4-12 MAGNETICALLY COUPLED CIRCUITS
(互感耦合电路)

1. Magnetically Coupled Inductance

Self-inductance

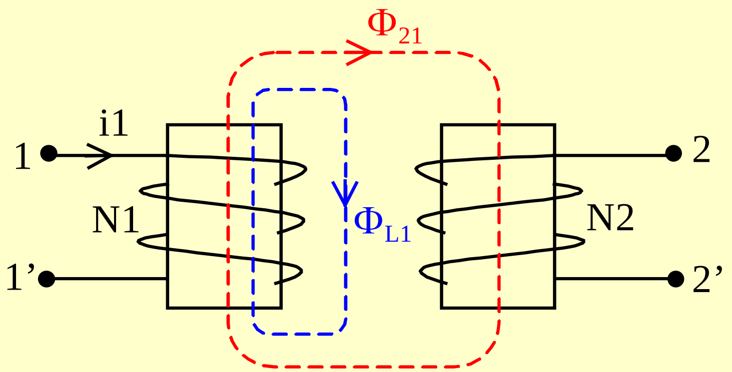
Φ_{L1} : magnetic flux links only coil 1

Φ_{21} : magnetic flux links both coils

$\Psi_{11}=N_1\Phi_{11}=N_1\ (\phi_{L1}+\phi_{21})$

$$L_1 \triangleq \frac{\psi_{11}}{i_1} = \frac{N_1\phi_{11}}{i_1}$$

$$L_2 \triangleq \frac{\psi_{22}}{i_2} = \frac{N_2\phi_{22}}{i_2} \quad (\phi_{22} = \phi_{L2} + \phi_{12})$$



Mutual inductance

M_{21} is the mutual inductance of coil 2 with respect to coil 1.

$$M_{21} \triangleq \frac{\psi_{21}}{i_1} = \frac{N_2 \phi_{21}}{i_1}$$

M_{12} is the mutual inductance of coil 1 with respect to coil 2.

$$M_{12} \triangleq \frac{\psi_{12}}{i_2} = \frac{N_1 \phi_{12}}{i_2}$$

$M = M_{12} = M_{21}$,

M is the mutual inductance between the two coils. Mutual inductance is the ability of one inductor to induce a voltage across a neighboring inductor, measured in henrys (H).

Coupling coefficient

$$k_1 = \phi_{21} / \phi_{11} \ , \quad k_2 = \phi_{12} / \phi_{22}$$

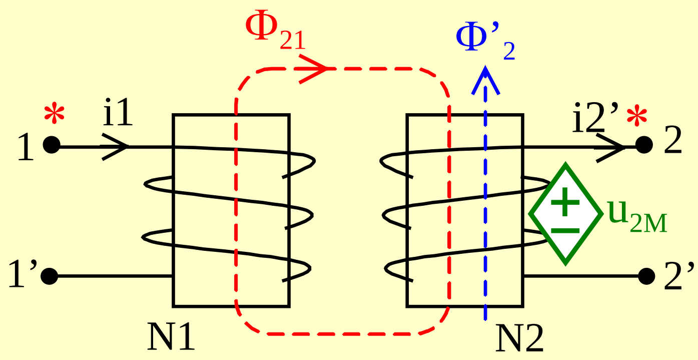
$$k_1 k_2 = \frac{\phi_{21}}{\phi_{11}} \frac{\phi_{12}}{\phi_{22}} = \frac{\phi_{21} \, {}^{N_2}/_{i_1}}{\phi_{11} \, {}^{N_1}/_{i_1}} \frac{\phi_{21} \, {}^{N_1}/_{i_2}}{\phi_{22} \, {}^{N_2}/_{i_2}} = \frac{M_{21}}{L_1} \frac{M_{12}}{L_2} = \frac{M^2}{L_1 L_2}$$

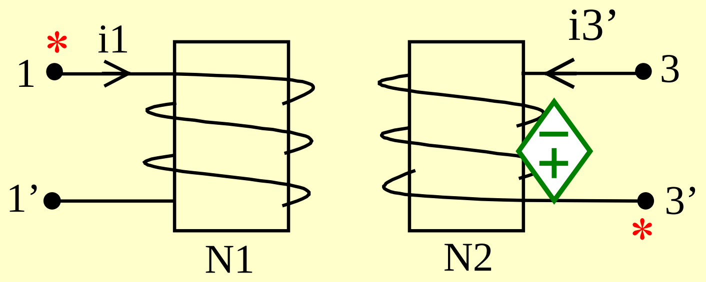
$$k^2 = k_1 k_2 = M^2 / L_1 L_2 \quad \Rightarrow \quad k = M / \sqrt{L_1 L_2}$$

The coupling coefficient k is a measure of the magnetic coupling between two coils，0<k<1.

k=1，perfectly coupled，全耦合电感器。

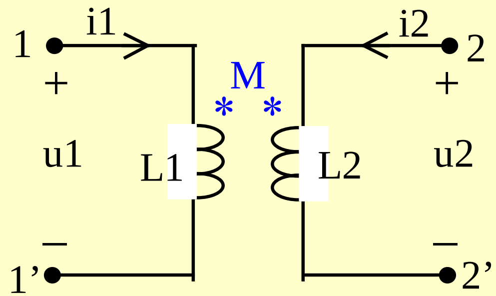
Mutual voltage





$$u_{2M}(t) = {d\psi_{21}(t)}\big/{dt} = M\frac{di_1(t)}{dt}$$

*, Δ, ○, ● ：the dotted terminal (同名端)



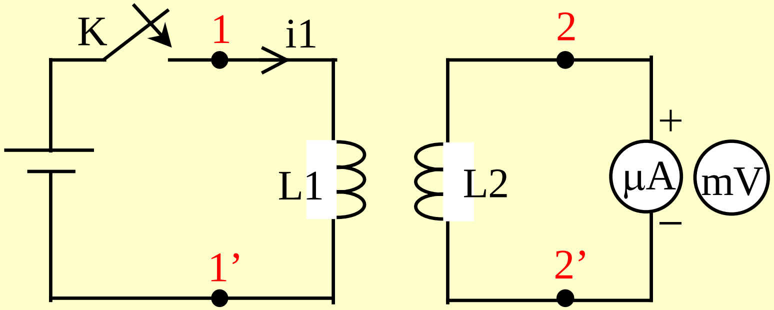
The dot convention

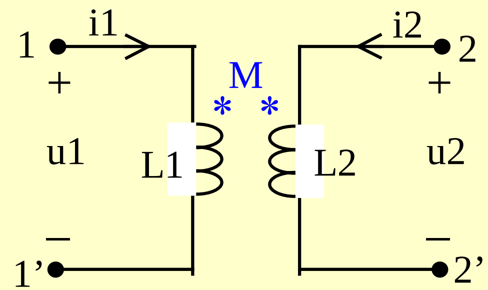
If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.

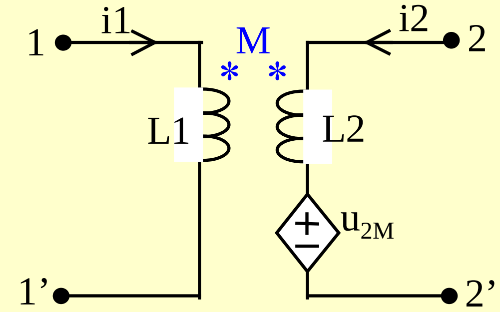
If a current leaves the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is negative at the dotted terminal of the second coil.

Check Your Understanding

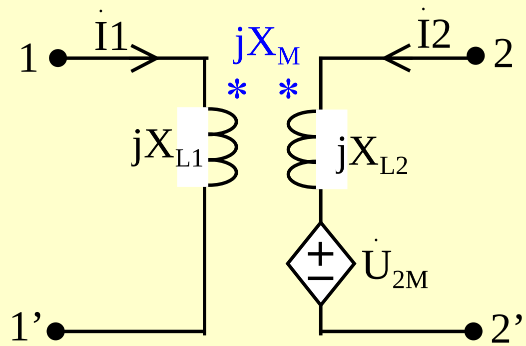
How to determine the dotted terminal?

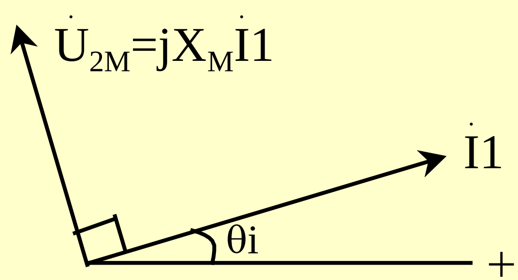






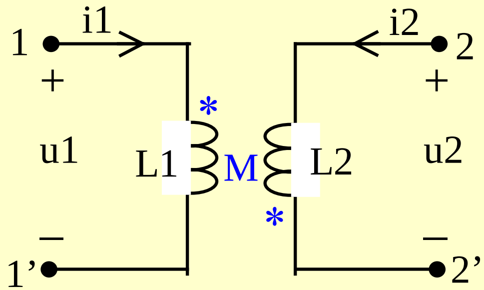
$$u_2 = u_{2L} + u_{2M} = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

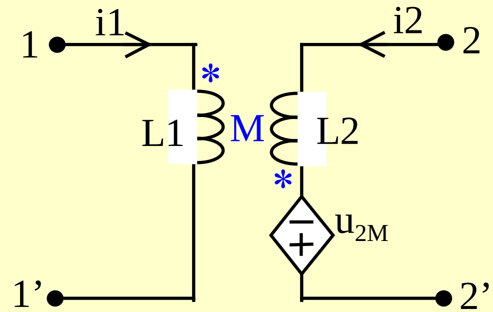




$$\dot{U}_2 = j\omega L_2 \dot{I}_2 + j\omega M \dot{I}_1 = jX_{L2} \dot{I}_2 + jX_M \dot{I}_1$$

$X_M \triangleq \omega M$: mutual inductance impedance，Ω



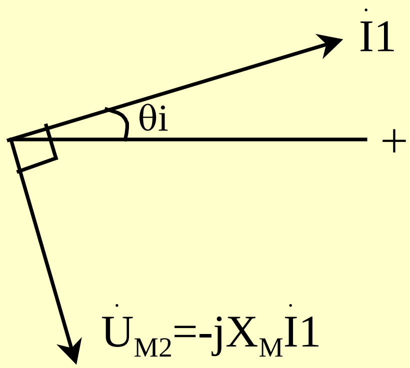


$$u_2 = u_{2L} - u_{2M}$$

$$u_{2M} = M\frac{di_1}{dt}$$

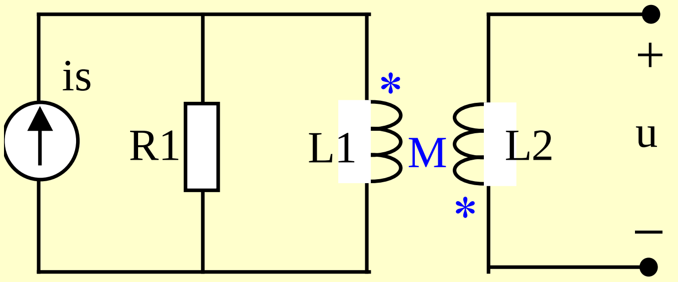
$$u_2 = L_2\frac{di_2}{dt} - M\frac{di_1}{dt}$$

$$u_1 = L_1\frac{di_1}{dt} - M\frac{di_2}{dt}$$



例：R_1=1Ω，L_1=1H，L_2=2H，M=0.5H，$i_S=10\sqrt{2}\sin t\ A$，求稳态开路电压 u。

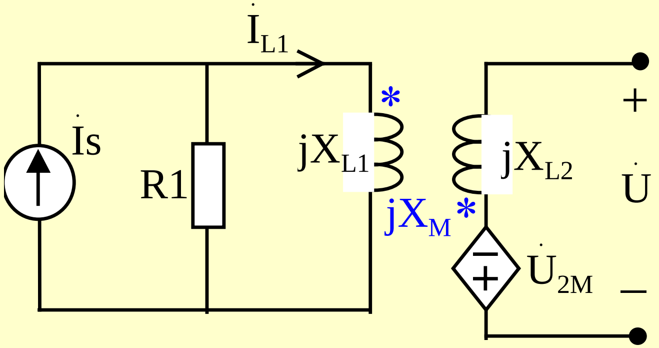
解：



$$i_S=10\sqrt{2}\sin t\ A \rightarrow \dot{I}_S=10\angle 0^\circ\ A$$

$$X_{L1}=\omega L_1=1\,\Omega,\ X_M=\omega M=0.5\,\Omega$$

$$\dot{I}_{L1}=\frac{R_1}{R_1+jXL_1}\dot{I}_S=5\sqrt{2}\angle 45^\circ\ A$$



$$\dot{U}_{2M}=jX_M\dot{I}_{L1}=2.5\sqrt{2}\angle 45^\circ\ V$$

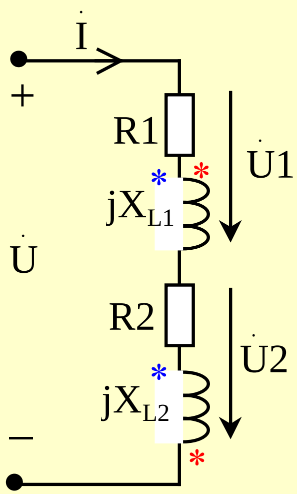
$$\therefore\quad \dot{U}=-\dot{U}_{2M}=2.5\sqrt{2}\angle -135^\circ\ V$$

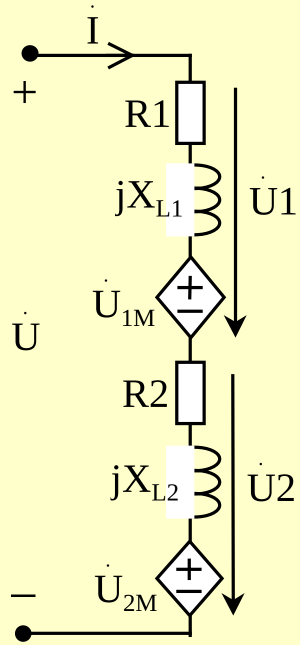
$$u=5\sin(t-135^\circ)\ V$$

2. Series And Parallel Connection

Series-aiding connection (顺向串联)；

Series-opposing connection (反向串联)。



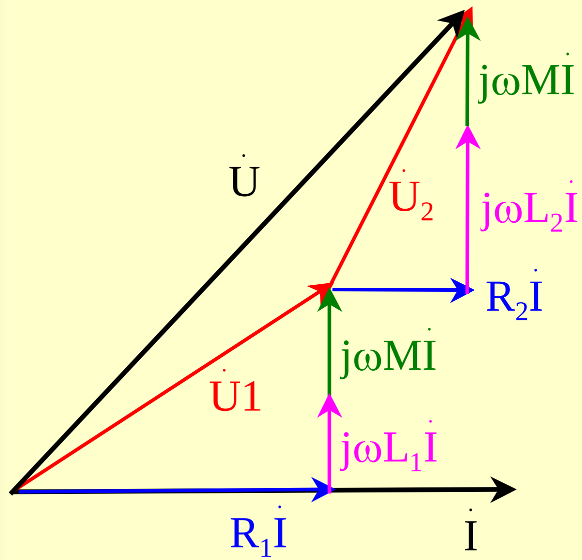


$$\dot{U}_1 = \dot{I}(R_1 + j\omega L_1 \pm j\omega M)$$

$$\dot{U}_2 = \dot{I}(R_2 + j\omega L_2 \pm j\omega M)$$

$$\dot{U} = \dot{U}_1 + \dot{U}_2$$

$$= \dot{I}\left[(R_1 + R_2) + j\omega(L_1 + L_2 \pm 2M)\right]$$

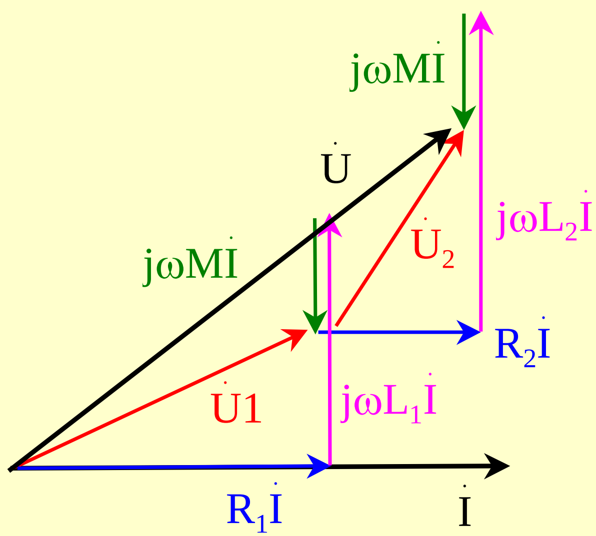


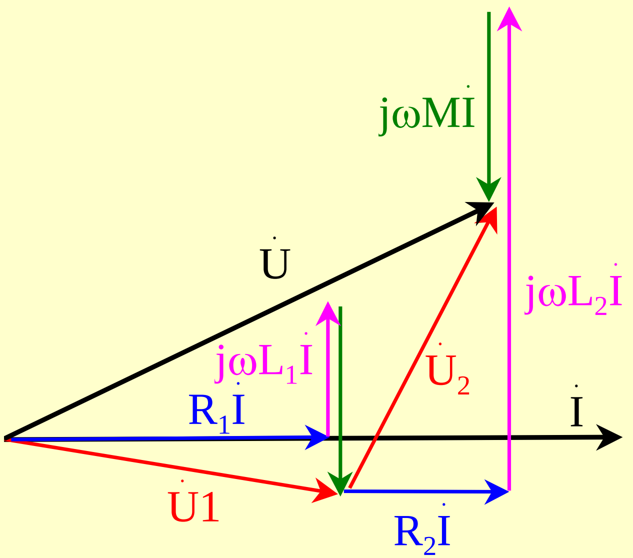
$$\dot{U} = \dot{U}_1 + \dot{U}_2$$

$$= \dot{I}\left[(R_1 + R_2) + j\omega(L_1 + L_2 \pm 2M)\right]$$

Series-aiding connection：“+”，$L > L_1 + L_2$；

Series-opposing connection：“—”，$L < L_1 + L_2$。





Parallel-aiding connection

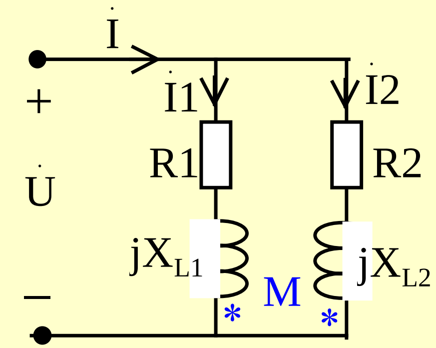
$$\dot{U}=\dot{I}_1(R_1+j\omega L_1)+j\omega M\dot{I}_2=Z_1\dot{I}_1+Z_M\dot{I}_2$$

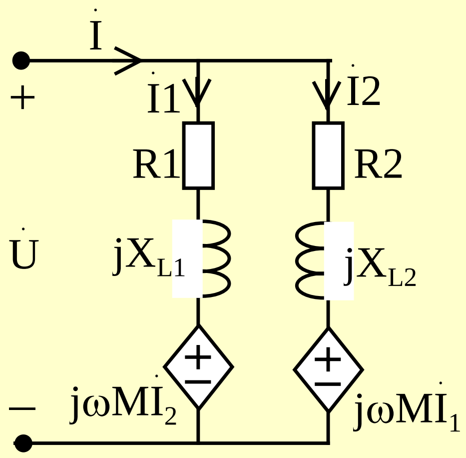
$$\dot{U}=\dot{I}_2(R_2+j\omega L_2)+j\omega M\dot{I}_1=Z_2\dot{I}_2+Z_M\dot{I}_1$$

$$\Rightarrow\ \dot{I}_1=\frac{Z_2-Z_M}{Z_1Z_2-Z_M^2}\dot{U}\ ,\quad \dot{I}_2=\frac{Z_1-Z_M}{Z_1Z_2-Z_M^2}\dot{U}$$

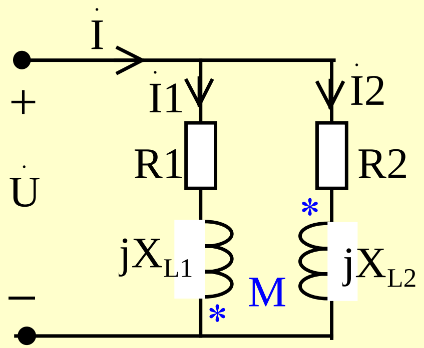
$$\dot{I}=\dot{I}_1+\dot{I}_2=\frac{Z_1+Z_2-2Z_M}{Z_1Z_2-Z_M^2}\dot{U}$$

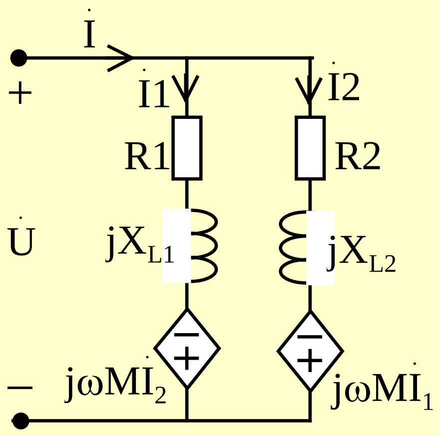
$$Z=\frac{\dot{U}}{\dot{I}}=\frac{Z_1Z_2-Z_M^2}{Z_1+Z_2-2Z_M}$$





Parallel-opposing connection

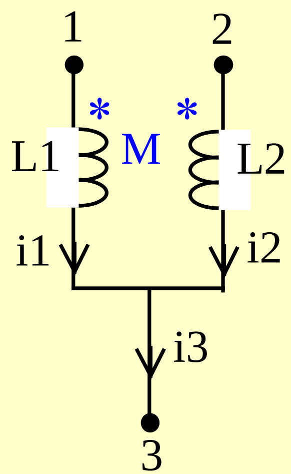


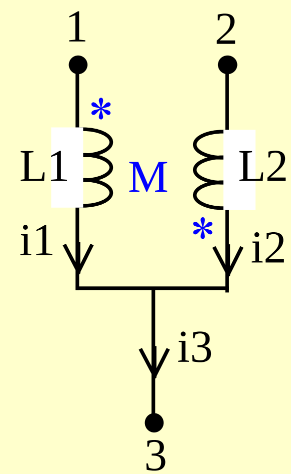


$$Z = \frac{\dot{U}}{\dot{I}} = \frac{Z_1 Z_2 - Z_M^2}{Z_1 + Z_2 + 2Z_M}$$

Replace a magnetically coupled circuit by an equivalent circuit with no magnetic coupling (去耦合)

When the terminals of a pair of mutually coupled inductors are connected, the T-equivalent circuit with three inductors is the model that is not magnetically coupled.



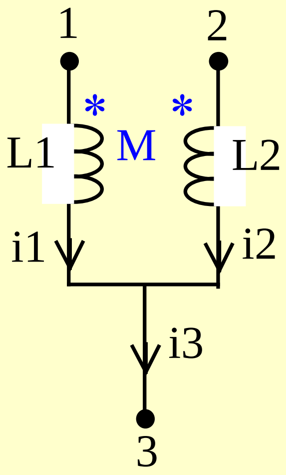


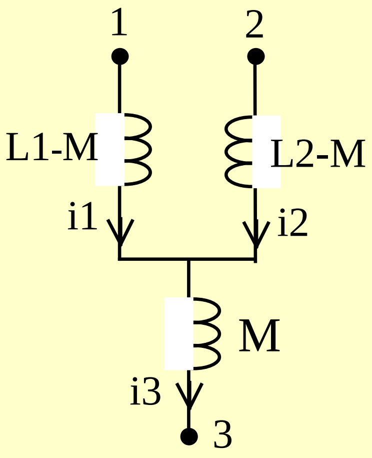
$$u_{13} = L_1 \frac{di_1}{dt} \pm M \frac{di_2}{dt}, \quad u_{23} = L_2 \frac{di_2}{dt} \pm M \frac{di_1}{dt}$$

$$\dot{U}_{13} = j\omega L_1 \dot{I}_1 \pm j\omega M \dot{I}_2, \quad \dot{U}_{23} = j\omega L_2 \dot{I}_2 \pm j\omega M \dot{I}_1$$

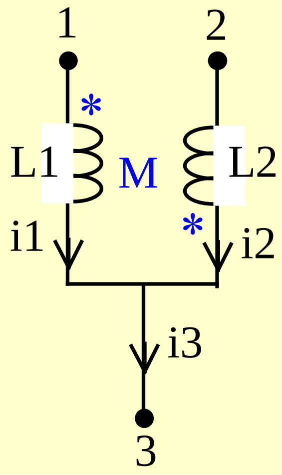
$$\because\ \dot{I}_1+\dot{I}_2=\dot{I}_3 \quad \Rightarrow \quad \begin{aligned}\dot{U}_{13}&=j\omega(L_1\mp M)\dot{I}_1\pm j\omega M\dot{I}_3\\ \dot{U}_{23}&=j\omega(L_2\mp M)\dot{I}_2\pm j\omega M\dot{I}_3\end{aligned}$$

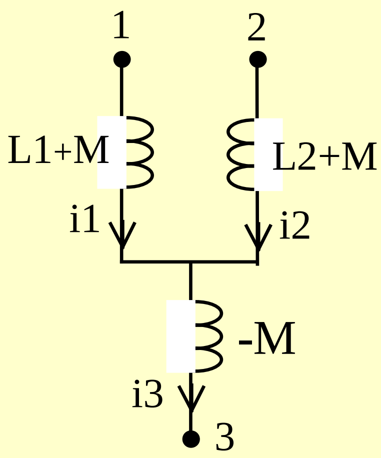
If either of the dots is placed on the same end of coil





If either of the dots is placed on the different end of its coil



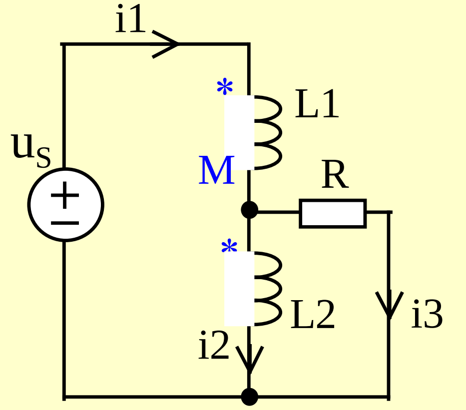


$$\dot{U}_{13} = j\omega(L_1 \mp M)\dot{I}_1 \pm j\omega M \dot{I}_3$$

$$\dot{U}_{23} = j\omega(L_2 \mp M)\dot{I}_2 \pm j\omega M \dot{I}_3$$

例：R=1Ω，$L_1=M=1H$，$L_2=2H$，$u_S=2\sqrt{2}\sin t\ V$，计算稳态响应 i_1，i_2，i_3。

解：

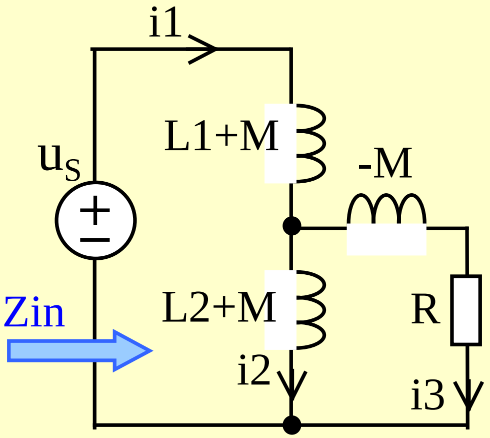


$$Z_{in}=j\omega(L_1+M)+\frac{j\omega(L_2+M)(R-j\omega M)}{j\omega(L_2+M)+(R-j\omega M)}=1.90\angle-18.4^\circ\ \Omega$$

$$\dot{I}_1=\frac{\dot{U}_S}{Z_{in}}=0.88\angle-37.9^\circ\ A$$

$$\dot{I}_3=\frac{j\omega(L_2+M)}{j\omega(L_2+M)+(R-j\omega M)}\dot{I}_1=1.18\angle-11.3^\circ\ A$$

$$\dot{I}_2=\dot{I}_1-\dot{I}_3=0.55\angle-146.3^\circ\ A$$

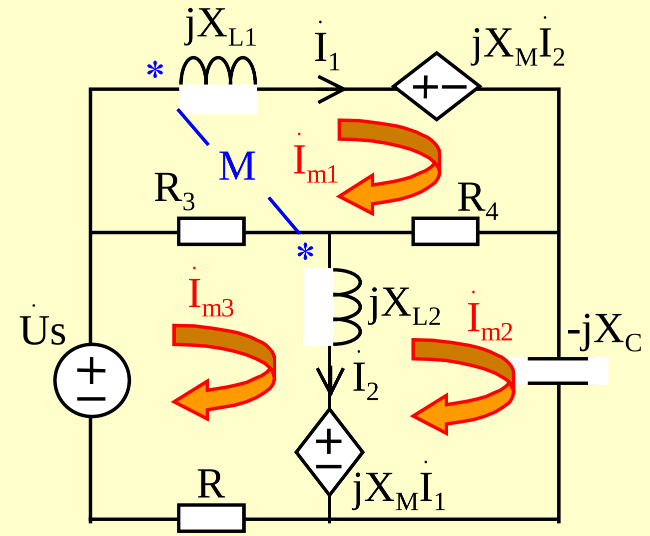


$$i_1=0.88\sqrt{2}\sin(t-37.9^\circ)\ A$$

$$i_2=0.55\sqrt{2}\sin(t-146.3^\circ)A$$

$$i_3=1.18\sqrt{2}\sin(t-11.3^\circ)\ A$$

例：网孔法



解：

$$\dot{I}_1=\dot{I}_{m1}\ ,\quad \dot{I}_2=-\dot{I}_{m2}+\dot{I}_{m3}$$

$\dot{I}_{m1}$ 网孔：

$$(R_3+R_4+jX_{L1})\dot{I}_{m1}-R_4\dot{I}_{m2}-R_3\dot{I}_{m3}+jX_M(-\dot{I}_{m2}+\dot{I}_{m3})=0$$

$\dot{I}_{m2}$ 网孔： $-R_4\dot{I}_{m1}+(R_4-jX_C+jX_{L2})\dot{I}_{m2}-jX_{L2}\dot{I}_{m3}-jX_M\dot{I}_{m1}=0$

$\dot{I}_{m3}$ 网孔： $-R_3\dot{I}_{m1}-jX_{L2}\dot{I}_{m2}+(R+R_3+jX_{L2})\dot{I}_{m3}+jX_M\dot{I}_{m1}=\dot{U}_S$

4-13 TRANSFORMERS (变压器)

A transformer is generally a four-terminal device comprising two (or more) magnetically coupled coils.

1. Air-core Transformers (空芯变压器)

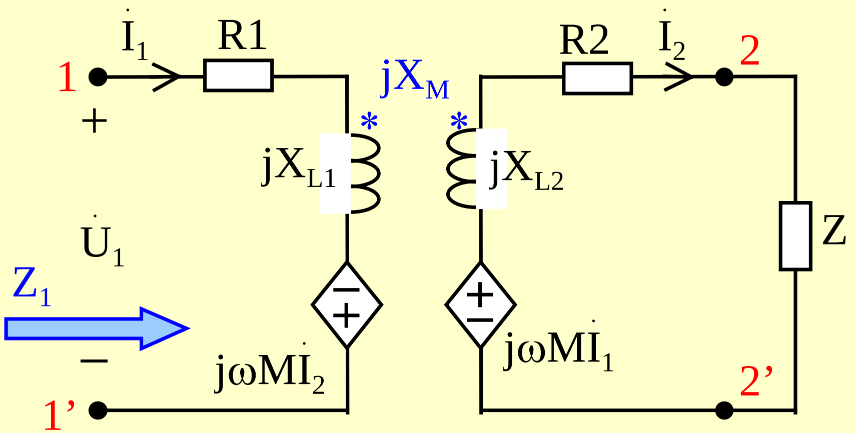
Air-core transformers are also called linear transformers, the coils are wound on a magnetically linear material.

The coil that is directly connected to the voltage source is called the primary winding. (原边线圈)

The coil connected to the load is called the secondary winding. (副边线圈)

$$(R_1 + jX_{L1})\dot{I}_1 - jX_M \dot{I}_2 = \dot{U}_1$$

$$[(R_2 + R) + j(X_{L2} + X)]\dot{I}_2 - jX_M \dot{I}_1 = 0$$



The input impedance：

$$Z_1 = \frac{\dot{U}_1}{\dot{I}_1}$$

$$= \left[R_1 + \frac{(\omega M)^2}{(R_2 + R)^2 + (X_{L2} + X)^2}(R_2 + R)\right]$$

$$+ j\left[\omega L_1 - \frac{(\omega M)^2}{(R_2 + R)^2 + (X_{L2} + X)^2}(X_{L2} + X)\right]$$

$$Z_1 = \frac{\dot{U}_1}{\dot{I}_1} = \left[R_1 + \frac{(\omega M)^2}{(R_2 + R)^2 + (X_{L2} + X)^2}(R_2 + R) \right]$$

$$+ j\left[\omega L_1 - \frac{(\omega M)^2}{(R_2 + R)^2 + (X_{L2} + X)^2}(X_{L2} + X) \right]$$

The primary impedance：$R_1 + j\omega L_1$

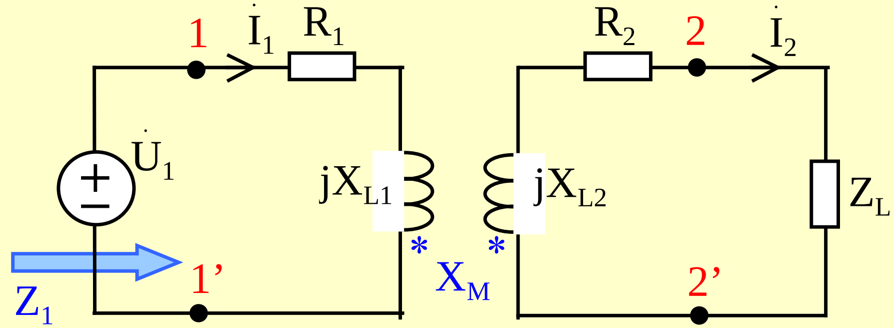
The reflected impedance (归算阻抗)：

$$Z' = \frac{(\omega M)^2}{R_{22}^2 + X_{22}^2}(R_{22} - jX_{22}) \qquad (R_{22} = R_2 + R，\ X_{22} = X_{L2} + X)$$

Inductive load：$X_{22}>0$；

Capacitive load：$X_{22}<0$

例：L_1=0.025H，R_1=50Ω，L_2=0.10H，R_2=200Ω，M=0.045H，U_1=100V，ω=100×$10^3$$rad/s$，$Z$=(1000+$j$500)Ω，求 I_1，电源输出 $\tilde{S}_1$，传输效率，变压器输出 $\tilde{S}_2$。



解：

$$R_{22} = R_2 + R = 1200\,\Omega,$$
$$X_{22} = \omega L_2 + X = 10500\,\Omega$$

$$Z' = \frac{(\omega M)^2}{R_{22}^2 + X_{22}^2}(R_{22} - jX_{22}) = (218 - j1904)\ \Omega$$

$$Z_1 = R_1 + j\omega L_1 + Z' = 268 + j596 = 653\angle 65.8^\circ\ \Omega$$

$$\dot{U}_1 = 100\angle 0^\circ\ V,\quad \dot{I}_1 = \frac{\dot{U}_1}{Z_1} = 0.153\angle -65.8^\circ\ A$$

$$\tilde{S}_1 = \dot{U}_1 \overset{*}{I}_1 = 15.31\angle 65.8^\circ\ VA = 6.28\,W + j13.96\ \text{var}$$

$$\dot{I}_2 = \frac{j\omega M}{R_{22} + jX_{22}} \dot{I}_1\ , \quad I_2 = \frac{\omega M}{\sqrt{R_{22}^2 + X_{22}^2}} I_1 = 0.0652\ \ A$$

$$\tilde{S}_2 = I_2^2 Z = 4.25\,W + j2.13\,\text{var}$$

$$\eta = \frac{P_2}{P_1} \times 100\% = 67.7\%$$

2. Ideal Transformers (理想变压器)

A transformer is said to be ideal if it has the following properties：

Coils have very large reactance (L_1，L_2，$M \to \infty$).

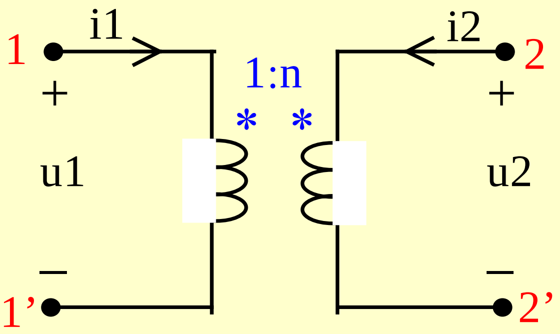
Coupling coefficient is equal to unity (k=1).

Primary and secondary coils are lossless ($R_1=R_2=0$).

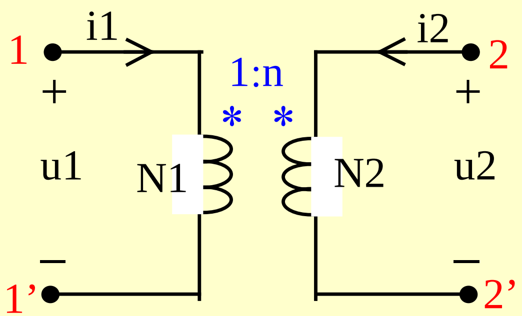
Iron-core transformers are close approximations to ideal transformers.

n is the turns ratio or transformation ratio.

变比



升压变压器：$N_2 > N_1$，$1:n$，$n = N_2 / N_1$



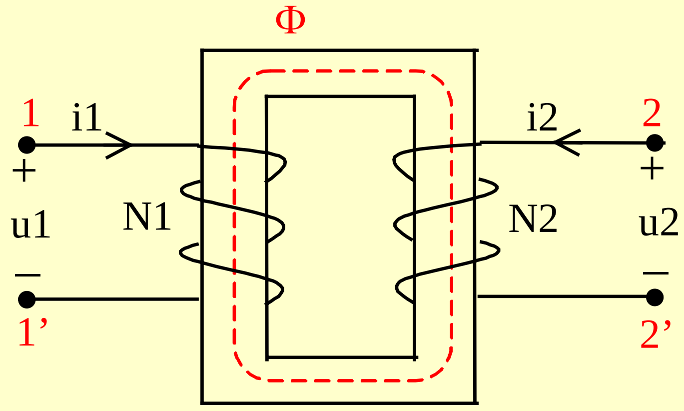
A step-up transformer is one whose secondary voltage is greater than its primary voltage.

降压变压器：$N_1 > N_2$，$n:1$，$n = N_1 / N_2$

A step-down transformer is one whose secondary voltage is less than its primary voltage.

$$\begin{cases} u_2 = n u_1 \\ i_2 = -\dfrac{1}{n} i_1 \end{cases} \qquad \begin{cases} u_1 = \dfrac{1}{n} u_2 \\ i_1 = -n i_2 \end{cases} \qquad \begin{cases} \dot{U}_2 = n \dot{U}_1 \\ \dot{I}_2 = -\dfrac{1}{n} \dot{I}_1 \end{cases}$$

$$u_1 = N_1 \frac{d\phi}{dt} \quad , \quad u_2 = N_2 \frac{d\phi}{dt}$$



$$\frac{u_2}{u_1} = \frac{N_2}{N_1} = n \, , \quad (N_2 > N_1)$$

$$L_1 = \frac{N_1\phi_{11}}{i_1} \, , \; L_2 = \frac{N_2\phi_{22}}{i_2} \, , \; M = \frac{N_2\phi_{21}}{i_1} \, , \; M = \frac{N_1\phi_{12}}{i_2}$$

$$\Rightarrow \quad \frac{M}{L_1} = \frac{N_2\phi_{21}}{N_1\phi_{11}} , \quad \frac{L_2}{M} = \frac{N_2\phi_{22}}{N_1\phi_{12}}$$

$$\because \; \phi_{11} = \phi_{21} \, , \; \phi_{22} = \phi_{12} \quad \therefore \; \frac{M}{L_1} = \frac{L_2}{M} = \frac{N_2}{N_1}$$

$$u_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}, \quad u_2 = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

$$\frac{u_1}{L_1} = \frac{di_1}{dt} + \frac{M}{L_1} \frac{di_2}{dt}$$

$$\because L_1 \to \infty, \quad \therefore \frac{di_1}{dt} = -\frac{M}{L_1} \frac{di_2}{dt}$$

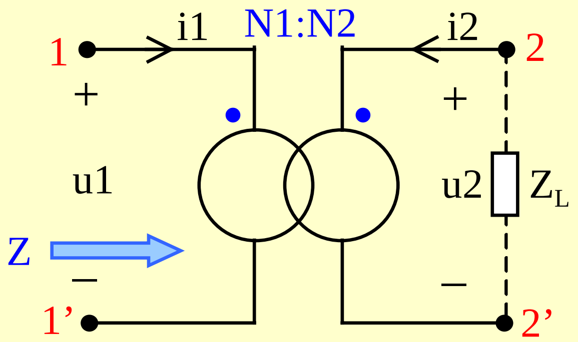
$$\Rightarrow \frac{i_2}{i_1} = -\frac{N_1}{N_2} = -\frac{1}{n}$$

$$\dot{U}_2 = -\dot{I}_2 Z_L$$

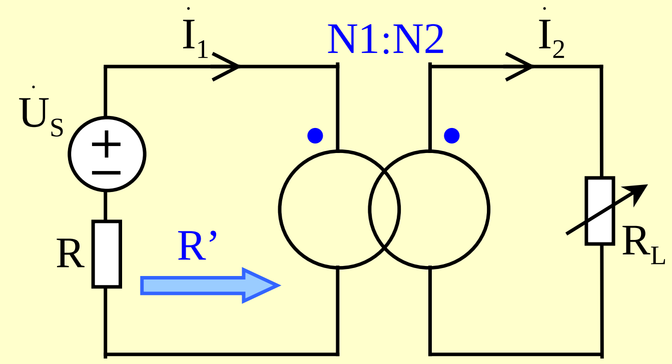
$$\frac{\dot{U}_1}{\dot{U}_2} = n = \frac{N_1}{N_2}, \quad \frac{\dot{I}_1}{\dot{I}_2} = -\frac{1}{n} = -\frac{N_2}{N_1}$$

$$\dot{U}_1 = (\frac{N_1}{N_2})^2 Z_L \dot{I}_1$$

$$Z = \frac{\dot{U}_1}{\dot{I}_1} = (\frac{N_1}{N_2})^2 Z_L = n^2 Z_L$$



例：信号源开路电压 U_S=3V，内阻 R=10Ω，负载电阻 R_L=90Ω，为接负载获最大功率，接入了一个变压器，求变比 $n = N_1/N_2$，负载端电压及流过的电流。



解：

$R' = R$时，变压器侧吸收最大功率

$$R' = n^2 R_L, \quad \Rightarrow \quad n^2 = \frac{R'}{R_L} = \frac{1}{9}$$

$$n = \frac{N_1}{N_2} = \frac{1}{3}$$

$$I_1 = \frac{U_S}{R + R'} = \frac{3}{20} = 0.15A, \quad I_2 = nI_1 = \frac{0.15}{3} = 0.05A$$

$$U_2 = I_2 R_L = 0.15 \times 90 = 4.5\ \mathrm{V}$$

3．Ideal Autotransformers（自耦变压器)

An autotransformer is a transformer in which both the primary and the secondary are in a single winding.

$$\frac{\dot{U}_1}{\dot{U}_2} = \frac{N_1 + N_2}{N_2} = 1 + \frac{N_1}{N_2}$$

$$\frac{\dot{I}_1}{\dot{I}_2} = -\frac{N_2}{N_1 + N_2}$$