CHAPTER 6 THE FOURIER SERIES

Jean Baptiste Joseph Fourier
(1768-1830)

6-1 Trigonometric Fourier Series

The Fourier series of a periodic function f(t) is a representation that resolves f(t) into a dc component and an ac component comprising an infinite series of harmonic sinusoids.

Dirichlet conditions：

f(t) is single-valued everywhere.

f(t) has a finite number of finite discontinuities in any one period.

f(t) has a finite number of maxima and minima in any one period.

The integral $\int_{t_0}^{t_0+T} |f(t)| dt < \infty$ for any t_0.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_1 t + b_n \sin n\omega_1 t)$$

a_0、a_n、b_n are the Fourier coefficients.

$$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt = \frac{2}{T}\int_{0}^{T} f(t)dt$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos n\omega_1 t dt = \frac{2}{T}\int_{0}^{T} f(t)\cos n\omega_1 t dt$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin n\omega_1 t dt = \frac{2}{T}\int_{0}^{T} f(t)\sin n\omega_1 t dt$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} B_{nm}\sin(n\omega_1 t + \theta_n) = \frac{a_0}{2} + \sum_{n=1}^{\infty} B_{nm}(\sin n\omega_1 t\cos\theta_n + \cos n\omega_1 t\sin\theta_n)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_{nm}\cos(n\omega_1 + \psi_n)$$

$$a_n = B_{nm}\sin\theta_n\ , \quad b_n = B_{nm}\cos\theta_n$$

$$B_{nm} = \sqrt{a_n^2 + b_n^2}\ ,\ \theta_n = tg^{-1}\frac{a_n}{b_n} \qquad A_{nm} = \sqrt{a_n^2 + b_n^2}\ ,\ \psi_n = tg^{-1}\frac{-b_n}{a_n}$$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}B_{nm}\sin(n\omega_1 t+\theta_n)$$

$\frac{a_0}{2}$ ：dc component.

$B_{1m}\sin(\omega_1 t+\theta_1)$ ：fundamental frequency.

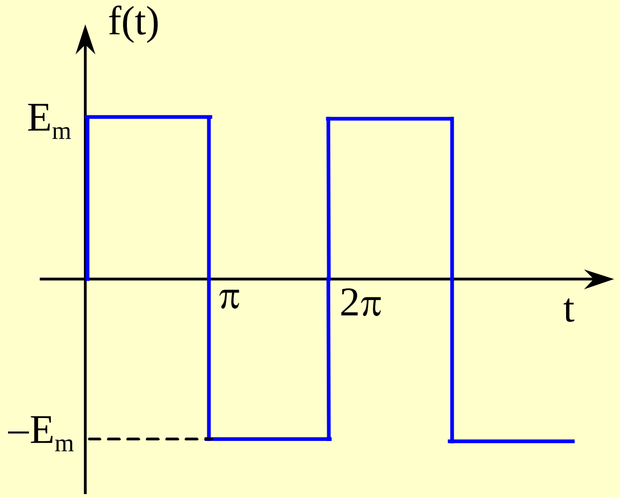
$B_{2m}\sin(2\omega_1 t+\theta_2)$ ：2-nd harmonic frequency

$B_{nm}\sin(n\omega_1 t+\theta_n)$ ：n-th harmonic frequency.

The process of determining the coefficients is called Fourier analysis.

- The plot of the amplitude of the harmonics versus $n\omega_0$ is the amplitude spectrum of f(t). (幅值频谱)
- The plot of the phase θ_n versus $n\omega_0$ is the phase spectrum of f(t). (初相频谱)
- Both the amplitude and phase spectra form the frequency spectrum of f(t).

$$\begin{cases} f(t) = E_m & 0 \le t \le \dfrac{T}{2} \\ f(t) = -E_m & \dfrac{T}{2} \le t \le T \end{cases}$$



$$a_0 = \frac{1}{T}\int_0^T f(t)dt = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(n\omega_1 t)d(\omega_1 t) = \frac{1}{\pi}[\int_0^{\pi} E_m \cos(n\omega_1 t)d(\omega_1 t)$$

$$-\int_{\pi}^{2\pi} E_m \cos(n\omega_1 t)d(\omega_1 t)] = \frac{2E_m}{\pi}\int_0^{\pi}\cos(n\omega_1 t)d(\omega_1 t) = 0$$

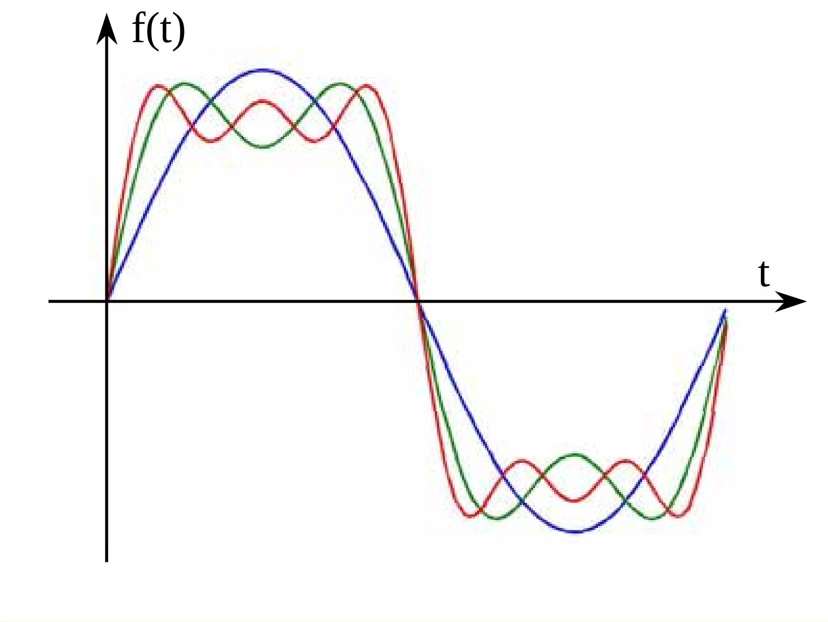
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(n\omega_1 t)d(\omega_1 t) = \frac{1}{\pi}[\int_0^{\pi} E_m \sin(n\omega_1 t)d(\omega_1 t) - \int_{\pi}^{2\pi} E_m \sin(n\omega_1 t)d(\omega_1 t)]$$

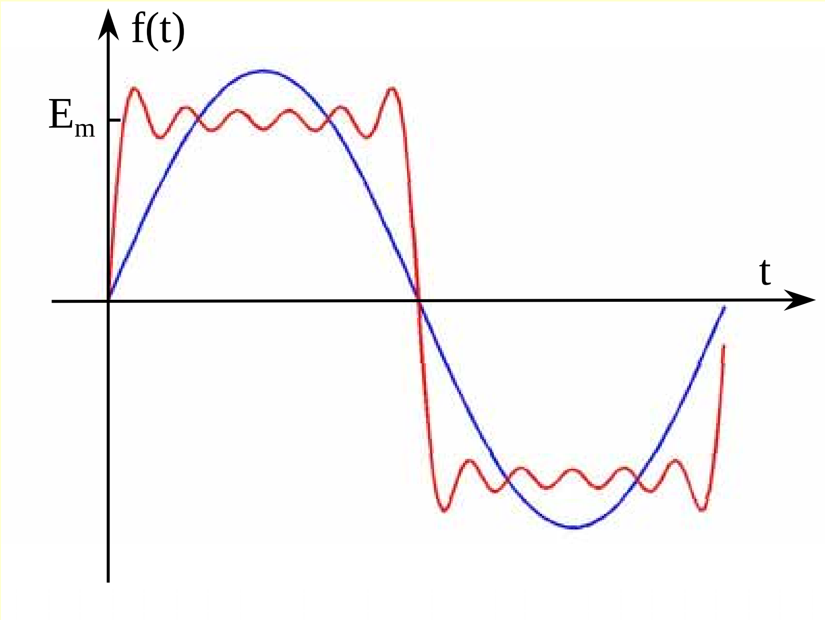
$$= \frac{2E_m}{\pi}\int_0^{\pi}\sin(n\omega_1 t)d(\omega_1 t) = \frac{2E_m}{\pi}\left[-\frac{1}{n}\cos(n\omega_1 t)\right]_0^{\pi} = \frac{2E_m}{n\pi}[1-\cos(n\pi)]$$

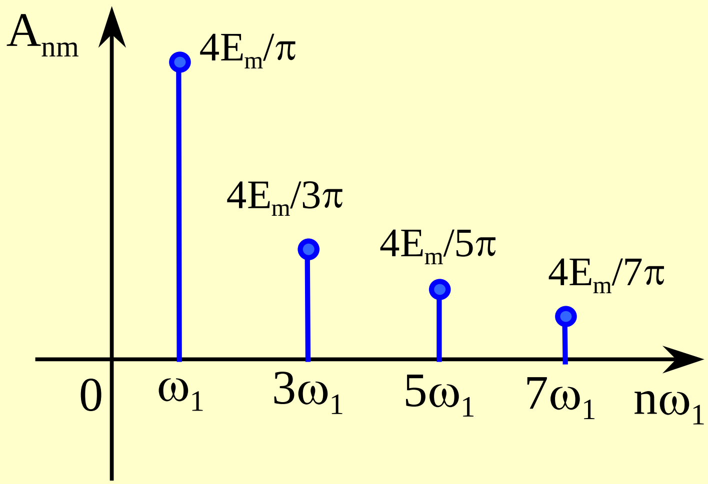
$$\cos(n\pi)=1, \quad b_n=0 \qquad \text{If n is even.}$$

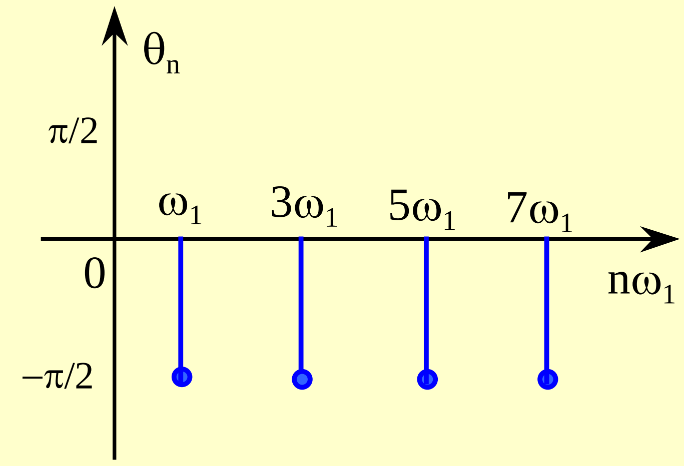
$$\cos(n\pi)=-1, \quad b_n=\frac{4E_m}{n\pi} \qquad \text{If n is odd.}$$

$$\therefore \; f(t)=\frac{4E_m}{\pi}[\sin(\omega_1 t)+\frac{1}{3}\sin(3\omega_1 t)+\frac{1}{5}\sin(5\omega_1 t)+\cdots]$$



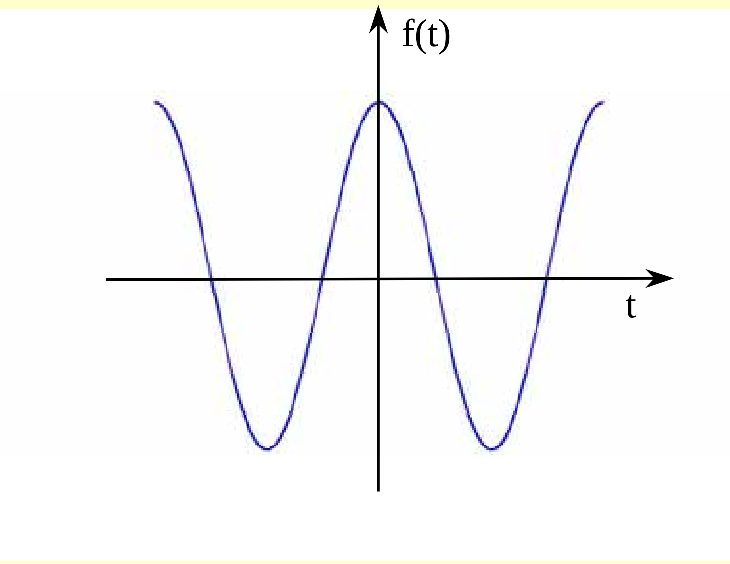






Symmetry Considerations

Even symmetry $f(t)=f(-t)$



$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega_1 t+b_n\sin n\omega_1 t)$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos n\omega_1(-t)+b_n\sin n\omega_1(-t)\right]$$

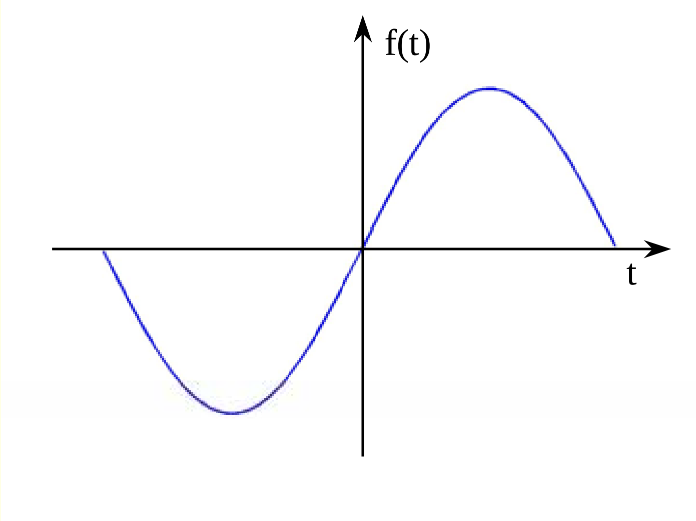
$$\Rightarrow f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_1 t \quad (b_n = 0)$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_1 t dt = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos n\omega_1 t dt$$

Odd symmetry $f(-t) = -f(t)$

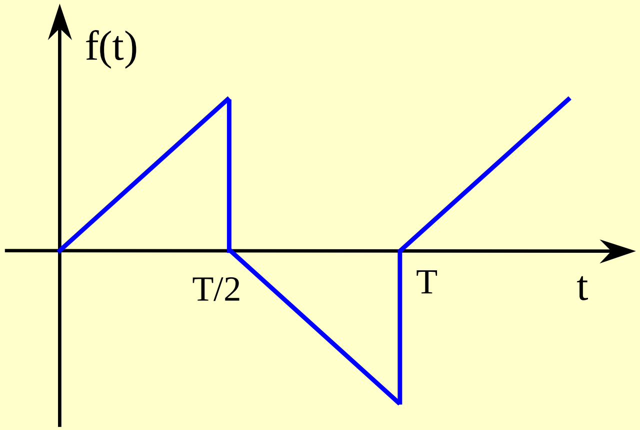
$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_1 t \qquad (a_0 = 0,\ a_n = 0)$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin n\omega_1 t dt$$



Half-wave symmetry $f(t) = -f(t + \frac{T}{2})$

$$f(t) = \sum_{n=1}^{\infty} A_{nm} \sin(n\omega_1 t + \theta_n) \quad n = 1, 3, 5 \ldots$$



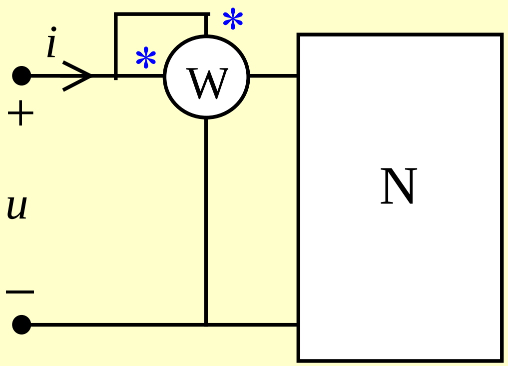
Average power and rms values

$$I = \sqrt{\frac{1}{T}\int_0^T i^2 dt} \qquad i = I_0 + \sum_{n=1}^{\infty} I_{nm} \sin(n\omega_1 + \theta_n)$$

$$I = \sqrt{\frac{1}{T}\int_0^T [I_0 + \sum_{n=1}^{\infty} I_{nm} \sin(n\omega_1 t + \theta_n)]^2 dt} = \sqrt{I_0^2 + \sum_{n=1}^{\infty} I_n^2}$$

$$u = U_0 + \sum_{n=1}^{\infty} U_{nm} \sin(n\omega_1 t + \theta_{un})$$

$$i = I_0 + \sum_{n=1}^{\infty} I_{nm} \sin(n\omega_1 t + \theta_{in})$$



The instantaneous power： $p(\mathrm{t})=u(t)i(t)$

The average power：

$$P = \frac{1}{T}\int_0^T p\,dt = U_0 I_0 + \sum_{n=1}^{\infty} U_n I_n \cos(\theta_{un} - \theta_{in}) = P_0 + \sum_{n=1}^{\infty} P_n$$

- The total average power is the sum of the average powers in each harmonically related voltage and current.

6-2 Circuit Applications

Steps for applying Fourier series：

Express the excitation as a Fourier series.

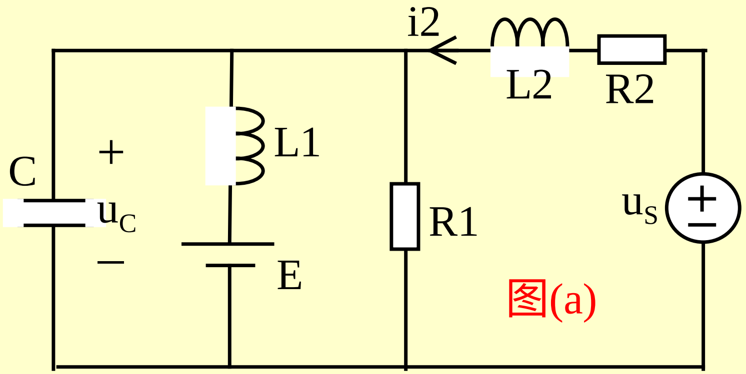
Find the response of each term in the Fourier series.

Add the individual responses using the superposition principle (time domain).

例：R_1=1Ω，R_2=2Ω，L_1=1H，L_2=2H，

$C=\frac{1}{4}F$ ，E=4V，$u_S=10\sqrt{2}\sin 2t\ V$ ，

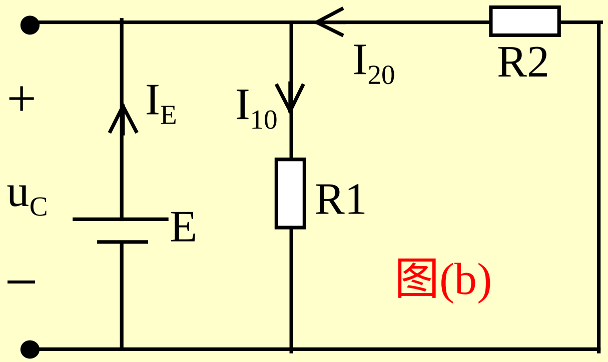
求：i_2，I_2；u_C ，P_{u_S} ，P_E，P_R。



图(a)

解：

(1) E单独激励：图(b)



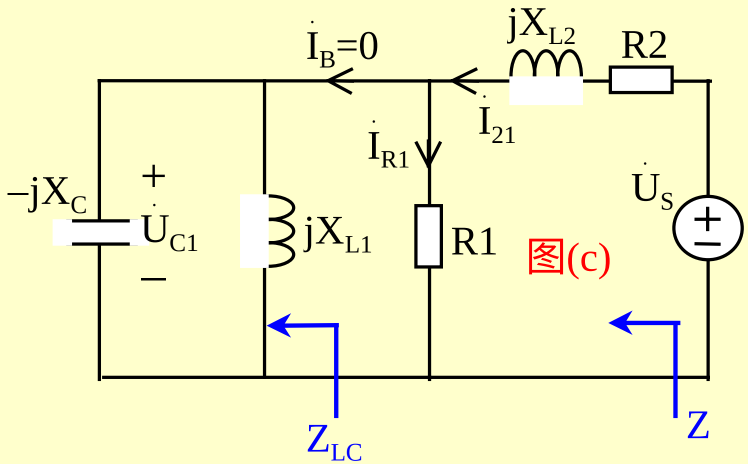
图(b)

$$I_{20} = -\frac{E}{R_2} = -2A,\ I_{10} = \frac{E}{R_1} = 4A$$

$$I_E = I_{10} - I_{20} = 6A,\quad u_{C0} = E = 4V$$

$P_{10}=R_1I_{10}^2=16W$，$P_{20}=R_2I_{20}^2=8W$；$P_E=EI_E=24W$

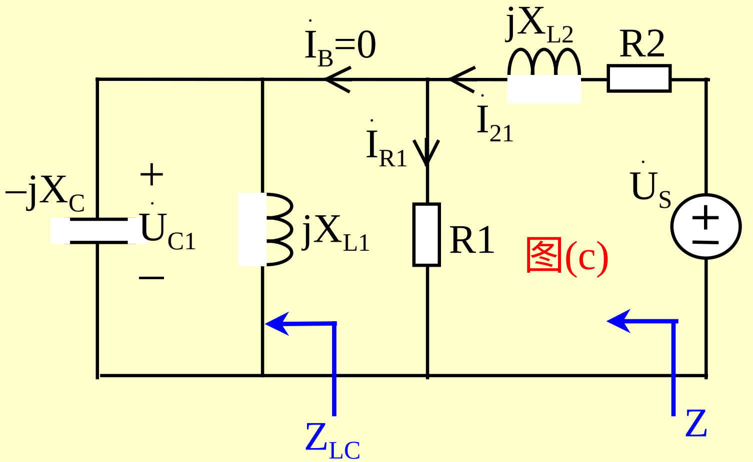
(2) u_S单独激励：图(c)



图(c)

$$\omega L_1 = 2\Omega,\ \frac{1}{\omega C} = 2\Omega,\ \omega L_2 = 4\Omega$$

$Z_{LC}=\infty$，并联谐振，$\dot{I}_B = 0$



$$Z=R_1+R_2+j\omega L_2=3+j4=5\angle 53.1^\circ\ \Omega$$

$$\dot{I}_{21}=\dot{I}_{R1}=\frac{\dot{U}_S}{Z}=2\angle-53.1^\circ\ A$$

$$\dot{U}_{C1}=R_1\dot{I}_{R1}=2\angle-53.1^\circ\ V$$

$$i_{21}=2\sqrt{2}\sin(2t-53.1^\circ)A,\ u_{C1}=2\sqrt{2}\sin(2t-53.1^\circ)V$$

$P_{11}=R_1I_{R1}{}^2=4W$，$P_{21}=R_2I_{21}{}^2=8W$，$P_{us}=U_SI_{21}\cos(\theta_{us}-\theta_{i21})=12W$

叠加：$P_E=24W$，$P_{us}=12W$，$P_R=P_{10}+P_{20}+P_{11}+P_{21}=36W$

$$i_2=I_{20}+i_{21}=-2+2\sqrt{2}\sin(2t-53.1^\circ)\ A,\ I_2=\sqrt{2^2+2^2}=\sqrt{8}=2.83A$$

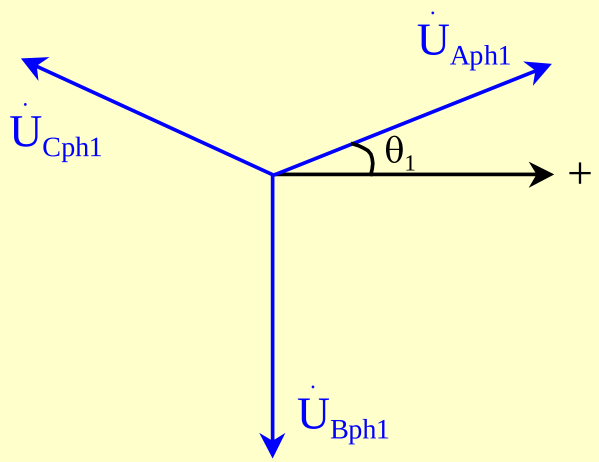
$$u_C=u_{C0}+u_{C1}=4+2\sqrt{2}\sin(2t-53.1^\circ)V$$

6.3 Balanced Three-Phase Circuits Excited By Nonsinusoidal Periodic Functions

Harmonics in balanced three-phase circuits

$$u_{Aph}=U_{ph1m}\sin(\omega_1 t+\theta_1)+U_{ph3m}\sin(3\omega_1 t+\theta_3)+U_{ph5m}\sin(5\omega_1 t+\theta_5)$$

Fundamental frequency：



$$u_{Aph1}=U_{ph1m}\sin(\omega_1 t+\theta_1)$$

$$u_{Bph1}=u_{Aph1}(t-\frac{T}{3})=U_{ph1m}\sin(\omega_1 t+\theta_1-120^{\circ})$$

$$\omega_1 T=2\pi$$

$$u_{Cph1}=u_{Aph1}(t+\frac{T}{3})=U_{ph1m}\sin(\omega_1 t+\theta_1+120^{\circ})$$

The abc sequence or positive sequence（正序）

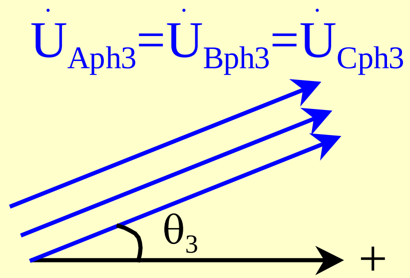
3-rd harmonic frequency：

$$u_{Aph3}(t)=U_{ph3m}\sin(3\omega_1 t+\theta_3)$$

$$u_{Bph3(t)}=u_{Aph3}(t-\frac{T}{3})=U_{ph3m}\sin\left[3\omega_1(t-\frac{T}{3})+\theta_3\right]$$

$$=U_{ph3m}\sin(3\omega_1 t+\theta_3)$$

$$u_{Cph3}(t)=u_{Aph3}(t+\frac{T}{3})=U_{ph3m}\sin(3\omega_1 t+\theta_3)$$



Zero sequence（零序)

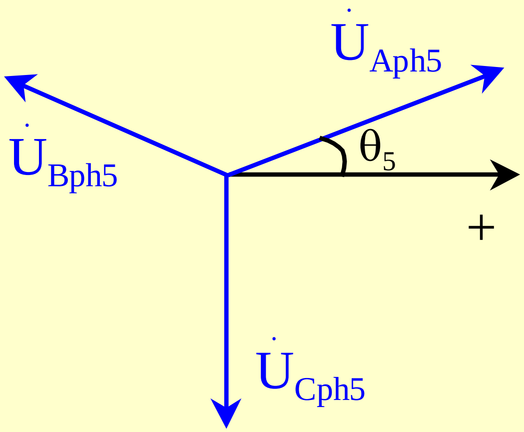
5-th harmonic frequency：

$$u_{Aph5}(t)=U_{ph5m}\sin(5\omega_1 t+\theta_5)$$

$$u_{Bph5(t)}=u_{Aph5}(t-\frac{T}{3})=U_{ph5m}\sin\left[5\omega_1(t-\frac{T}{3})+\theta_5\right]$$

$$=U_{ph5m}\sin(5\omega_1 t+\theta_5+120^\circ)$$

$$u_{Cph5(t)}=u_{Cph5}(t+\frac{T}{3})=U_{ph5m}\sin(5\omega_1 t+\theta_5-120^\circ)$$



The acb sequence or negative sequence（负序)

(6k+1)-th (k=0,1,2……) harmonic：positive sequence；

(6k–1)-th (k=1,2,3……) harmonic：negative sequence；

3k-th (k=1,3,5……) harmonic：zero sequence.

Balanced three-phase system with nonsinusoidal source

(6k+1)-th (k=0,1,2……) and (6k–1)-th (k=1,2,3……) harmonic

Y-connected source：

$\dot{U}_{AB(6k+1)} = \sqrt{3}\dot{U}_{A(6k+1)}\angle 30^\circ$ *positive sequence*

$\dot{U}_{AB(6k-1)} = \sqrt{3}\dot{U}_{A(6k-1)}\angle -30^\circ$ *negative sequence*

Δ-connected source：

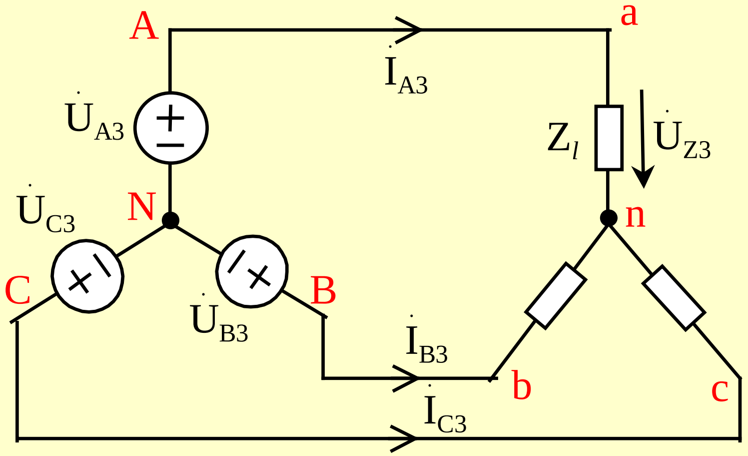
$\dot{I}_{A(6k+1)} = \sqrt{3}\dot{I}_{AB(6k+1)}\angle -30^\circ$ *positive sequence*

$\dot{I}_{A(6k-1)} = \sqrt{3}\dot{I}_{AB(6k-1)}\angle 30^\circ$ *negative sequence*

3k-th (k=1,3,5……) harmonic

$$\dot{U}_{A(3k)} = \dot{U}_{B(3k)} = \dot{U}_{C(3k)}$$

Y-Y connection system



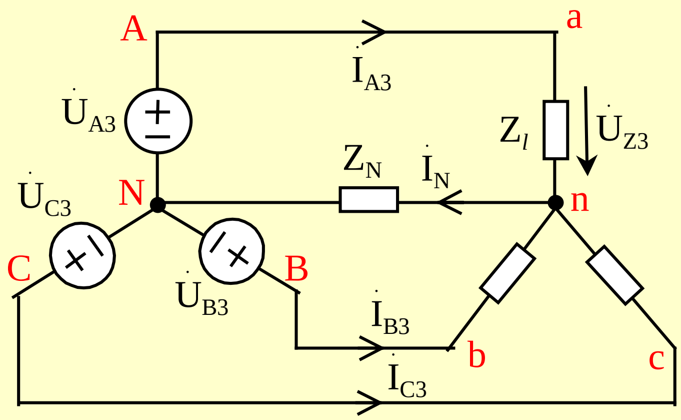
$$\dot{U}_{nN3} = \frac{\dot{U}_{A3}/Z_l + \dot{U}_{B3}/Z_l + \dot{U}_{c3}/Z_l}{\frac{1}{Z_l}\cdot 3} = \dot{U}_{A3}$$

$$\dot{I}_{A3} = \frac{\dot{U}_{A3} - \dot{U}_{nN3}}{Z_l} = 0$$

$$\dot{I}_{B3} = \dot{I}_{C3} = 0 \qquad \dot{U}_{Z_l} = 0$$

$$\dot{U}_{AB3} = \dot{U}_{A3} - \dot{U}_{B3} = 0, \quad \dot{U}_{BC3} = \dot{U}_{CA3} = 0$$

$$\dot{U}_{nN3} = \frac{\dot{U}_{A3}/Z_l + \dot{U}_{B3}/Z_l + \dot{U}_{C3}/Z_l}{\frac{3}{Z_l} + \frac{1}{Z_N}} = \frac{3\dot{U}_A Z_N}{3Z_N + Z_l}$$



$$\dot{I}_{A3} = \frac{\dot{U}_{A3} - \dot{U}_{nN}}{Z_l} = \frac{\dot{U}_{A3}}{3Z_N + Z_l}$$

$$\dot{I}_{B3} = \dot{I}_{C3} = \dot{I}_{A3} \qquad \dot{U}_{AZ} = \dot{U}_{BZ} = \dot{U}_{CZ} = Z_l \dot{I}_{A3}$$

$$\dot{I}_{N3} = \dot{I}_{A3} + \dot{I}_{B3} + \dot{I}_{C3} = 3\dot{I}_{A3}$$

$$\dot{U}_{AB3} = \dot{U}_{A3} - \dot{U}_{B3} = 0, \quad \dot{U}_{BC} = \dot{U}_{CA} = 0$$

6-4 Exponential Fourier Series

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\omega_1 t+b_n\sin n\omega_1 t)$$

Euler's identity： $\cos n\omega_1 t=\dfrac{e^{jn\omega_1 t}+e^{-jn\omega_1 t}}{2}$, $\sin n\omega_1 t=\dfrac{e^{jn\omega_1 t}-e^{-jn\omega_1 t}}{2j}$

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{jn\omega_1 t}+e^{-jn\omega_1 t}}{2}\right)+b_n\left(\frac{e^{jn\omega_1 t}-e^{-jn\omega_1 t}}{2j}\right)\right]$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[\left(\frac{a_n-jb_n}{2}\right)e^{jn\omega_1 t}+\left(\frac{a_n+jb_n}{2}\right)e^{-jn\omega_1 t}\right]$$

$$b_n=\frac{2}{T}\int_0^T f(t)\sin n\omega_1 t\,dt$$

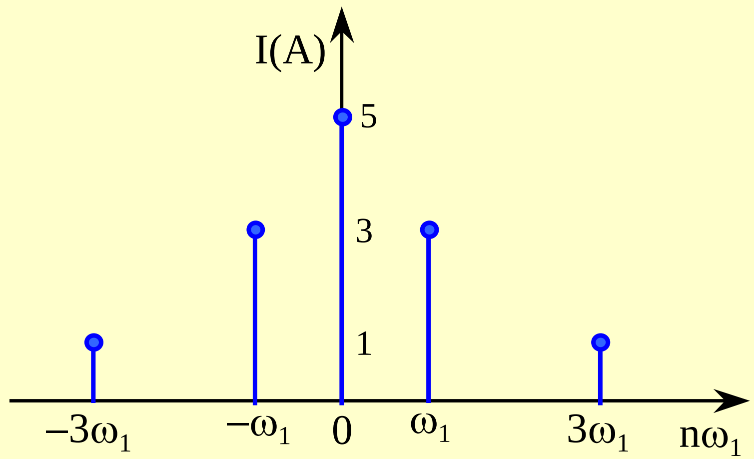
$$\sum_{n=1}^{\infty}\frac{a_n+jb_n}{2}e^{-jn\omega_1 t}=\sum_{n=-1}^{-\infty}\frac{a_n-jb_n}{2}e^{jn\omega_1 t}$$

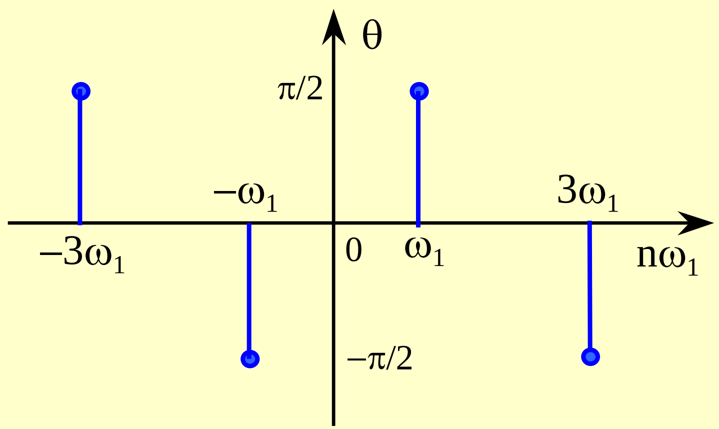
$$f(t)=\sum_{n=-\infty}^{+\infty}\frac{a_n-jb_n}{2}e^{jn\omega_1 t}=\sum_{n=-\infty}^{+\infty}\dot{F}_n e^{jn\omega_1 t}$$

$$\dot{F}_n=\frac{a_n-jb_n}{2}=\frac{1}{T}[\int_0^T f(t)(\cos n\omega_1 t-j\sin n\omega_1 t)dt]=\frac{1}{T}[\int_0^T f(t)e^{-jn\omega_1 t}dt]$$

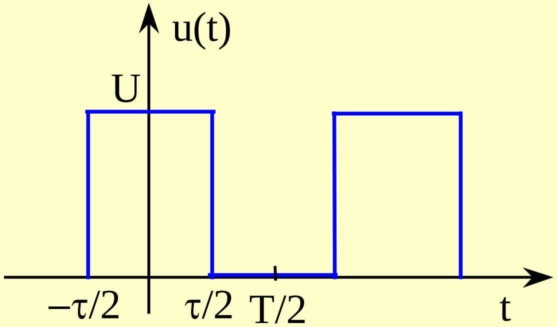
The exponential Fourier series of a periodic function f(t) describes the spectrum of f(t) in terms of the amplitude and phase angle of ac components at positive and negative harmonic frequencies.

$$i=[5+6\cos(\omega_1 t+\frac{\pi}{2})+2\cos(3\omega_1 t-\frac{\pi}{2})]$$





Fourier transform



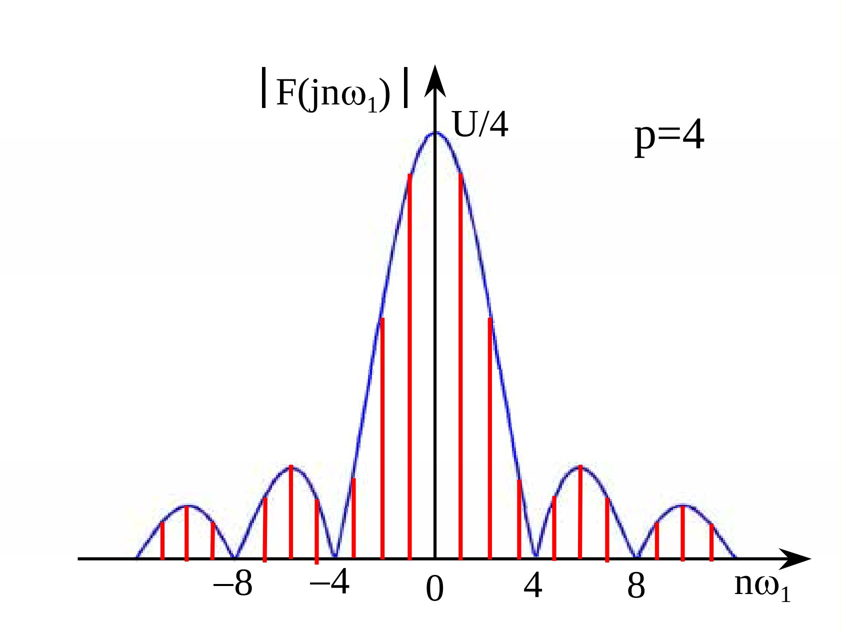
$$u(t)=\begin{cases}0 & -\frac{T}{2}\le t\le -\frac{\tau}{2}\\ U & -\frac{\tau}{2}\le t\le \frac{\tau}{2}\\ 0 & \frac{\tau}{2}\le t\le \frac{T}{2}\end{cases}$$

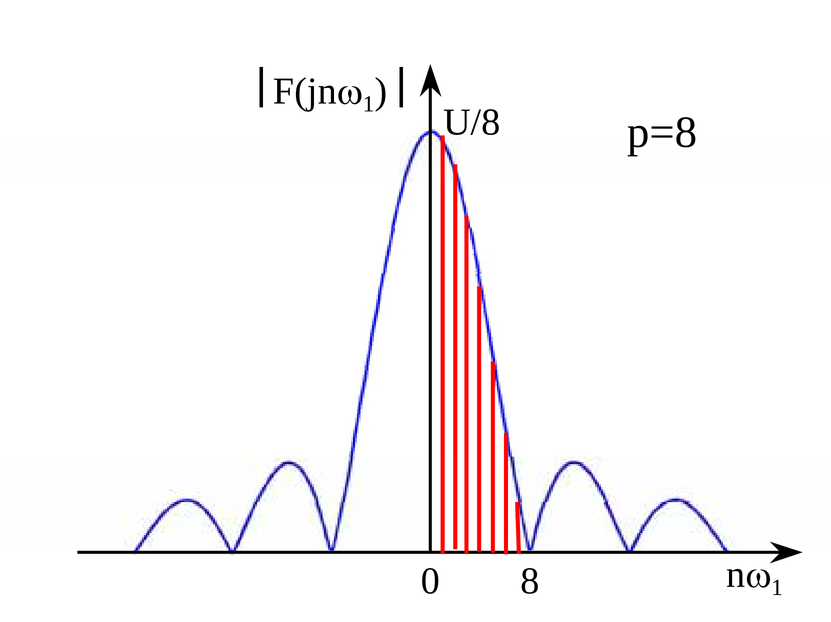
$$\dot{U}(n\omega_1)=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}u(t)e^{-jn\omega_1 t}dt=\frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}Ue^{-jn\omega_1 t}dt$$

$$=\frac{U}{T}\frac{e^{-jn\omega_1\frac{\tau}{2}}-e^{jn\omega_1\frac{\tau}{2}}}{-jn\omega_1}=\frac{\tau U}{T}\frac{\sin\frac{n\omega_1\tau}{2}}{\frac{n\omega_1\tau}{2}}$$

$$\frac{\omega_1\tau}{2}=\frac{\pi}{p}\quad (p=\frac{T}{\tau})$$

$$\dot{U}(n\omega_1)=\frac{U}{p}\frac{\sin\frac{n\pi}{p}}{\frac{n\pi}{p}},\quad \omega_1=\frac{2\pi}{T}$$





$$\dot{F}(j\omega) = \lim_{T\to\infty} \frac{\dot{F}_n}{\frac{1}{T}} = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-jn\omega_1 t} dt \qquad (\omega_1 = \frac{2\pi}{T})$$

The Fourier transform： $F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$

The inverse Fourier transform： $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} dt$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} U e^{-j\omega t} dt = \frac{U}{-j\omega}(e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}}) = \frac{U}{\omega} \sin\omega\frac{\tau}{2} = U\tau \frac{\sin\omega\frac{\tau}{2}}{\omega\frac{\tau}{2}}$$