frac{t}{T}\right)=\begin{cases}1 & |t|\le T/2\\ 0 & |t|>T/2\end{cases}$$

Sa(x) function

$$Sa(x)=\frac{\sin x}{x}$$

Triangular function

$$\Lambda\left(\frac{t}{T}\right)=\begin{cases}1-|t|/T & |t|\le T\\ 0 & |t|>T\end{cases}$$

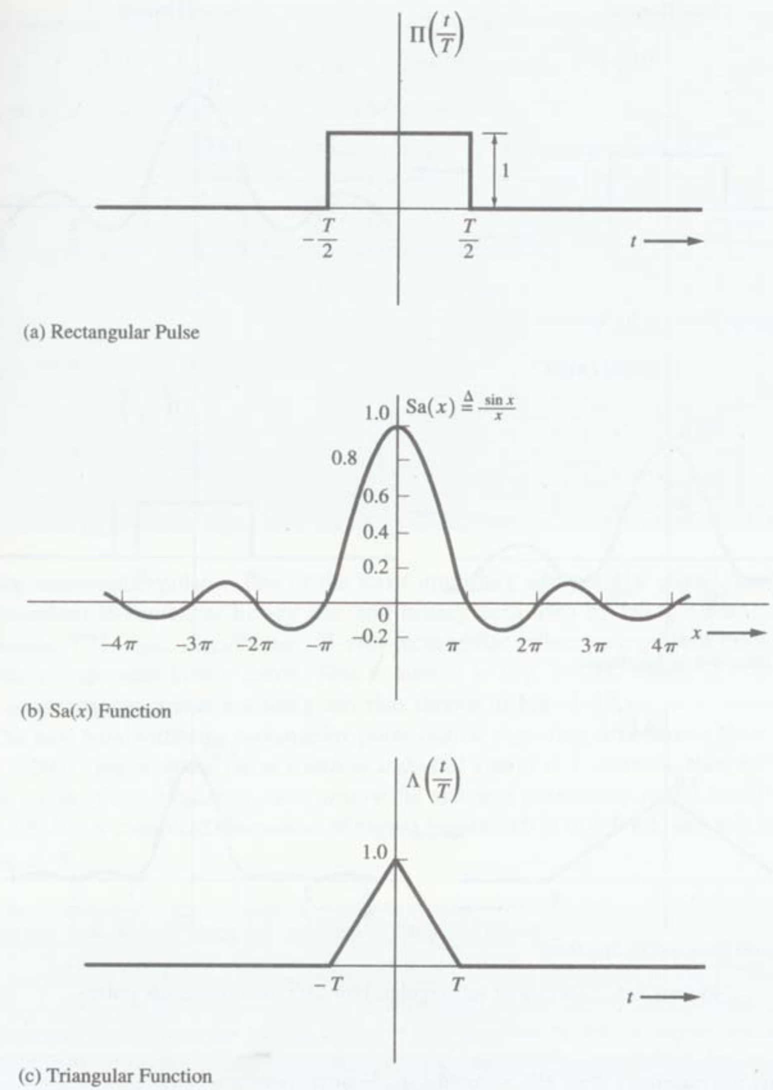

Figure 2–5 Waveshapes and corresponding symbolic notation.

Time average – Dc value of a waveform w(t):

$$W_{dc} = \langle w(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} w(t)dt$$

Power:

$$P = \langle w^2(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} w^2(t)dt$$

Energy:

$$E = \lim_{T \to \infty} \int_{-T/2}^{T/2} w^2(t)dt$$

Decibel

The decibel is a base 10 logarithmic measure of
Power ratios:

The decibel gain of a circuit:

$$dB = 10\log\left(\frac{average\ power\ out}{average\ power\ in}\right) = 10\log\left(\frac{P_{out}}{P_{in}}\right)$$

$$P = \frac{V_{rms}^2}{R} = I_{rms}^2 R$$

$$R = 1\Omega$$

$$dB = 20\log\left(\frac{V_{rms\ out}}{V_{rms\ in}}\right) = 20\log\left(\frac{I_{rms\ out}}{I_{rms\ in}}\right)$$

Decibel

The decibel signal-to-noise ratio:

$$dB = 10\log\left(\frac{P_{signal}}{P_{noise}}\right) = 10\log\left(\frac{\langle s^2(t)\rangle}{\langle n^2(t)\rangle}\right) = 20\log\left(\frac{V_{rms\ signal}}{V_{rms\ \ noise}}\right)$$

The decibel power level with respect to 1 mw :

$$dBm = 10\log\left(\frac{actual\ \ power\ level\ (watts)}{10^{-3}}\right)$$

other decibel measures: dBW, dBk ……

2.2 Fourier Transform and Spectra

The Fourier Transform of a waveform w(t) is:

$$W(f) = \mathcal{F}[w(t)] = \int_{-\infty}^{\infty} w(t) e^{-j2\pi ft} dt$$

The inverse Fourier transform:

$$w(t) = \mathcal{F}^{-1}[W(f)] = \int_{-\infty}^{\infty} W(f) e^{j2\pi ft} dt$$

Shorthand notation for a Fourier transform pair:

$$w(t) \leftrightarrow W(f)$$

Some useful pulses

$$\Pi\left(\frac{t}{T}\right) \leftrightarrow T\,Sa(\pi f T)$$

$$T\,Sa(\pi T t) \leftrightarrow \Pi\left(\frac{f}{T}\right)$$

$$\Lambda\left(\frac{t}{T}\right) \leftrightarrow T\,Sa^2(\pi f T)$$

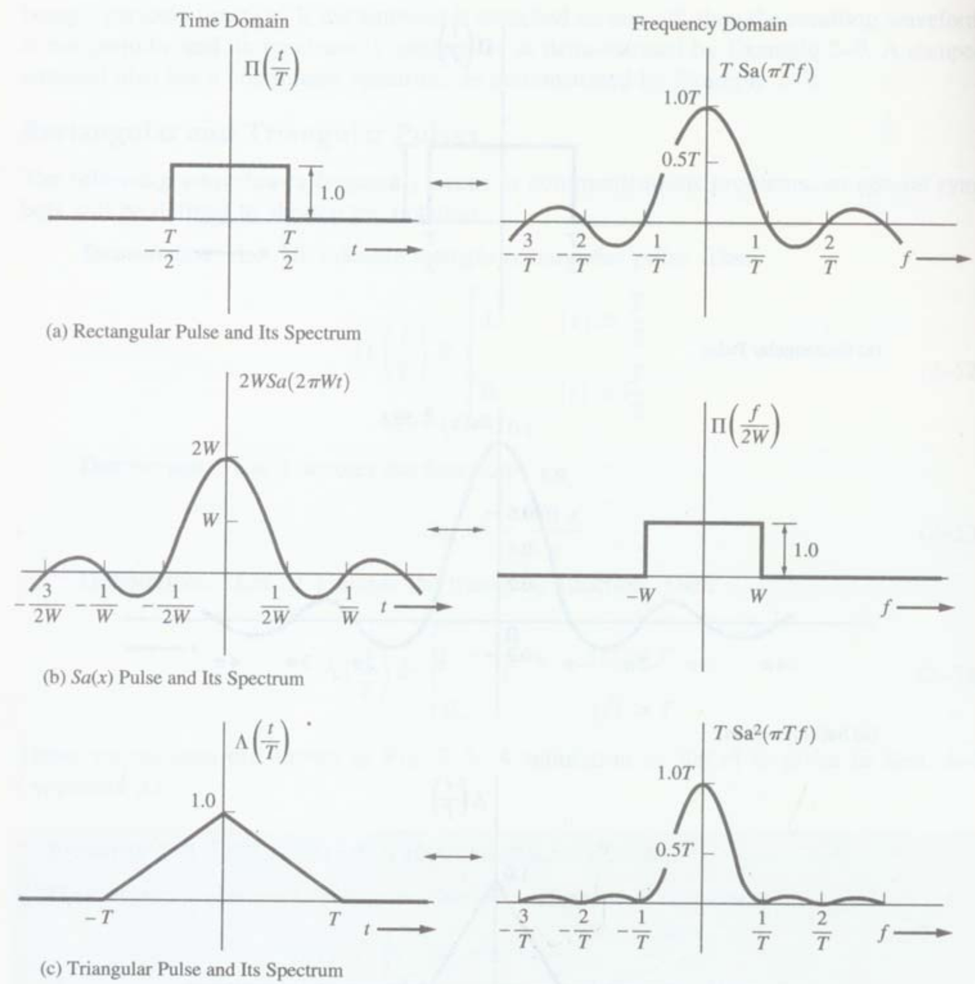

Figure 2–6 Spectra of rectangular, (sin x)/x, and triangular pulses.

Properties

- Linearity, time delay, scale change, conjugation, ……

- Spectral symmetry of real signals

$$W(-f) = W^*(f)$$

- Parseval's theorem

$$\int_{-\infty}^{\infty} w_1(t) w_2^*(t)\,dt = \int_{-\infty}^{\infty} W_1(f) W_2^*(f)\,df$$

2.2 Fourier Transform and Spectra

Proof:

$$\int_{-\infty}^{\infty} w_1(t)w_2^*(t)dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} W_1(f)e^{j2\pi ft}df\right]w_2^*(t)dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_1(f)w_2^*(t)e^{j2\pi ft}dfdt$$

$$= \int_{-\infty}^{\infty} W_1(f)\left[\int_{-\infty}^{\infty} w_2(t)e^{-j2\pi ft}dt\right]^* df = \int_{-\infty}^{\infty} W_1(f)W_2^*(f)df$$

when $w(t) = w_1(t) = w_2(t)$

$$E = \int_{-\infty}^{\infty} |w_1(t)|^2 dt = \int_{-\infty}^{\infty} |W(f)|^2 df$$

Rayleigh's energy theorem

Energy spectrum density is: $\mathcal{E}(f) = |W(f)|^2$

2.3 PSD and Autocorrelation Function

The average normalized power in **time domain description** is:

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} w^2(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} w_T{}^2(t)dt$$

where $w_T(t) = w(t)\Pi(t/T)$ is the truncated version of the waveform. By the use of Parseval's theorem, we have:

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} |W_T(f)|^2 df = \int_{-\infty}^{\infty} \left(\lim_{T\to\infty} \frac{|W_T(f)|^2}{T} \right) df \quad (2\text{-}65)$$

2.3 PSD and Autocorrelation Function

The power spectral density (PSD) for a deterministic power waveform is:

$$p_w(f) = \lim_{T\to\infty} \frac{|W_T(f)|^2}{T}$$

For a real waveform the autocorrelation function is:

$$R_\omega(\tau) = \langle w(t)w(t+\tau)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} w(t)w(t+\tau)dt$$

The PSD and the autocorrelation function are Fourier transform pairs:

$$R_w(\tau) \leftrightarrow \mathcal{P}_w(f)$$

2.3 PSD and Autocorrelation Function

- **The PSD can be obtained by either of the following two methods:**

1) Direct method:

$$\mathcal{P}_w(f) = \lim_{T\to\infty} \frac{|W(f)|^2}{T}$$

2) InDirect method:

$$\mathcal{P}_w(f) = \mathcal{F}\left(R_w(\tau)\right)$$

2.3 PSD and Autocorrelation Function

The **total average normalized power** for the waveform $w(t)$ can be evaluated by using any of the **four techniques** embedded in the following equation:

$$
\begin{aligned}
P &= \left\langle w^2(t) \right\rangle \\
&= \int_{-\infty}^{\infty} \mathcal{P}_w(f)\,df \\
&= W_{rms}^2 \\
&= R_w(0)
\end{aligned}
$$

Orthogonal function

If the functions in the set { $\phi_n(t)$ } are orthogonal, then they satisfy the relation

$$\int_a^b \varphi_n(t)\varphi_m^*(t)dt = \begin{cases} 0 & m \neq n \\ K_n & m = n \end{cases} = K_n\delta_{mn}$$

where $\delta_{mn} = \begin{cases} 0 & \mathrm{n} \neq \mathrm{m} \\ 1 & \mathrm{n} = \mathrm{m} \end{cases}$

if the constants K_n are all equal to 1, the $\phi_n(t)$ are said to be orthonormal functions

Orthogonal Series

$w(t)$ can be represented over the interval (a,b) by the series:

$$w(t) = \sum_n a_n \varphi_n(t) \qquad (2\text{-}83)$$

where the orthogonal coefficients are given by:

$$a_n = \frac{1}{K_n} \int_a^b w(t) \varphi_n^*(t) dt \qquad (2\text{-}84)$$

Note:

The orthogonal function set $\{\varphi_n(t)\}$ has to be complete

- **complex exponential Fourier series form:**

A physical waveform may be represented over the interval $a<t<a+T_0$ by the complex exponential Fourier series:

$$w(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

where the complex (phasor) Fourier coefficients are:

$$c_n = \frac{1}{T_0}\int_a^{a+T_0} w(t)e^{-jn\omega_0 t}\,dt$$

and $\omega_0 = 2\pi f_0 = 2\pi / T_0$

2.5 Fourier Series

- **the spectrum of the periodic waveform**

  The spectrum of the waveform w(t) with period T_0 is :

$$W(f) = \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_0)$$

- **PSD for Periodic waveform**

  For a periodic waveform, the PSD is given by:

$$\mathcal{P}_w(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0)$$

the normalized power : $P_w = \langle w^2(t) \rangle = \sum_{n=-\infty}^{\infty} |c_n|^2$

2.9 Bandwidth of Signals

- The **spectral width of signals and noise** in communication systems is a very important concept.

- In engineering definitions, **the bandwidth** is taken to be the width of a **positive frequency band**. In other words, the bandwidth is:

$$f_2 - f_1$$

where $f_2 > f_1 \geqslant 0$

Some definitions of bandwidth:

- **Absolute bandwidth**: $f_2 - f_1$
- **Null-to-null bandwidth** (or zero-crossing bandwidth): $f_2 - f_1$
- **Equivalent noise bandwidth**:

$$\int_0^{\infty} p_s(f)df = B_{eq}\, p_s(f_0)$$

$$B_{eq} = \frac{1}{|H(f_0)|^2}\int_0^{\infty} |H(f_0)|^2 df$$

- **3-dB bandwidth, 6-dB bandwidth**, ……

2.9 Bandwidth of Signals

Example2-18 Bandwidths for a BPSK signal (p106)

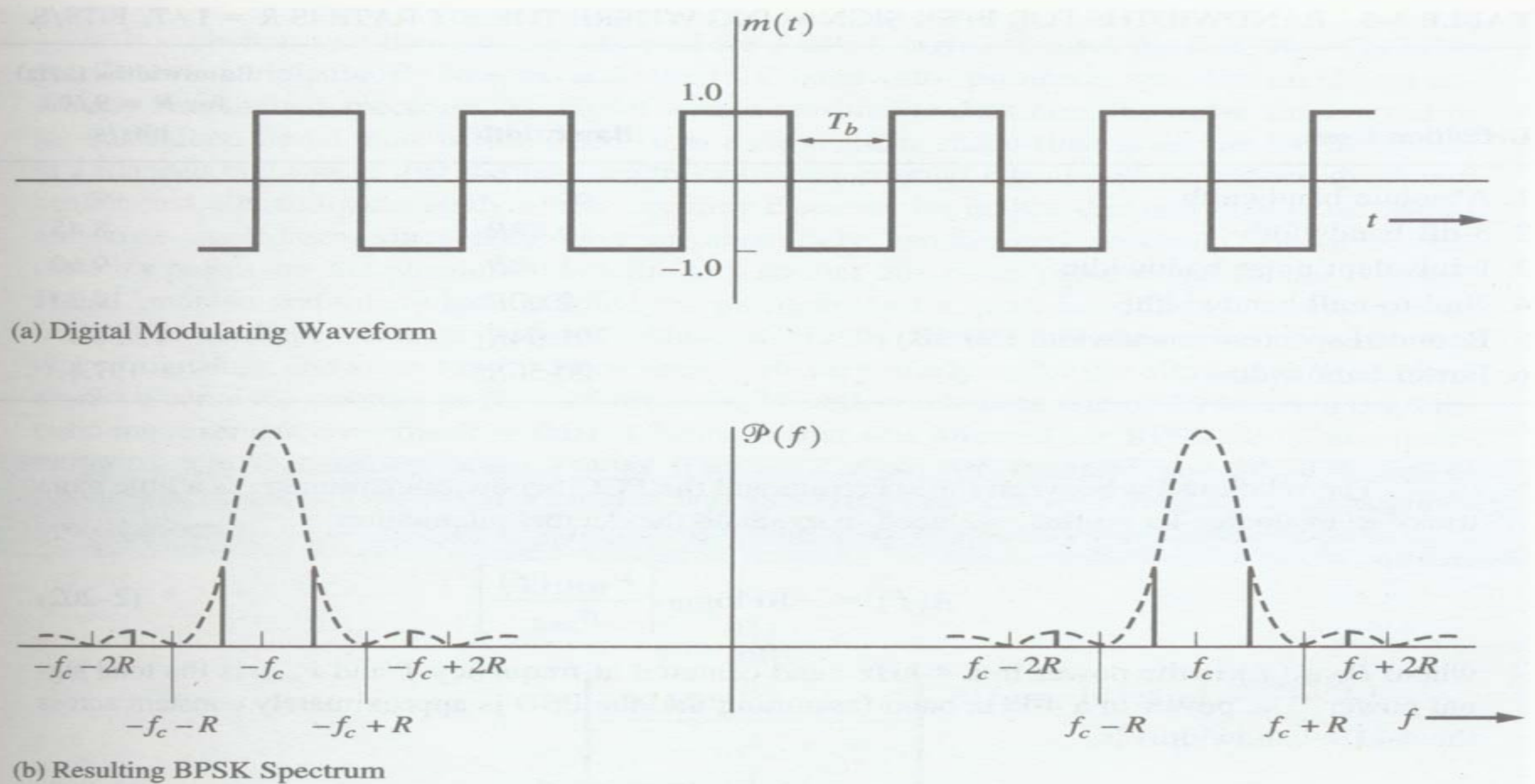

Figure 2–23 Spectrum of a BPSK signal.