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PROBABILITY DENSITY FUNCTIONS OF THE EMPIRICAL WAVELET COEFFICIENTS OF A WAVELET ESTIMATOR OF MULTIDIMENSIONAL POISSON INTENSITIES

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Abstract. We determine the probability density functions of the empirical wavelet coefficient estimator $\hat{\beta}_\eta = \int \psi_\eta dN$ in the wavelet series expansion $\hat{p} = \sum \hat{\beta}_\eta \psi_\eta$ of non homogeneous multidimensional Poisson processes intensity functions.

1. Introduction

Estimation of non homogeneous Poisson intensities is a research subject of both theoretical and practical interest given the importance of Poisson processes in point processes theory as well as its use in a large number of practical applications. Parametric, semi parametric, non parametric and bayesian methods have been used to estimate Poisson intensities. We cite the first fourteen references. In our works (de Miranda, 2003 and de Miranda and Morettin, 2005, 2006) we have studied the wavelet estimator $\hat{p} = \sum \hat{\beta}_\eta \psi_\eta$ of p, the intensity of a general point process. For the class of non internally correlated (NIC) point processes, a broader then Poisson class first defined in de Miranda & Morettin (2005, 2006), we have obtained inferential sequences, first defined in de Miranda (2003, 2005), for both the coefficients $\hat{\beta}_\eta$ and the intensity $\hat{p}$, therefore, as a particular case, their variance and variance function were determined. The probability density functions of these coefficients are not accessible under NIC conditions since they depend on the point process probability structure. In this way we used Tchebichev's inequality to draw confidence intervals for $\hat{\beta}_\eta$ and pointwise confidence bands for $\hat{p}$ since the conclusions that steam from this inequality are valid whatever the probability involved is. In this short article we specialize to non homogeneous Poisson processes. This restriction on the set of probability structures is strong enough to let us obtain the probability density function of the wavelet coefficients $\hat{\beta}_\eta$ and yet not too strong as to forbid its practical and theoretical use as seen above. The key feature here is the independence in Poisson process internal probability structure.

Key words and phrases. Characteristic function, intensity, Poisson process, probability density function, wavelet coefficient.

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This article is organized as follows. In Section 2 we present some basics and notations, in Section 3 we state and prove the main results and in Section 4 we make some comments.

2. Some Basics and Notations

Let N be a point process on $\mathbb{R}^d$, with unknown intensity p. Let $\{\psi_{j,i}|i, j \in \mathbb{Z}\}$ be an orthonormal wavelet basis of $L^2(\mathbb{R})$ of the form $\psi_{j,i}(t) = 2^{j/2}\psi(2^j t - i)$ or $\psi_{j,i}(t) = 2^{j/2}\psi(2^j(t-t_1)+t_1-iT)$ for some mother wavelet ψ obtained, if necessary by the composition of a standard wavelet with an affine transformation, such that its support is $[t_1, t_2]$ with $T = t_2 - t_1$. Here i corresponds to translations and j to dilations. Let ϕ be the father wavelet corresponding to ψ. Similarly, let $\{\phi_{\ell i,k}, \psi_{j,i} : i, k \in \mathbb{Z},\ j \geq \ell i, j,\ \ell i \in \mathbb{Z}\}$ be an orthonormal wavelet basis that contains all the scales beyond some fixed extended integer ℓi. It is pleasant to adopt the following notation. Let ${}_d\mathbb{Z} = \{z \in \mathbb{Z} : z \geq d\}, d \in \mathbb{Z} \cup \{-\infty\}$ and define $Ze(\ell i) = \begin{cases} \mathbb{Z} \cup ({}_{\ell i}\mathbb{Z} \times \mathbb{Z}) \text{ if } \ell i \in \mathbb{Z}, \\ \mathbb{Z}^2 \text{ if } \ell i = -\infty. \end{cases}$

Let us use Greek letters for indexes in $Ze(\ell i)$ and we shall write $\psi_\eta = \phi_{\ell i,\eta}$ if and only if $\eta \in \mathbb{Z}$ and $\psi_\eta = \psi_{j,i}$ if and only if $\eta = (j, i) \in \mathbb{Z}^2$. Thus, the wavelet expansions $f(t) = \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \delta_{ji}\psi_{j,i}(t)$ and $f(t) = \sum_{k\in\mathbb{Z}} \gamma_k\phi_{\ell i,k}(t) + \sum_{i\in\mathbb{Z}} \sum_{j\in_{\ell i}\mathbb{Z}} \delta_{ji}\psi_{j,i}(t)$ will be simply written $f = \sum_{\eta\in Ze(\ell i)} \alpha_\eta\psi_\eta$, for α_η given by $\int_{-\infty}^{\infty} f\psi_\eta dt = \int_{\mathbb{R}}(\sum_\xi \alpha_\xi\psi_\xi)\psi_\eta dt = \sum_\xi \int_{\mathbb{R}} \alpha_\xi\psi_\xi\psi_\eta dt = \sum_\xi \alpha_\xi < \psi_\xi, \psi_\eta >= \alpha_\eta$. Let for all n, $1 \leq n \leq d$, $\{\psi_{n,j,i}|i, j \in \mathbb{Z}\}$, $\psi_{n,j,i}(t) = 2^{j/2}\psi_n(2^j t - i)$ or $\psi_{n,j,i}(t) = 2^{j/2}\psi_n(2^j(t - a_n) + a_n - iT_n)$ and $\{\phi_{n,\ell i_n,k}, \psi_{n,j,i} : i, k \in \mathbb{Z},\ j \geq \ell i_n, j,\ \ell i_n \in \mathbb{Z}\}$ be orthonormal wavelet bases of $L^2(\mathbb{R})$ as above where $\operatorname{supp}\psi_n = [a_n, b_n]$ and $T_n = b_n - a_n$. These bases are simply written as $\{\psi_{n,\eta_n}|\eta_n \in Ze(\ell i_n)\}$ and they are, under restriction, also orthonormal bases of $L^2[a_n, b_n]$, $1 \leq n \leq d$. Taking tensor products we form the orthonormal basis $\{\tilde\psi_{\tilde\eta}|\tilde\psi_{\tilde\eta} = \otimes_{n=1}^d \psi_{n,\eta_n}, \tilde\eta = (\eta_1, \ldots, \eta_d) \in \prod_{n=1}^d Ze(\ell i_n)\}$ of $L^2(\mathbb{R}^d)$ and also, under restriction, of $L^2(\prod_{n=1}^d [a_n, b_n])$. Denote $\prod_{n=1}^d Ze(\ell i_n)$ by $\mathcal{Z}e(\tilde{\ell i})$; $\tilde{\ell i} = (\ell i_1, ..., \ell i_d)$. From now on we will drop the tilde and use simple notation for vectors in $\mathbb{R}^d$, tensor product wavelets and d-tuples in $\mathcal{Z}e(\tilde{\ell i})$. In this way if $f \in L^2(\mathbb{R}^d)$ we have $f = \sum_{\eta\in\mathcal{Z}e(\ell i)} \alpha_\eta\psi_\eta$ with $\alpha_\eta = \int_{\mathbb{R}^d} f\psi_\eta d\ell$.

Frequently we want to obtain the restriction of p to $\prod_{n=1}^d [a_n, b_n] = [a, b] = O$, an observation region, based on the points of a trajectory of the process that are contained in this $\mathbb{R}^d$ interval.

From now on we assume that p is locally square integrable. Therefore for the wavelet expansion of p restricted to bounded $\mathbb{R}^d$ interval observation regions, we have

$$p = \sum_\eta \beta_\eta\psi_\eta, \tag{2.1}$$

with

$$\beta_\eta = \int_{\mathbb{R}^d} p\psi_\eta d\ell. \tag{2.2}$$

The main estimation purpose is to obtain p through the expansion (2.1) and for this we need to estimate the wavelet coefficients β_η given by (2.2). The unbiased estimator we use is $\hat{\beta}_\eta = \int \psi_\eta dN$.

We use $O_F = (0, \ldots, 0) \in \mathcal{Z}e(\ell i)$, $O_M = ((\ell i_1, 0), \ldots, (\ell i_d, 0)) \in \mathcal{Z}e(\ell i)$. We write for $\eta \in Ze(\ell i)$, $j(\eta) = \ell i$ if $\eta \in \mathbb{Z}$ and $j(\eta) = j$ if $\eta = (j, i)$. Also, if $\eta \in \mathcal{Z}e(\ell i)$, $j(\eta) = (j(\eta_1), \ldots, j(\eta_m))$ and $|j(\eta)| = \sum_{\ell=1}^{m} j(\eta_i)$.

3. Main results

In this section we present the central results of this paper. Theorem 1 tells us how to obtain the probability density function of the empirical wavelet coefficient $\hat{\beta}_\eta$. Note that this function depends on both the wavelet ψ_η and the Poisson intensity $p(x)$. Corollary 1 presents the series expansion of the characteristic function of $\hat{\beta}_\eta$ and Corollary 2 gives formulas for the first four centered moments of $\hat{\beta}_\eta$ as well as its asymmetry and kurtosis coefficients. Theorem 2 is the analog of Theorem 1 for the specific case of Haar wavelets.

Theorem 1. *Let N be a Poisson process on $\mathbb{R}^d$ with intensity function $p : \mathbb{R}^d \to \mathbb{R}_+$. Suppose the wavelet ψ_η is compactly supported and continuous. Then $f_\eta : \mathbb{R} \to \mathbb{R}_+$, the probability density function of $\hat{\beta}_\eta = \int_{\mathbb{R}^d} \psi_\eta dN$, is given by the principal value $f_\eta(y) =$*

$$\frac{1}{2\pi} \int_{\mathbb{R}} exp\left(\int_{supp\psi_\eta} p(x) \left(cos\left(w\psi_\eta(x) \right) - 1 \right) dx \right) cos\left(\int_{supp\psi_\eta} p(x) sin\left(w\psi_\eta(x) \right) dx - wy \right) dw.$$

Proof. First observe that $\hat{\beta}_\eta = \int_{supp\psi_\eta} \psi_\eta dN$ and divide $supp\psi_\eta$ into n^d congruent pieces by taking a product partition generated by partitions of each of its edges into n equal segments. Denote these pieces by R_i, $1 \le i \le n^d$. Due to continuity of ψ_η and compactness of its support, ψ_η is uniformly continuous on $supp\psi_\eta$ so that $\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n > n_0\ \forall j, 1 \le j \le n^d$, $\forall x_j \in R_j\ \psi_\eta(R_j) \subset \psi_\eta(x_j) + (-\epsilon/2, \epsilon/2)$ and, for all $\omega \in \Omega$, the point process probability space, we have

$$\sum_{j=1}^{n^d} (\psi_\eta(x_j) - \epsilon/2) N(\omega)(R_j) \le \int \psi_\eta dN(\omega) \le \sum_{j=1}^{n^d} (\psi_\eta(x_j) + \epsilon/2) N(\omega)(R_j)$$

from which $\forall \omega \in \Omega$

$$\int \psi_\eta dN(\omega) = \lim_{n \to \infty} \sum_{j=1}^{n^d} \psi_\eta(x_j) N(\omega)(R_j);$$

i.e. $\int \psi_\eta dN = \lim_{n \to \infty} \sum_{j=1}^{n^d} \psi_\eta(x_j) N(R_j)$. Now, since N is Poisson we have $N(R_j) \sim Poisson(\int_{R_j} pd\ell) := X_j$; $\mathbb{E}X_j = \int_{R_j} pd\ell$.

Let $B_n = \sum_{j=1}^{n^d} \psi_\eta(x_j) X_j$. Thus,

$$\mathbb{E}(e^{iwB_n}) = \mathbb{E}(e^{iw\sum_{j=1}^{n^d} \psi_\eta(x_j)X_j}) = \mathbb{E}(\prod_{j=1}^{n^d} exp(iw\psi_\eta(x_j)X_j)).$$

The independence feature of Poisson processes implies then

$$\mathbb{E}(e^{iwB_n}) = \prod_{j=1}^{n^d} \mathbb{E}(exp(iw\psi_\eta(x_j)X_j)) = \prod_{j=1}^{n^d} exp((\int_{R_j} p(x)dx)(e^{iw\psi_\eta(x_j)} - 1)) =$$

$$exp(\sum_{j=1}^{n^d}(\int_{R_j} p(x)dx)(e^{iw\psi_\eta(x_j)} - 1))$$

Again, due to uniform continuity, $\forall w \in \mathbb{R}\ \forall \epsilon_1 > 0\ \exists n_1 \in \mathbb{N}\ \forall n > n_1\ \forall j\,, 1 \le j \le n^d$,

$$|(\int_{R_j} p(x)dx)(e^{iw\psi_\eta(x_j)} - 1) - \int_{R_j} p(x)(e^{iw\psi_\eta(x)} - 1)dx| < \epsilon_1 \int_{R_j} p(x)dx.$$

Noting that

$$\sum_{j=1}^{n^d} \int_{R_j} p(x)(e^{iw\psi_\eta(x)} - 1)dx = \int_{supp\psi_\eta} p(x)(e^{iw\psi_\eta(x)} - 1)dx$$

we write for sufficiently large n

$$|\sum_{j=1}^{n^d}[(\int_{R_j} p(x)dx)(e^{iw\psi_\eta(x_j)} - 1)] - \int_{supp\psi_\eta} p(x)(e^{iw\psi_\eta(x)} - 1)dx| < \epsilon_1 \int_{supp\psi_\eta} p(x)dx \text{ and}$$

$$\mathbb{E}(e^{iwB_n}) = exp(\int_{supp\psi_\eta} p(x)(e^{iw\psi_\eta(x)} - 1)dx)e^{\delta}\text{ , where } |\delta| < \epsilon_1 \int_{supp\psi_\eta} p(x)dx.$$

Taking the limits $\epsilon_1 \to 0$ and $n \to \infty$ we obtain by the dominated convergence theorem

$$\mathbb{E}(e^{iw\hat{\beta}_\eta}) = \mathbb{E}(e^{iw \lim_{n\to\infty} B_n}) = \lim_{n\to\infty} \mathbb{E}(e^{iwB_n}) = exp(\int_{supp\psi_\eta} p(x)(e^{iw\psi_\eta(x)} - 1)dx)$$

that is,

$$\int_{\mathbb{R}} e^{iwy} f_\eta(y)dy = exp(\int_{supp\psi_\eta} p(x)(e^{iw\psi_\eta(x)} - 1)dx).$$

Now, applying Fourier's inversion formula we have

$$f_\eta(y) = \frac{1}{2\pi} \int_{\mathbb{R}} exp(\int_{supp\psi_\eta} p(x)(e^{iw\psi_\eta(x)} - 1)dx)e^{-iwy}dw.$$

Since $f_\eta(y)$ is a real valued function, the result follows from

$$Re(exp(\int_{supp\psi_\eta} p(x)(e^{iw\psi_\eta(x)} - 1)dx - iwy)) =$$

$$exp(\int_{supp\psi_\eta} p(x)(cos(w\psi_\eta(x)) - 1)dx)cos(\int_{supp\psi_\eta} p(x)sin(w\psi_\eta(x))dx - wy).$$

□

From the proof of Theorem 1, the characteristic function of $\hat{\beta}_\eta$ is given by the following:

Corollary 1. *Under Theorem's 1 hypothesis we have*

$$\mathbb{E}(e^{iw\hat{\beta}_\eta}) = 1 + \sum_{n=1}^{\infty} \sum_{(\sum_{m=1}^{\infty} i_m = n)} \{ \frac{(iw)^{\sum_{m=1}^{\infty} m i_m}}{\prod_{m=1}^{\infty}(i_m!(m!)^{i_m})} \prod_{m=1}^{\infty} (\int_{supp\psi_\eta} p(x)\psi_\eta^m(x)dx)^{i_m} \}$$

Proof.

$$\mathbb{E}(e^{iw\hat{\beta}_\eta}) = exp(\int_{supp\psi_\eta} p(x)(e^{iw\psi_\eta(x)} - 1)dx) =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} (\int_{supp\psi_\eta} p(x) \sum_{m=1}^{\infty} \frac{(iw\psi_\eta(x))^m}{m!} dx)^n = \sum_{n=0}^{\infty} \frac{1}{n!} (\sum_{m=1}^{\infty} \frac{(iw)^m}{m!} \int_{supp\psi_\eta} p(x)\psi_\eta^m(x)dx)^n =$$

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!} (\sum_{(\sum_{m=1}^{\infty} i_m = n)} \binom{n}{i_1, i_2, ...} \frac{(iw)^{\sum_{m=1}^{\infty} m i_m}}{\prod_{m=1}^{\infty}(m!)^{i_m}} \prod_{m=1}^{\infty} (\int_{supp\psi_\eta} p(x)\psi_\eta^m(x)dx)^{i_m}),$$

by the multinomial formula, where $\binom{n}{i_1, i_2, ...} = \frac{n!}{\prod_{m=1}^{\infty} i_m!}$ and $\sum_{(\sum_{m=1}^{\infty} i_m = n)}$ means that the summation is made over all sequences of non negative integers i_m such that their sum is equal to n. □

Observe that, for a real random variable Y, the central moment of order r, $\mu_r(Y) = \mathbb{E}(Y - \mathbb{E}(Y))^r$ and the moments $\mathbb{E}(Y^j)$, $r, j \in \mathbb{N}$, obey the following relation $\mu_r(Y) = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \mathbb{E}(Y^{r-j})(\mathbb{E}(Y))^j$. Since we have the series expansion of the characteristic function of $\hat{\beta}_\eta$ its moments are easily obtained. The variance, asymmetry and kurtosis of the wavelet coefficient distributions is the subject of the following:

Corollary 2. *Under Theorem's 1 hypothesis, we have*

$$\beta_\eta = \mathbb{E}(\hat{\beta}_\eta) = \int \psi_\eta p d\ell, \; var(\hat{\beta}_\eta) = \int \psi_\eta^2 p d\ell,$$

$$\mu_3(\hat{\beta}_\eta) = \int \psi_\eta^3 p d\ell \quad and \quad \mu_4(\hat{\beta}_\eta) = \int \psi_\eta^4 p d\ell + 3(\int \psi_\eta^2 p d\ell)^2$$

so that the coefficients of asymmetry α_3 and kurtosis α_4 are given by:

$$\alpha_3(\eta) = \frac{\int \psi_\eta^3 p d\ell}{(\int \psi_\eta^2 p d\ell)^{(3/2)}} \; and \; \alpha_4(\eta) = 3 + \frac{\int \psi_\eta^4 p d\ell}{(\int \psi_\eta^2 p d\ell)^2}.$$

Proof. The r^{th} moment of $\hat{\beta}_\eta$ is the coefficient of $\frac{(iw)^r}{r!}$ in the series expansion of $\mathbb{E}(e^{iw\hat{\beta}_\eta})$. By exhaustion, one verifies that the only possibilities for the indexes that appear in Corollary 1 for the first four positive powers of (iw) are:
for $(iw)^1$, $i_1 = 1$ and $\forall k > 1 \; i_k = 0$;
for $(iw)^2$, $i_2 = 1$ and $\forall k \neq 2 \; i_k = 0$ or $i_1 = 2$ and $\forall k \neq 1 \; i_k = 0$;
for $(iw)^3$, $i_3 = 1$ and $\forall k \neq 3 \; i_k = 0$ or $i_1 = i_2 = 1$ and $\forall k > 2 \; i_k = 0$ or $i_1 = 3$ and $\forall k \neq 1 \; i_k = 0$;
for $(iw)^4$, $i_4 = 1$ and $\forall k \neq 4 \; i_k = 0$ or $i_1 = i_3 = 1$ and $\forall k \; 1 \neq k \neq 3 \; i_k = 0$ or $i_2 = 2$ and $\forall k \neq 2 \; i_k = 0$ or $i_1 = 2, i_2 = 1$ and $\forall k > 2 \; i_k = 0$ or $i_1 = 4$ and $\forall k > 1 \; i_k = 0$.

In this way,

$$\mathbb{E}(\hat{\beta}_\eta) = \int_{supp\psi_\eta} p(x)\psi_\eta(x)dx = \int \psi_\eta p d\ell = \beta_\eta \; ; \mathbb{E}(\hat{\beta}_\eta^2) = \int \psi_\eta^2 p d\ell + (\int \psi_\eta p d\ell)^2;$$

$$\mathbb{E}(\hat{\beta}_\eta^3) = \int \psi_\eta^3 p d\ell + 3\int \psi_\eta p d\ell \int \psi_\eta^2 p d\ell + (\int \psi_\eta p d\ell)^3$$

and

$$\begin{aligned} \mathbb{E}(\hat{\beta}_\eta^4) &= \int \psi_\eta^4 p d\ell + 4\int \psi_\eta p d\ell \int \psi_\eta^3 p d\ell + 3(\int \psi_\eta^2 p d\ell)^2 \\ &+ 6(\int \psi_\eta p d\ell)^2 \int \psi_\eta^2 p d\ell + (\int \psi_\eta p d\ell)^4. \end{aligned}$$

Now we have

$$\mathbb{E}(\hat{\beta}_\eta - \mathbb{E}(\hat{\beta}_\eta))^2 = \mathbb{E}\hat{\beta}_\eta^2 - (\mathbb{E}\hat{\beta}_\eta)^2 = \int \psi_\eta^2 p d\ell$$

$$\mathbb{E}(\hat{\beta}_\eta - \mathbb{E}(\hat{\beta}_\eta))^3 = \mathbb{E}\hat{\beta}_\eta^3 - 3\mathbb{E}\hat{\beta}_\eta^2\mathbb{E}\hat{\beta}_\eta + 2(\mathbb{E}\hat{\beta}_\eta)^3 = \int \psi_\eta^3 p d\ell$$

$$\mathbb{E}(\hat{\beta}_\eta - \mathbb{E}(\hat{\beta}_\eta))^4 = \mathbb{E}\hat{\beta}_\eta^4 - 4\mathbb{E}\hat{\beta}_\eta^3\mathbb{E}\hat{\beta}_\eta + 6\mathbb{E}\hat{\beta}_\eta^2(\mathbb{E}\hat{\beta}_\eta)^2 - 3(\mathbb{E}\hat{\beta}_\eta)^4 = \int \psi_\eta^4 p d\ell + 3(\int \psi_\eta^2 p d\ell)^2.$$

The asymmetry and kurtosis coefficients are thus

$$\alpha_3(\eta) = \frac{\int \psi_\eta^3 p d\ell}{(\int \psi_\eta^2 p d\ell)^{(3/2)}} \text{ and } \alpha_4(\eta) = 3 + \frac{\int \psi_\eta^4 p d\ell}{(\int \psi_\eta^2 p d\ell)^2}.$$

□

One of the most important and used wavelet families is the Haar family. This is a consequence of the extremely simple forms of its scale function and mother wavelet that makes it computationally easier to use Haar wavelets instead of other more elaborated ones. However, Haar wavelets are not continuous and Theorem 1 does not apply to them. In this way, we present the following:

Theorem 2. *Let N be a Poisson process on $\mathbb{R}^d$ with intensity function $p : \mathbb{R}^d \to \mathbb{R}_+$. Denote $\psi_\eta^+ = (|\psi_\eta| + \psi_\eta)/2$ and $\psi_\eta^- = (|\psi_\eta| - \psi_\eta)/2$. Suppose the wavelet family used is Haar, that is, the wavelets in this family are tensor products of one dimensional Haar wavelets only. Then $\hat{\beta}_\eta \sim ||\psi_\eta||_\infty(X^+ - X^-)$,where X^+ and X^- are independent Poisson random variables with means $\lambda_\eta^+ = \int_{supp\psi_\eta^+} p d\ell$ and $\lambda_\eta^- = \int_{supp\psi_\eta^-} p d\ell$. The probability function of $\hat{\beta}_\eta$, $f_\eta : ||\psi_\eta||_\infty\mathbb{Z} \to \mathbb{R}_+$, is given by:*

$$f_\eta(||\psi_\eta||_\infty z) = exp(-\int_{supp\psi_\eta} p d\ell) \sum_{k \geq max\{0,z\}} \frac{(\lambda_\eta^+)^k(\lambda_\eta^-)^{k-z}}{k!(k-z)!}.$$

Proof.

$$\hat{\beta}_\eta = \int \psi_\eta dN = \int_{supp\psi_\eta^+} \psi_\eta^+ dN - \int_{supp\psi_\eta^-} \psi_\eta^- dN = ||\psi_\eta||_\infty(\int_{supp\psi_\eta^+} dN - \int_{supp\psi_\eta^-} dN).$$

Thus, due to the internal independence of Poisson processes, $\hat{\beta}_\eta \sim ||\psi_\eta||_\infty(X^+ - X^-)$.

$$\mathbb{P}(X^+ - X^- = z) = \sum_k \mathbb{P}(X^+ = k, X^- = k - z) = \sum_k \mathbb{P}(X^+ = k)\mathbb{P}(X^- = k - z) =$$

$$\sum_{k \geq max\{0,z\}} \mathbb{P}(X^+ = k)\mathbb{P}(X^- = k - z) = \sum_{k \geq max\{0,z\}} \frac{e^{-\lambda_\eta^+}(\lambda_\eta^+)^k}{k!} \frac{e^{-\lambda_\eta^-}(\lambda_\eta^-)^{k-z}}{(k-z)!}.$$

□

4. Comments

We remark that if the intensity may be regarded as constant on $supp\psi_\eta$ then we can write the following approximations:

$$\alpha_3(\eta) = \frac{\int \psi_\eta^3 p d\ell}{(\int \psi_\eta^2 p d\ell)^{(3/2)}} \cong \frac{2^{|j(\eta)|/2}}{p^{\frac{1}{2}}} \int \psi_{z(\eta)}^3 d\ell$$

and

$$\alpha_4(\eta) = 3 + \frac{\int \psi_\eta^4 p d\ell}{(\int \psi_\eta^2 p d\ell)^2} \cong 3 + \frac{2^{|j(\eta)|}}{p} \int \psi_{z(\eta)}^4 d\ell,$$

where $\psi_{z(\eta)}$ is any re-scaled wavelet that corresponds to ψ_η such that $j(z(\eta)) = 0 \in \mathbb{Z}^d$. Since, for all $\eta \in \mathcal{Z}e(\ell i)$, $\int \psi_{z(\eta)}^3 d\ell$ and $\int \psi_{z(\eta)}^4 d\ell$ are limited by constants, we observe that for continuous intensities the kurtosis coefficient will increase without bound as $|j(\eta)|$ goes to infinity; and the same will happen to the absolute value of the asymmetry coefficient in case the wavelet has non vanishing integral of its third power. Note that one can have all $\alpha_3(\eta)$'s, but $\alpha(O_F)$, equal to zero if the multidimensional wavelet basis is formed by tensor products of one dimensional wavelets such that the integral of the third power of each of these wavelets is zero.

It is also worth noting that in case we have n independent replications of the Poisson process, i.e. we have n independent trajectories of the process, we can form the estimators $\tilde{\beta}_\eta = \frac{1}{n}\sum_{i=1}^n \hat{\beta}_\eta(i)$, $\tilde{p} = \frac{1}{n}\sum_{i=1}^n \hat{p}(i)$, where $\hat{\beta}_\eta(i)$ and $\hat{p}(i)$ are the estimated wavelet coefficient and intensity obtained from the i^{th} observation. These estimators inherit the unbiasedness characteristics of $\hat{\beta}_\eta(i)$ and $\hat{p}(i)$. Moreover, $\tilde{\beta}_\eta$ also presents the desired feature of asymptotical normality as a consequence of the finiteness of the first and second moments of the wavelet coefficient estimators $\hat{\beta}_\eta(i)$ that guarantees the central limit theorem can be applied to the independent sum $\tilde{\beta}_\eta$. As a matter of fact the asymptotic normality of $\tilde{\beta}_\eta$ is not restricted to Poisson process setting; in de Miranda (2003) and de Miranda & Morettin (2005, 2006) we have also shown that the finiteness requirements mentioned above are also valid for NIC point processes so that they will also exhibit this feature in case of independent replications.

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