Construction of Cubic Triangular Patches with C^1 Continuity around a Corner

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Abstract

This paper presents a novel approach for constructing a piecewise triangular cubic polynomial surface with C^1 continuity around a common corner vertex. A C^1 continuity condition between two cubic triangular patches is first derived using mixed directional derivatives. An approach for constructing a surface with C^1 continuity around a corner is then developed. Our approach is easy and fast with the virtue of cubic reproduction, local shape controllability, C^2 continuous at the corner vertex. Some experimental results are presented to show the applicability and flexibility of the approach.

Key Words: Triangular patches, vertex consistency problem, C^1 continuity, interpolation

1. Introduction

Surface design is an important field in computer aided geometric design and computer graphics. A widely accepted and popular way in surface reconstruction from the scattered data is the use of smoothly joined triangular Bernstein-Bézier patches or B-spline patches. The resulting surface must be visually smooth, that is, the patch boundary and across-boundary data must agree with the given values and this provides C or G continuity for the overall surface. The compositions of Bézier triangles that meet with G^1 continuity have been developed by many researchers [5,10,11]. The twist compatibility problem [18] or the vertex enclosure/consistency problem [12] which arises when joining some polynomial patches with G^1 continuity around a common vertex is a difficult problem. For example, let 5 triangular patches $T_i, i = 1, 2, \cdots 5$ meet at a corner as in Figure1. Starting from the first patch T_1, patch T_{i+1} could be determined by the continuity conditions with T_i along their common boundary, $i = 1, 2, 3, 4$. But satisfying the continuity condition along the common boundary between patch T_5 and T_1 will present serious difficulties.

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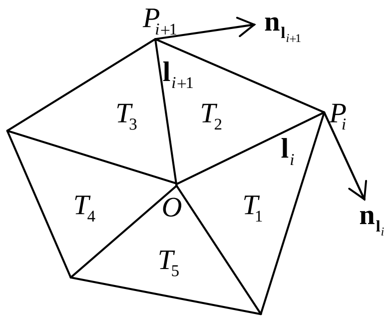

Fig. 1. Triangular patches around a corner.

The earliest schemes that addressed the vertex consistency problem are Clough-Tocher-like domain splitting methods [1,13,16]. The triangles are divided into three sub-triangles and quartic G^1 patch per sub-triangle is produced to interpolate positions and normals. The free parameters are employed in order to control the shapes.

The Gregory technique [3,4] seeks to construct patches on the faces formed by a net of intersecting curves in space. It uses the curves themselves and cross-boundary tangent information. From the given information, sub-patches are formed at each corner of a face and these are then blended to form the full patch that join together with tangent plane continuity. A number of variants and extensions to the basic method have been investigated by several researchers [15,19]. The extensions of the technique to give higher order continuity are studied in [7,8].

Loop presents a piecewise G^1 spline surface composed of sextic triangular Bézier patches, one per triangle [9]. Optional shape parameters are available for additional local control over the shape of the surface. But unwanted surface undulations occur due to severe constraints on the second derivatives along boundary curves at each end-point. The recent work of [5] presents an interpolating quintic G^1 triangular spline surface, which is a generalization of Loop's scheme [9]. The basic idea is to use a regular 4-split of each triangle so that the constraints between each end-point of the boundary curves are relaxed and an interpolating curve network can be built without unwanted undulations. Both strategies lead to linear systems of equations with a circulant matrix, which will not give a solution in general if an even number of patches meet at a corner [5,9].

Furthermore some special approaches can be found in [6,14,17], in which various restrictions are made on the input data, thus their methods are not general enough in practice.

In this paper we present a novel approach to construct a piecewise triangular polynomial surfaces with C^1 continuity around a corner vertex. For easy evaluation and manipulation one often aims at a local method which uses low degree polynomial patches. Cubic polynomials are used in our method. The basic idea is to use and keep the mixed directional derivatives along their common boundary between adjacent patches. The result surface is piecewise cubic triangular polynomials and is C^1

continuous across the boundaries between different patches and is C^2 continuous at the corner vertex. Our approach is easy to use, the result surfaces can be quickly obtained by solving a simple 4×4 linear system which is always non-degenerate. The user can adjust the input values to control the shape of the surface interactively, which makes it a new and useful tool for shape design in CAD.

2. Preliminaries

2.1 Representations of triangular surfaces

Let T be a non-degenerate triangle in the plane with vertices $T_i=(x_i,y_i), i=1,2,3$. Any point $P=(x,y)$ within T can be expressed uniquely as

$$P=uT_1+vT_2+wT_3$$

in terms of barycentric coordinates $(u,v,w), u+v+w=1, u\geq 0, v\geq 0, w\geq 0$ that can be obtained by solving the following equations

$$\begin{cases} x &= ux_1+vx_2+wx_3 \\ y &= uy_1+vy_2+wy_3 \\ 1 &= u+v+w \end{cases} \tag{1}$$

The Bernstein-Bézier polynomial or the Bézier triangular surface of degree n over triangle domain T has the form[2]

$$T^n(P)=T^n(u,v,w)=\sum_{i+j+k=n} B^n_{i,j,k}(u,v,w)T_{i,j,k}$$

where $B^n_{i,j,k}(u,v,w)=\dfrac{n!}{i!\,j!\,k!}u^i v^j w^k, T_{i,j,k}\in R.$ Let

$$\prod_n:=\left\{\sum_{i=0}^{n}\sum_{j=0}^{n-i} a_{i,j}x^i y^j, a_{i,j}\in R\right\}$$

be the polynomial set with degree no larger than n. It is known that the degree n Bernstein-Bézier polynomials and the polynomials in $\prod_n$ can be converted into each other using Eq. 1[2].

2.2 Directional derivatives

Let $u(x,y)$ be a bivariate continuous function with continuous second order partial derivatives. Let $\mathbf{l}$ be a vector in the plane. The *directional derivative* of $u(x,y)$ according to direction $\mathbf{l}$ is defined by

$$\frac{\partial u}{\partial \mathbf{l}} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha ,$$

where α is the anti-clockwise orientational angle from x-axis to the vector $\mathbf{l}$, see Figure 1 [2].

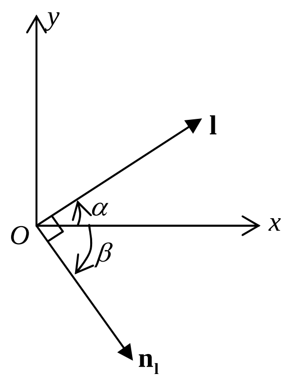

Fig. 2. Vector **l** and its normal vector $\mathbf{n}_\mathbf{l}$ in the plane.

The *mixed directional derivative* may be obtained from a generalization of the above definition of direction derivative: let $\mathbf{l}$ and $\mathbf{m}$ be two independent vectors in the plane. Then the mixed directional derivative of $u(x, y)$ according to directions , $\mathbf{m}$ is defined by the directional derivative of $\frac{\partial u}{\partial \mathbf{l}}$ according to vector $\mathbf{m}$ as:

$$\frac{\partial^2 u}{\partial \mathbf{l} \partial \mathbf{m}} = \frac{\partial}{\partial \mathbf{m}}\left(\frac{\partial}{\partial \mathbf{l}}\right).$$

It is easily seen that

$$\frac{\partial^2 u}{\partial \mathbf{l} \partial \mathbf{m}} = \frac{\partial^2 u}{\partial \mathbf{m} \partial \mathbf{l}} .$$

Denote $\mathbf{n}_\mathbf{l}$ as the vector that is orthogonal to $\mathbf{l}$ such that the anti-clockwise angle from $\mathbf{n}_\mathbf{l}$ to $\mathbf{l}$ is $\frac{\pi}{2}$, see Figure 2. The vector $\mathbf{n}_\mathbf{l}$ is called *the normal vector* of $\mathbf{l}$.

Given a vector $\mathbf{l}$ having angle α with x axis, we can obtain the mixed directional derivative of $u(x, y)$ according to vector $\mathbf{l}$ and its normal vector $\mathbf{n}_\mathbf{l}$ by simple computation as

$$\begin{aligned} \frac{\partial^2 u}{\partial \mathbf{l} \partial \mathbf{m}} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha\right)\cos\beta + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha\right)\sin\beta \\ &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha\right)\sin\alpha - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\sin\alpha\right)\cos\beta \qquad (2) \\ &= \frac{\partial^2 u}{\partial x^2}\sin\alpha\cos\alpha + \frac{\partial^2 u}{\partial x \partial y}\left(\sin^2\alpha - \cos^2\alpha\right) - \frac{\partial^2 u}{\partial y^2}\sin\alpha\cos\alpha, \end{aligned}$$

where $\beta = \alpha - \pi/2$ is the anti-clockwise angle from x axis to $\mathbf{n}_\mathbf{l}$, see Figure 2. Similarly, the second order directional derivative of $u(x,y)$ according to $\mathbf{l}$ is derived by

$$\frac{\partial^2 u}{\partial \mathbf{l}^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\alpha + \frac{\partial^2 u}{\partial x \partial y}\sin 2\alpha + \frac{\partial^2 u}{\partial y^2}\sin^2\alpha. \tag{3}$$

It is seen from Eqs. 2 and 3 that the second order directional derivatives of $u(x,y)$ according to the vector $\mathbf{l}$ are dependent on the second order partial derivatives of $u(x,y)$ and the angle between the vector $\mathbf{l}$ and x axis.

3. Main results and proof

First, we have the following lemma.

Lemma 1. *Let* $\mathbf{l}$ *be a vector in the plane and* $u(x,y)$ *be a bivariate function defined on the plane. A rotation transformation is applied so that the* y' *axis of the new coordinate frame* $ox'y'$ *is coincident with the vector* $\mathbf{l}$ *and the new bivariate function is denoted by* $\overline{u}(x',y')$. *Then*

$$\frac{\partial u}{\partial \mathbf{l}} = \frac{\partial \overline{u}}{\partial y'}, \frac{\partial^2 u}{\partial \mathbf{l}^2} = \frac{\partial^2 \overline{u}}{\partial y'^2}, \frac{\partial^2 u}{\partial \mathbf{l} \partial \mathbf{n}_\mathbf{l}} = \frac{\partial^2 \overline{u}}{\partial x' \partial y'}.$$

Proof. The conclusions can be easily shown by noting that the rotation transformation has the following form

$$\begin{cases} x = x'\sin\alpha + y'\cos\alpha \\ y = -x'\cos\alpha + y'\sin\alpha \end{cases}.$$

We are now ready to prove the main result in this paper. The continuity conditions between two cubic bivariate polynomial triangular patches along their common boundary are given by the following theorem.

Theorem 1. *Consider two adjacent cubic polynomial triangular patches* $z = u(x,y)$ *and* $z = v(x,y)$ *that share a common boundary* $\mathbf{l} = \overrightarrow{op}$ *with* $O = (x_0, y_0)$, $P = (x_1, y_1)$, *see Figure 3. If* $u(x,y)$ *and* $v(x,y)$ *satisfy the following conditions:*

(I) $u(x_0, y_0) = v(x_0, y_0)$,

(II) $u(x_1, y_1) = v(x_1, y_1)$,

(III) $\frac{\partial u}{\partial \mathbf{n}_\mathbf{l}}(x_1, y_1) = \frac{\partial v}{\partial \mathbf{n}_\mathbf{l}}(x_1, y_1)$,

(IV) $\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0)$,

(V) $\frac{\partial u}{\partial y}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$,

(VI) $\frac{\partial^2 u}{\partial \mathbf{l}^2}(x_0, y_0) = \frac{\partial^2 v}{\partial \mathbf{l}^2}(x_0, y_0)$,

(VII) $\frac{\partial^2 u}{\partial \mathbf{l} \partial \mathbf{n}_{\mathbf{l}}}(x_0, y_0) = \frac{\partial^2 v}{\partial \mathbf{l} \partial \mathbf{n}_{\mathbf{l}}}(x_0, y_0)$,

then the two patches $u(x, y)$ *and* $v(x, y)$ *join at the common boundary* $\mathbf{l}$ *with* C^1 *continuity.*

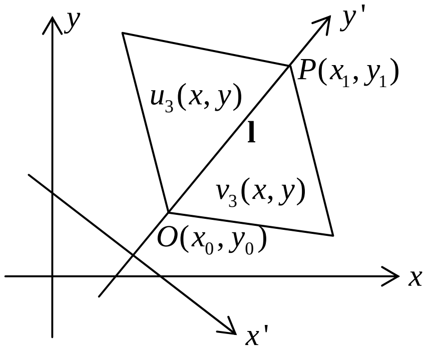

Fig. 3. Two triangular patches $u(x, y)$ and $v(x, y)$ with a common boundary

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Proof. Without loss of generality, we assume that the point O on the common boundary $\mathbf{l}$ lies on y axis because any translation transformation will not change the given conditions and conclusions. We then apply a rotation transformation to make the y' axis of the new coordinate frame coincident with the common boundary $\mathbf{l} = \overrightarrow{OP}$. For simplicity, the new coordinate axis x', y' are still denoted as x, y and the transformed point (x'_*, y'_*) of (x_*, y_*) are still denoted as (x_*, y_*). It can be easily shown by simple computation that under the above rotation transformation the conditions of the theorem are converted into:

(I) $u(0, y_0) = v(0, y_0)$,

(II) $u(0, y_1) = v(0, y_1)$,

(III) $\frac{\partial u}{\partial x}(0, y_1) = \frac{\partial v}{\partial x}(0, y_1)$,

(IV) $\frac{\partial u}{\partial x}(0, y_0) = \frac{\partial v}{\partial x}(0, y_0)$,

(V) $\frac{\partial u}{\partial y}(0, y_0) = \frac{\partial v}{\partial y}(0, y_0)$,

(VI) $\frac{\partial^2 u}{\partial y^2}(0, y_0) = \frac{\partial^2 v}{\partial y^2}(0, y_0)$,

(VII) $\frac{\partial^2 u}{\partial x \partial y}(0, y_0) = \frac{\partial^2 v}{\partial x \partial y}(0, y_0)$,

The cubic bivariate polynomial $u(x, y)$ can be rewritten as

$$u(x,y) = u_2(x,y)x + p_3(y), \tag{4}$$

where $u_2(x,y)$ is a quadric bivariate polynomial, $p_3(y)$ is cubic polynomial on y. Noting that $u(x,y)\big|_{x=0} = p_3(y)$, it can be shown that a unique cubic polynomial $p_3(y)$ can be determined by the four interpolation conditions (I), (II), (V) and (VI). Therefore, we have

$$u(x,y)\big|_{x=0} = v(x,y)\big|_{x=0} = p_3(y),$$

which indicates that the two patches $u(x,y)$ and $v(x,y)$ are C^0 continuous along their common boundary.

We now try to prove that they are C^1 continuous along their common boundary. Following Eq. 4, we have

$$\left.\frac{\partial u}{\partial y}\right|_{x=0} = \left.\frac{\partial v}{\partial y}\right|_{x=0} = p_3'(y) \tag{5}$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial u_2(x,y)}{\partial x} \cdot x + u_2(x,y) := u_1(x,y)x + p_2(y),$$

where $u_1(x,y)$ is a linear bivariate polynomial, and $p_2(y)$ is a quadratic polynomial with y. It can be derived from the above equation that

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \frac{\partial u_1(x,y)}{\partial y} \cdot x + p_2'(y),$$

thus we have

$$\left.\frac{\partial u(x,y)}{\partial x}\right|_{x=0} = p_2(y), \text{ and } \left.\frac{\partial^2 u(x,y)}{\partial x \partial y}\right|_{x=0} = p_2'(y).$$

It can be proven that the above quadratic polynomial $p_2(y)$ can be uniquely determined by the 3 conditions (III), (IV), and (VII). Thus we have

$$\left.\frac{\partial u(x,y)}{\partial x}\right|_{x=0} = \left.\frac{\partial v(x,y)}{\partial x}\right|_{x=0} = p_2(y), \tag{6}$$

The conclusion of the theorem is thus obtained following Eqs. 5 and 6.

4. Construction of surface with C^1 continuity around a corner

It is known that we need 10 independent conditions to determine a cubic bivariate polynomial surface or a cubic Bézier surface over a triangular patch.

Considering a corner vertex O of order n, with neighbor vertices P_i, where the subscripts are always taken modulo n, we define a surface patch $T_i^3(P) = T_i^3(x, y)$ over each triangle ΔOP_iP_{i+1}, $i = 1, 2, \cdots, n$, see Figure 4. Let $\mathbf{l}_i$ and $\mathbf{n}_{\mathbf{l}_i}$ be the edge vector OP_i and its corresponding normal vector.

Now we have the following theorem.

Theorem 2. *Consider the surface patch* $T_i^3(x, y)$ *over triangle* ΔOP_iP_{i+1} *for a specific index* i. *If* $\lambda_{i,j}, j = 0, 1, \cdots, 9,$ *are given, then a cubic bivariate polynomial surface* $T_i^3(x, y)$ *can be uniquely obtained by the following 10 conditions in two bundles:*

(I) $$T_i^3(O) = \lambda_{i,0}, \frac{\partial T_i^3}{\partial x}(O) = \lambda_{i,1}, \frac{\partial T_i^3}{\partial y}(O) = \lambda_{i,2}, \frac{\partial^2 T_i^3}{\partial x^2}(O) = \lambda_{i,3}, \frac{\partial^2 T_i^3}{\partial x \partial y}(O) = \lambda_{i,4}, \frac{\partial^2 T_i^3}{\partial y^2}(O) = \lambda_{i,5},$$

(II) $$T_i^3(P_i) = \lambda_{i,6}, T_i^3(P_{i+1}) = \lambda_{i,7}, \frac{\partial T_i^3}{\partial \mathbf{n}_{\mathbf{l}_i}}(P_i) = \lambda_{i,8}, \frac{\partial T_i^3}{\partial \mathbf{n}_{\mathbf{l}_{i+1}}}(P_{i+1}) = \lambda_{i,9}.$$

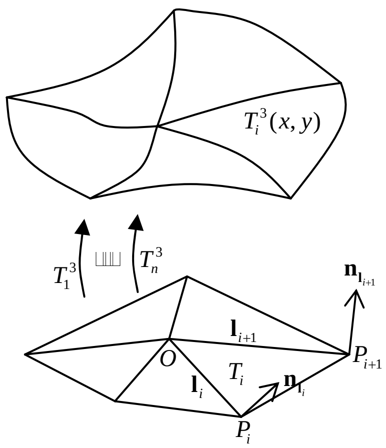


Fig. 4. Cubic polynomial surface patches defined on the triangles around a corner vertex *O*.

Proof. Without loss of generality, the point O is assumed to be $(0,0)$. Let the cubic bivariate polynomial be represented by

$$T_i^3(x, y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3.$$

We can determine the coefficients $a_0, a_1, a_2, a_3, a_4, a_5$ from condition bundle (I) as

$$a_0 = \lambda_{i,0}, a_1 = \lambda_{i,1}, a_2 = \lambda_{i,2}, a_3 = \frac{\lambda_{i,3}}{2}, a_4 = \lambda_{i,4}, a_5 = \frac{\lambda_{i,5}}{2}. \tag{7}$$

We now prove that the other coefficients can be determined by condition bundle **(II)**. Suppose $P_i = (x_i, y_i)$ The unit vector of $\mathbf{l}_i$ is

$$\left(\frac{x_i}{\sqrt{x_i^2 + y_i^2}}, \frac{y_i}{\sqrt{x_i^2 + y_i^2}} \right) := \left(r_1, r_2 \right),$$

and the unit vector of $\mathbf{n}_{\mathbf{l}_i}$ is $\left(r_2, -r_1 \right)$. Thus we have

$$\frac{\partial T_i^3}{\partial \mathbf{n}_{\mathbf{l}_i}} = \frac{\partial T_i^3}{\partial x} \cdot \frac{y_i}{\sqrt{x_i^2 + y_i^2}} - \frac{\partial T_i^3}{\partial y} \cdot \frac{x_i}{\sqrt{x_i^2 + y_i^2}}. \tag{8}$$

The first two equations $T_i^3(P_i) = \lambda_{i,6}$ and $T_i^3(P_{i+1}) = \lambda_{i,7}$ of condition bundle (II) can be respectively converted into

$$x_i^3 a_6 + x_i^2 y_i a_7 + x_i y_i^2 a_8 + y_i^3 a_9 = A_1, \tag{9}$$

and

$$x_{i+1}^3 a_6 + x_{i+1}^2 y_{i+1} a_7 + x_{i+1} y_{i+1}^2 a_8 + y_{i+1}^3 a_9 = A_2 \tag{10}$$

where

$$A_1 = \lambda_{i,6} - \left(a_0 + a_1 x_i + a_2 y_i + a_3 x_i^2 + a_4 x_i y_i + a_5 y_i^2 \right),$$
$$A_2 = \lambda_{i,7} - \left(a_0 + a_1 x_{i+1} + a_2 y_{i+1} + a_3 x_{i+1}^2 + a_4 x_{i+1} y_{i+1} + a_5 y_{i+1}^2 \right).$$

From Eq.8 the last two equations $\frac{\partial T_i^3}{\partial n_{l_i}}(P_i) = \lambda_{i,8}$ and $\frac{\partial T_i^3}{\partial n_{l_{i+1}}}(P_{i+1}) = \lambda_{i,9}$ can be respectively converted into

$$\frac{3x_i^2 y_i}{\sqrt{x_i^2 + y_i^2}} a_6 + \frac{2x_i y_i^2 - x_i^3}{\sqrt{x_i^2 + y_i^2}} a_7 + \frac{y_i^3 - 2x_i^2 y_i}{\sqrt{x_i^2 + y_i^2}} a_8 - \frac{3x_i y_i^2}{\sqrt{x_i^2 + y_i^2}} a_9 = A_3, \tag{11}$$

and

$$\frac{3x_{i+1}^2 y_{i+1}}{\sqrt{x_{i+1}^2 + y_{i+1}^2}} a_6 + \frac{2x_{i+1} y_{i+1}^2 - x_{i+1}^3}{\sqrt{x_{i+1}^2 + y_{i+1}^2}} a_7 + \frac{y_{i+1}^3 - 2x_{i+1}^2 y_{i+1}}{\sqrt{x_{i+1}^2 + y_{i+1}^2}} a_8 - \frac{3x_{i+1} y_{i+1}^2}{\sqrt{x_{i+1}^2 + y_{i+1}^2}} a_9 = A_4, \tag{12}$$

where

$$A_3 = \lambda_{i,8} - \frac{1}{\sqrt{x_i^2 + y_i^2}}\left[-a_2 x_i + a_1 y_i - a_4 x_i^2 + 2\left(a_3 - a_5\right) x_i y_i + a_4 y_i^2\right],$$

$$A_4 = \lambda_{i,9} - \frac{1}{\sqrt{x_{i+1}^2 + y_{i+1}^2}}\left[-a_2 x_{i+1} + a_1 y_{i+1} - a_4 x_{i+1}^2 + 2\left(a_3 - a_5\right) x_{i+1} y_{i+1} + a_4 y_{i+1}^2\right].$$

Eqs. 9, 10, 11, 12 form a linear system with unknown coefficients a_6, a_7, a_8, a_9. It has unique solution if and only if the following determinant:

$$D = \begin{vmatrix} x_i^3 & x_i^2 y_i & x_i y_i^2 & y_i^3 \\ x_{i+1}^3 & x_{i+1}^2 y_{i+1} & x_{i+1} y_{i+1}^2 & y_{i+1}^3 \\ \frac{3x_i^2 y_i}{\sqrt{x_i^2 + y_i^2}} & \frac{2x_i y_i^2 - x_i^3}{\sqrt{x_i^2 + y_i^2}} & \frac{y_i^3 - 2x_i^2 y_i}{\sqrt{x_i^2 + y_i^2}} & -\frac{3x_i y_i^2}{\sqrt{x_i^2 + y_i^2}} \\ \frac{3x_{i+1}^2 y_{i+1}}{\sqrt{x_{i+1}^2 + y_{i+1}^2}} & \frac{2x_{i+1} y_{i+1}^2 - x_{i+1}^3}{\sqrt{x_{i+1}^2 + y_{i+1}^2}} & \frac{y_{i+1}^3 - 2x_{i+1}^2 y_{i+1}}{\sqrt{x_{i+1}^2 + y_{i+1}^2}} & -\frac{3x_{i+1} y_{i+1}^2}{\sqrt{x_{i+1}^2 + y_{i+1}^2}} \end{vmatrix} \neq 0$$

or the following determinant

$$D = \begin{vmatrix} x_i^3 & x_i^2 y_i & x_i y_i^2 & y_i^3 \\ x_{i+1}^3 & x_{i+1}^2 y_{i+1} & x_{i+1} y_{i+1}^2 & y_{i+1}^3 \\ 3x_i^2 y_i & 2x_i y_i^2 - x_i^3 & y_i^3 - 2x_i^2 y_i & -3x_i y_i^2 \\ 3x_{i+1}^2 y_{i+1} & 2x_{i+1} y_{i+1}^2 - x_{i+1}^3 & y_{i+1}^3 - 2x_{i+1}^2 y_{i+1} & -3x_{i+1} y_{i+1}^2 \end{vmatrix} \neq 0$$

We now prove that the determinant $D \neq 0$. There are at most two among the four numbers $x_i, x_{i+1}, y_i, y_{i+1}$ that are equal to 0 as the three points O, P_i, P_{i+1} form a triangle. So there are three cases as follows:

(a) There are two zeros in $\{x_i, x_{i+1}, y_i, y_{i+1}\}$. The possibilities are that $x_i = y_{i+1}$ or $x_{i+1} = y_i$. Obviously the determinant $D \neq 0$ for both possibilities.

(b) There is only one zero in $\{x_i, x_{i+1}, y_i, y_{i+1}\}$. Without loss of generality, let $y_i = 0$, then

$$D = \begin{vmatrix} x_i^3 & 0 & 0 & 0 \\ x_{i+1}^3 & x_{i+1}^2 y_{i+1} & x_{i+1} y_{i+1}^2 & y_{i+1}^3 \\ 0 & -x_i^3 & 0 & 0 \\ 3x_{i+1}^2 y_{i+1} & 2x_{i+1} y_{i+1}^2 - x_{i+1}^3 & y_{i+1}^3 - 2x_{i+1}^2 y_{i+1} & -3x_{i+1} y_{i+1}^2 \end{vmatrix} = x_i^6 \left(x_{i+1}^2 + y_{i+1}^2\right) y_{i+1}^4 \neq 0$$

(c) None of the four numbers $\{x_i, x_{i+1}, y_i, y_{i+1}\}$ is zero. Let $y_i = k_i x_i$, $y_{i+1} = k_{i+1} x_{i+1}$. Therefore,

$$D=\begin{vmatrix} x_i^3 & x_i^2 y_i & x_i y_i^2 & y_i^3 \\ x_{i+1}^3 & x_{i+1}^2 y_{i+1} & x_{i+1} y_{i+1}^2 & y_{i+1}^3 \\ 3x_i^2 y_i & 2x_i y_i^2 - x_i^3 & y_i^3 - 2x_i^2 y_i & -3x_i y_i^2 \\ 3x_{i+1}^2 y_{i+1} & 2x_{i+1} y_{i+1}^2 - x_{i+1}^3 & y_{i+1}^3 - 2x_{i+1}^2 y_{i+1} & -3x_{i+1} y_{i+1}^2 \end{vmatrix}$$

$$= x_i^6 y_{i+1}^6 \begin{vmatrix} 1 & k_i & k_i^2 & k_i^3 \\ 1 & k_{i+1} & k_{i+1}^2 & k_{i+1}^3 \\ 3k_i & 2k_i^2 - 1 & k_i^3 - 2k_i & -3k_i^2 \\ 3k_{i+1} & 2k_{i+1}^2 - 1 & k_{i+1}^3 - 2k_{i+1} & -3k_{i+1}^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & k_i & k_i^2 & k_i^3 \\ 0 & k_{i+1} - k_i & k_{i+1}^2 - k_i^2 & k_{i+1}^3 - k_i^3 \\ 0 & -k_i^2 - 1 & -2k_i^3 - 2k_i & -3k_i^2 - 3k_i^4 \\ 0 & -k_{i+1}^2 - 1 & -2k_{i+1}^3 - 2k_{i+1} & -3k_{i+1}^2 - 3k_{i+1}^4 \end{vmatrix} = -x_i^6 y_{i+1}^6 \left(k_i - k_{i+1}\right)^4 \left(k_i^2 + 1\right)\left(k_{i+1}^2 + 1\right) \neq 0\,,$$

by the fact $k_i - k_{i+1} \neq 0$ as three points O, P_i, P_{i+1} form a non-degenerate triangle.

Therefore we have completed the proof of Theorem 2.

For a patch with corner vertex O and n neighbor points P_i for $i = 1, 2, \cdots, n$, see Figure 4, we construct a piecewise cubic polynomial surface $T(x, y)$ over the patches as follows. First we set 6 scalar values at the corner point O, that is, the position value $T(O)$, 2 values of the partial derivatives $\frac{\partial T}{\partial x}(O)$, $\frac{\partial T}{\partial y}(O)$ and 3 values of the second order partial derivatives $\frac{\partial^2 T}{\partial x^2}(O)$, $\frac{\partial^2 T}{\partial x \partial y}(O)$, $\frac{\partial^2 T}{\partial y^2}(O)$. At each end-point P_i for $i = 1, 2, \cdots, n$, we set 2 scalar values, one is its position value $T(P_i)$ and the other is the value of cross-edge directional derivative $\frac{\partial T_i}{\partial \mathbf{n}_{\mathbf{l}_i}}(P_i)$. Thus we have $6 + 2n$ values to determine the shape of the surface over the patches.

At each triangle ΔOP_iP_{i+1}, there are a total of 10 values, six values at the vertex O plus two values at P_i and P_{i+1} respectively. From Theorem 2 the cubic triangular surface patch $T_i^3(x, y)$ over ΔOP_iP_{i+1} can be uniquely determined for all $i = 1, 2, \cdots, n$.

It is easily shown that the adjacent patches $T_i^3(x, y)$ and $T_{i+1}^3(x, y)$ join at their common boundary with C^1 continuity by Eq. 2, Eq. 3 and Theorem 1. As each triangular patch $T_i^3(x, y)$ uses the same values at the common corner O, thus the composition surface over the patches is C^2 continuous at the corner vertex.

Thus a continuous piecewise cubic triangular polynomial surface over the n patches could be constructed from the given $6 + 2n$ values with C^1 continuous along the boundary curves between adjacent triangular patches and C^2 continuous at the corner.

It is worthwhile to note that changes of the six values at the corner will affect the shape over all the triangular patches while changes of the two values at P_i would locally affect the shapes of the triangles T_i and T_{i+1}.

Our algorithm is cubic reproduction. That is, if the values of the vertices of the patch and the corner are computed from a cubic polynomial surface, the result of our approach is exactly the same as the original cubic polynomial surface. This can be easily seen from the proof of Theorem 2.

We now give the algorithm for designing a C^1 continuous surface over the triangular patches around a corner in the following.

Algorithm 1.

Input: 6 scalar values at the corner vertex, 2 scalar values at each patch vertex, totally $6+2n$ values.

Output: a C^1 continuous piecewise triangular polynomial surface around the corner.

Steps: For each triangular patch, do

Step 1. Compute $a_0, a_1, \cdots, a_5$ by Eq. 7.

Step 2. Compute $a_6, \cdots, a_9$ by solving the 4×4 linear system given by Eqs. 9-12.

Step 3. Convert the cubic triangular polynomial into Bernstein-Bézier form.

5. Experimental results

In this section we show several examples of construction of cubic triangular patches around a corner that illustrate the behaviors of our approach. All examples in the paper are the results of an implementation of the proposed algorithm in a 3D user interface design system in C++ developed at our lab. The user can explicitly change the number and shapes of the patches around the corner and control the values at the corner and the patch vertices to adjust the shape of the surfaces interactively.

Example 1. We use a sphere surface as a ground truth sample and approximate a part of the sphere surface $z(x,y)=\sqrt{1-x^2-y^2}$ using 6 triangular cubic polynomial patches. The domain edge points are $P_1=\left(\frac{\sqrt{2}}{2},0\right)$, $P_2=\left(\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4}\right)$, $P_3=\left(-\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4}\right)$ $P_4=\left(-\frac{\sqrt{2}}{2},0\right)$, $P_5=\left(-\frac{\sqrt{2}}{4},-\frac{\sqrt{6}}{4}\right)$, $P_6=\left(\frac{\sqrt{2}}{4},-\frac{\sqrt{6}}{4}\right)$ and the corner is $O=(0,0)$, see Figure 5(a). We set the values at the corner as $(u,u_x,u_y,u_{xx},u_{xy},u_{yy})\big|_O=(1,0,0,-1,0,-1)$ and the values at patch vertices $(u,u_{\mathbf{n}_1})\big|_{P_i}=\left(\frac{\sqrt{2}}{2},0\right)$, $i=1,2,\cdots,6$, computed following the equation of sphere surface. The piecewise cubic triangular patches generated by our approach are shown in Figure 5(b). The maximum approximation error is 0.006, and the approximation result looks good.

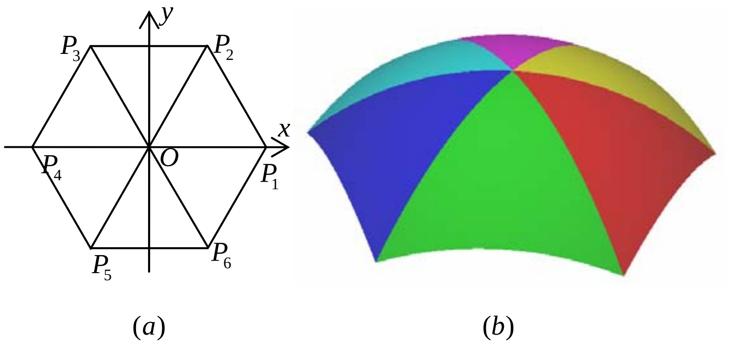


Fig. 5. Polynomial approximation to part of the sphere surface using 6 triangular patches: (a) domain patches; (b) piecewise cubic triangular polynomial surfaces.

Example 2. We use two surfaces with degree not larger than 3 for testing the cubic reproduction property of our approach. One surface is a saddle surface $z = xy$, see Figure 6(a), and the other is a cubic surface $z = x^3 - 3xy^2 + 2xy + y$, see Figure 6(b). We use 4 triangular patches in our tests. The reconstructed surface patches are exactly the same as the original surfaces in both cases.

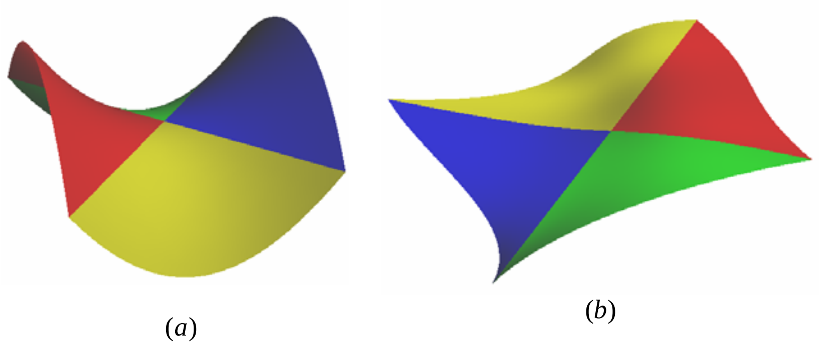


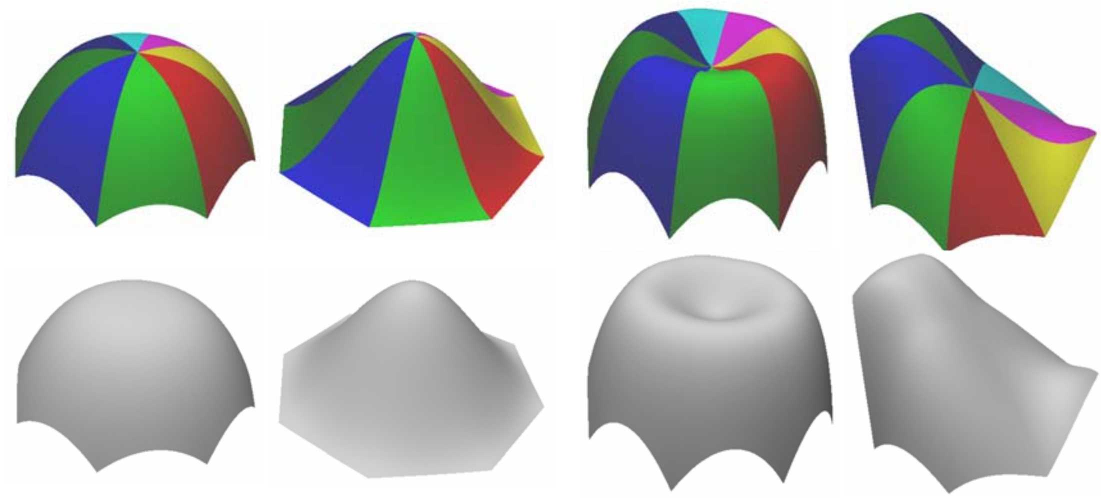
Fig. 6. Our approach can reproduce exact cubic surfaces: (a) a saddle surface; (b) a cubic surface.

Example 3. To illustrate the flexibility of our approach, we use an example with eight patches around the corner. We can easily adjust the values used in the algorithm to control the shape of the piecewise surface. The result of approximating a sphere surface is shown in Figure 7(a). We then adjust the values of mixed derivatives of the corner vertex, with large positive values, see Figure 7(b) or with large negative values, see Figure 7(c). Figure 7(d) shows a result when the user adjusts the position of one boundary point and the partial derivatives of the corner vertex. The corresponding smooth rendering effects are shown on the right side for each of the example.

6. Conclusions

In this paper, we proposed a novel approach for constructing C^1 continuous surface over arbitrary triangular patches around a corner. The approach is derived based on the mixed directional derivatives between the common boundaries between two adjacent patches. The result surface is piecewise cubic polynomials with the advantage of cubic reproduction. The approach is simple and fast. The user can easily control the shape of the interpolation surface by adjusting the input values. We demonstrate the applicability and flexibility of the approach by several experimental results.

The presented approach still has much to do for improvements and extensions. We should consider the extension to piecewise parametric surface construction over triangulation with arbitrary topology. It is also much worthwhile to extend our approach to build high order continuous surface over patches around a corner vertex. We believe that this extension is feasible but not straightforward.



(*a*) (*b*) (*c*) (*d*)

Fig. 7. Different examples using 8 triangular patches with different user inputs: (a) approximation to a sphere surface; (b) large positive mixed derivative value for corner vertex; (c) large negative mixed derivative value for corner vertex; (d) adjustment of the position of one boundary point and the partial derivative value of the corner vertex.

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