*Brazilian Journal of Probability and Statistics*
Volume 22, Number 1, Pages 69–83
ISSN 0103-0752

# FRACTIONAL INTEGRATION WITH BLOOMFIELD EXPONENTIAL SPECTRAL DISTURBANCES: A MONTE CARLO EXPERIMENT AND AN APPLICATION

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Abstract. We show in this article that fractionally integrated processes with Bloomfield (1973) exponential spectral disturbances can be well approximations to fractional models with AR disturbances. In fact, when testing $I(d)$ statistical models with the tests of Robinson (1994), the autoregressive structure underlying the $I(0)$ disturbances can be distorting the order of integration of the series, because of the roots being close to the unit root circle. In that respect, the short-run dynamics may well be approximated by the Bloomfield (1973) exponential spectral model. This is illustrated with several Monte Carlo experiments. An empirical application, showing the performance of this type of model, is also carried out at the end of the article.

## 1. Introduction

For the purpose of the present paper, we define an $I(0)$ process $\{u_t,\ t = 0, \pm 1, \ldots\}$ as a covariance stationary process with spectral density function that is positive and finite at the zero frequency. In this context, we say that $\{x_t,\ t = 0, \pm 1, \ldots\}$ is $I(d)$ if:

$$\begin{aligned} (1 - L)^d x_t &= u_t, \quad t = 1, 2, \ldots \\ x_t &= 0, \qquad t \le 0, \end{aligned} \tag{1.1}$$

where $L$ is the lag operator ($Lx_t = x_{t-1}$) and the polynomial in (1.1) can be expressed in terms of its Binomial expansion such that:

$$(1 - L)^d = \sum_{j=0}^{\infty} \binom{d}{j} (-1)^j L^j = 1 - d\,L + \frac{d(d-1)}{2} L^2 - \cdots ,$$

for any real $d$. This type of model was introduced by Granger and Joyeux (1980), Granger (1980, 1981) and Hosking (1981), (although earlier work by Adenstedt, 1974 and Taqqu, 1975, shows an awareness of the representation), and it was theoretically justified in terms of aggregation of ARMA series with randomly varying coefficients by Robinson (1978) and Granger (1980), and more recently, in terms of the duration of shocks by Parke (1999). Similarly, Cioczek-Georges and Mandelbrot (1995), Taqqu et al. (1997), Chambers (1998) and Lippi and Zaffaroni (1999) also use aggregation to motivate long memory

*Key words and phrases.* Bloomfield disturbances, fractional integration, long memory.
Received May/2006, May/2007.

processes, while Diebold and Inoue (2001) propose another source of long memory based on regime-switching models.

Robinson (1994) proposed a Lagrange Multiplier (LM) test of the null hypothesis:

$$H_0 : d = d_0 \,, \tag{1.2}$$

in a model given by (1.1) for any given real value $d_0$. The test has standard null and local limit distributions and it is parametric, in the sense that we have to specify the functional form of the $I(0)$ disturbances $u_t$ in (1.1), which may include, for example, autoregressive (AR) models. Bloomfield (1973) showed that the AR specification can be non-parametrically well approximated in terms of its spectral density function. Like the stationary AR($p$) case, this model also has exponentially decaying autocorrelations and is very easy to implement in the context of the tests of Robinson (1994).

We show in this article that fractionally integrated processes with Bloomfield (1973) exponential spectral disturbances can be well approximations to fractional models with AR disturbances. There are several advantages in this approach. Firstly, computationally, the derivation of the tests of Robinson (1994) greatly simplifies in the context of Bloomfield's (1973) disturbances, (see, Robinson, 1994). Secondly and more important, there is a drawback in the performance of Robinson's (1994) tests when AR disturbances are entertained, in that the roots of the AR polynomial can be arbitrarily close to the unit root case and thus, they may be competing with the order of integration in describing the nonstationarity, invalidating the test statistic. This is sorted out when using Bloomfield's (1973) exponential spectral model, which is always stationary across the whole range of values of its parameters.

The structure of the article is as follows: Section 2 briefly describes the tests of Robinson (1994) and the Bloomfield (1973) exponential spectral model. Section 3 reports some Monte Carlo simulations comparing the performance of Robinson's (1994) tests when using both AR and Bloomfield (1973) disturbances. An empirical application is carried out in Section 4, while Section 5 contains some concluding comments.

## 2. The tests of Robinson (1994) and the Bloomfield (1973) exponential spectral model

Let us suppose that $\{x_t, t = 1, 2, \ldots, T\}$ is the time series we observe. The statistic proposed by Robinson (1994) for testing $H_0$ (1.2) in (1.1) is given by:

$$\hat{r} = \left(\frac{T}{\hat{A}}\right)^{1/2} \frac{\hat{a}}{\hat{\sigma}^2} \,, \tag{2.1}$$

where

$$\hat{a} = -\frac{2\pi}{T} \sum_{j=1}^{T-1} \psi(\lambda_j) g(\lambda_j; \hat{\tau})^{-1} I(\lambda_j) \,; \quad \hat{\sigma}^2 = \sigma^2(\hat{\tau}) = \frac{2\pi}{T} \sum_{j=1}^{T-1} g(\lambda_j; \hat{\tau})^{-1} I(\lambda_j) \,;$$

$$\hat{A} = \frac{2}{T}\left(\sum_{j=1}^{T-1}\psi(\lambda_j)^2 - \sum_{j=1}^{T-1}\psi(\lambda_j)\hat{\varepsilon}(\lambda_j)' \times \left(\sum_{j=1}^{T-1}\hat{\varepsilon}(\lambda_j)\hat{\varepsilon}(\lambda_j)'\right)^{-1} \times \sum_{j=1}^{T-1}\hat{\varepsilon}(\lambda_j)\psi(\lambda_j)\right) ;$$

$$\psi(\lambda_j) = \log\left|2\sin\frac{\lambda_j}{2}\right| ; \quad \hat{\varepsilon}(\lambda_j) = \frac{\partial}{\partial\tau}\log g(\lambda_j;\hat{\tau}) ; \quad \lambda_j = \frac{2\pi j}{T} ;$$

$\hat{\tau} = \arg\min \sigma^2(\tau)$, $I(\lambda_j)$ is the periodogram of $\hat{u}_t = (1-L)^{d_0}x_t$, and the function $g$ above is a known function obtained from the spectral density function of $u_t$,

$$f(\lambda;\sigma^2;\tau) = \frac{\sigma^2}{2\pi}g(\lambda;\tau) , \quad -\pi < \lambda < \pi .$$

Note that these tests are purely parametric and therefore, they require specific modelling assumptions regarding the short memory specification of $u_t$. Thus, if $u_t$ is white noise, then $g \equiv 1$, ($\hat{\varepsilon}(\lambda_j) = 0$), and if $u_t$ is an AR process of form $\phi(L)u_t = \varepsilon_t$, then $g = |\phi(e^{i\lambda})|^{-2}$, with $\sigma^2 = V(\varepsilon_t)$, so that the AR coefficients are a function of $\tau$.

Robinson (1994) showed that under certain regularity conditions,[1]

$$\hat{r} \to_d N(0,1) \text{ as } T \to \infty . \tag{2.2}$$

Thus, an approximate one-sided 100%-level test of $H_0$ (1.2) against the alternative: $H_a$ : $d > d_0$ ($d < d_0$) will reject $H_0$ (1.2) if $\hat{r} > z_\alpha$ ($\hat{r} < -z_\alpha$), where the probability that a standard normal variate exceeds $z_\alpha$ is $\alpha$. Furthermore, he shows that the above test is efficient in the Pitman sense, i.e., that against local alternatives of form: $H_a$ : $d = d_0 + \delta T^{-1/2}$, with $\delta \neq 0$, the limit distribution is normal with variance 1 and mean which cannot (when $u_t$ is Gaussian) be exceeded in absolute value by that of any rival regular statistic. Empirical applications based on this version of Robinson's (1994) tests can be found in Gil-Alana and Robinson (1997) and Gil-Alana (2000) and, other versions of his tests, based on seasonal (quarterly and monthly) and cyclical data are respectively Gil-Alana and Robinson (2001) and Gil-Alana (1999, 2001a).

Note that the above test permits us to consider the unit root model as a particular case if $d_0$ in (1.2) is equal to 1. However, unlike most classic unit root tests (Dickey and Fuller, 1979; Phillips and Perron, 1988; etc.), which are embedded in autoregressive (AR) alternatives, Robinson's (1994) tests are nested in a fractional form, this being the reason for its standard null limit distribution.[2] Consider, for instance, the AR model $(1-\rho L)x_t = u_t$. Clearly, if $\rho = 1$, we obtain the same null unit root model as in $(1-L)^d x_t = u_t$ with $d = 1$. However, in the former specification, if $|\rho| < 1$, $x_t$ is covariance stationary: if $\rho = 1$ we have a unit root, which is nonstationary though non-explosive, and if $|\rho| > 1$ it implies a nonstationary explosive behaviour. Thus, we observe an abrupt change in the limit behaviour around $\rho = 1$. On the other hand, in the fractional specification, the limit behaviour is smooth around $d = 1$, and the boundary line between stationarity and nonstationarity is now around $d = 0.5$. Nevertheless, the behaviour in the fractional

[1] These conditions are very mild regarding technical assumptions that are satisfied by model (1). Moreover, they impose a martingale difference on $u_t$, which is a condition substantially weaker than Gaussianity.

[2] In a recent paper, Phillips and Magdalinos (2007) points out that the limit theory for moderate deviations from a unit root is quite different from stationary processes (with roots far away from the unit circle).

model is "smooth" in the sense that we do not observe an abrupt change with small changes in the parameters.

The AR modelling of the $I(0)$ disturbances $u_t$ is very conventional, but there exist many other types of $I(0)$ processes, including ones outside the stationary and invertible AR(MA) case. One model that seems especially relevant and convenient in the context of the present tests is that proposed by Bloomfield (1973). This model is non-parametric and is exclusively specified in terms of its spectral density function, which is given by:

$$f(\lambda_j; \sigma^2; \tau) = \frac{\sigma^2}{2\pi} \exp\left(2 \sum_{l=0}^{p} \tau_l \cos(\lambda_j l)\right) .$$

Then, the function $g$ above is given by:

$$g(\lambda_j; \tau) = \exp\left(2 \sum_{l=0}^{p} \tau_j \cos(\lambda_j l)\right) . \tag{2.3}$$

Formulae for Newton-type iteration for estimating the $\tau_l$ are very simple (involving no matrix inversion), updating formulae when $p$ is increased are also simple, and we can replace $\hat{A}$ below (2.1) (in the functional form of the test statistic) by the population quantity:

$$\sum_{l=p+1}^{\infty} l^{-2} = \frac{\pi^2}{6} - \sum_{l=1}^{p} l^{-2} ,$$

which indeed is constant with respect to the $\tau_l$ (unlike what happens in the AR case). To see this, let us first consider a pure AR($p$) process of form:

$$u_t = \sum_{l=1}^{p} \tau_l u_{t-l} + \varepsilon_t .$$

The function $g$ takes then the form:

$$g(\lambda_j; \tau) = \left| 1 - \sum_{l=1}^{p} \tau_l e^{il\lambda_j} \right|^{-2} ,$$

and noting that

$$\varepsilon(\lambda_j) = \frac{\partial}{\partial \tau} \log g(\lambda_j; \tau) ,$$

then, $\varepsilon(\lambda_j)$ will be a $(p \times 1)$ vector with $k^{\text{th}}$-element of form:

$$\varepsilon_k(\lambda_j) = \left(2 \left[\cos k\lambda_j - \sum_{l=1}^{p} \tau_l \cos(k-l)\lambda_j\right] g(\lambda_j; \tau)\right) . \tag{2.4}$$

Using now the model of Bloomfield (1973) and its corresponding $g$-function in (2.3), the $k^{\text{th}}$-element of $\varepsilon(\lambda_j)$ adopts the form:

$$\varepsilon_k(\lambda_j) = 2\cos(k\lambda_j) , \tag{2.5}$$

which is clearly simpler than (2.4). Moreover, the expression

$$\frac{2}{T}\sum_{j=1}^{T}\varepsilon(\lambda_j)\varepsilon(\lambda_j)'$$

can be approximated by $4I_p$, and thus, no matrix inversion is required in the computation of the test statistic.

The intuition behind the model of Bloomfield (1973) is the following. Suppose that $u_t$ follows an ARMA process of form:

$$u_t = \sum_{r=1}^{p}\phi_r u_{t-r} + \varepsilon_t + \sum_{r=1}^{q}\theta_r\varepsilon_{t-r}\ ,$$

where $\varepsilon_t$ is a white noise process and all zeros of $\phi(L) = (1 - \phi_1 L - \cdots - \phi_p L^p)$ lying outside the unit circle and all zeros of $\theta(L) = (1 + \theta_1 L + \cdots + \theta_q L^q)$ lying outside or on the unit circle. Clearly, the function $g$ of this process is then given by:

$$g(\lambda_j;\tau) = \left|\frac{1+\sum_{r=1}^{q}\theta_r e^{i\lambda_j r}}{1-\sum_{r=1}^{p}\phi_r e^{i\lambda_j r}}\right|^2\ . \tag{2.6}$$

Bloomfield (1973) showed that the logarithm of the above function is a fairly well behaved function and can thus be approximated by a truncated Fourier series. He showed that (2.3) approximates (2.6) well where $p$ and $q$ are small values, which usually happens in economics. Like the stationary AR($p$) case, this model has exponentially decaying autocorrelations and thus, using this specification, we do not need to rely on so many parameters as in the ARMA processes, which always results tedious in terms of estimation, testing and model specification. Moreover, this approximation remains valid even if the roots of the AR polynomial are close to the unit circle.

The Bloomfield (1973) model combined with fractional integration has not been very much used in econometrics though the Bloomfield model itself is a well-known model in other disciplines (see, e.g., Beran, 1993). Amongst the few empirical applications found in the literature are Gil-Alana and Robinson (1997), Velasco and Robinson (2000) and Gil-Alana (2001b). Beran (1995) proposed an approximation to the likelihood function to estimate the parameters which are involved in a fractional model with Bloomfield (1973) disturbances. However, unlike that procedure, Robinson's (1994) tests do not require estimation of the fractional differencing parameter, since it is based on the null differenced model, which is supposed to be $I(0)$. In the following section, several Monte Carlo experiments are conducted to show that the Bloomfield (1973) exponential spectral model can be a competitive model for the autoregressive disturbances in the context of fractionally integrated models.

## 3. A Monte Carlo experiment

Across this section we look at the rejection frequencies of the tests of Robinson (1994) assuming that the true model is given by (1.1) with AR(1) and AR(2) disturbances, and perform the tests using both AR and Bloomfield (1973) disturbances. We use Gaussian series generated by the routines GASDEV and RAN of Press, Flannery, Teukolsky and

Table 1. Rejection frequencies of Robinson's (1994) tests with AR(1) disturbances. In bold: the sizes of the tests. The nominal size is 0.050.

| True model: $(1-L)^d x_t = u_t;\ u_t = \phi u_{t-1} + \varepsilon_t;\ d = 1$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Alternatives: $(1-L)^d x_t = u_t;\ u_t \sim$ AR(1) and Bloomfield (1) | | | | | | | | | | | | |
| | | | $H_a : d > d_0$ | | | | | $H_a : d < d_0$ | | | | |
| Size | $\phi$ | $u_t/d$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 |
| | | AR(1) | 0.575 | 0.125 | 0.010 | 0.050 | **0.016** | **0.221** | 0.560 | 0.912 | 0.995 | 0.999 |
| | 0.25 | Bloomfield | 0.998 | 0.983 | 0.856 | 0.407 | 0.038 | 0.104 | 0.528 | 0.918 | 0.995 | 1.000 |
| | | AR(1) | 0.741 | 0.743 | 0.238 | 0.024 | **0.010** | **0.329** | 0.446 | 0.723 | 0.937 | 0.994 |
| 100 | 0.50 | Bloomfield | 0.999 | 0.995 | 0.945 | 0.667 | 0.161 | 0.018 | 0.207 | 0.683 | 0.952 | 0.997 |
| | | AR(1) | 0.721 | 0.886 | 0.859 | 0.306 | **0.019** | **0.189** | 0.405 | 0.362 | 0.555 | 0.847 |
| | 0.75 | Bloomfield | 0.999 | 0.999 | 0.996 | 0.960 | 0.733 | 0.030 | 0.088 | 0.100 | 0.488 | 0.873 |
| | | AR(1) | 0.442 | 0.632 | 0.802 | 0.751 | **0.101** | **0.037** | 0.414 | 0.186 | 0.006 | 0.265 |
| | 0.95 | Bloomfield | 1.000 | 1.000 | 0.999 | 0.999 | 0.993 | 0.006 | 0.010 | 0.102 | 0.025 | 0.256 |
| | | AR(1) | 0.680 | 0.093 | 0.016 | 0.207 | **0.025** | **0.148** | 0.738 | 0.995 | 1.000 | 1.000 |
| | 0.25 | Bloomfield | 1.000 | 0.999 | 0.994 | 0.767 | 0.057 | 0.066 | 0.721 | 0.995 | 1.000 | 1.000 |
| | | AR(1) | 0.820 | 0.878 | 0.303 | 0.057 | **0.020** | **0.205** | 0.521 | 0.921 | 0.998 | 1.000 |
| 200 | 0.50 | Bloomfield | 1.000 | 1.000 | 0.999 | 0.943 | 0.317 | 0.003 | 0.233 | 0.905 | 0.999 | 1.000 |
| | | AR(1) | 0.800 | 0.964 | 0.970 | 0.440 | **0.021** | **0.145** | 0.356 | 0.416 | 0.775 | 0.985 |
| | 0.75 | Bloomfield | 1.000 | 1.000 | 1.000 | 0.999 | 0.961 | 0.007 | 0.006 | 0.100 | 0.725 | 0.991 |
| | | AR(1) | 0.538 | 0.778 | 0.941 | 0.925 | **0.080** | **0.041** | 0.257 | 0.380 | 0.423 | 0.418 |
| | 0.95 | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.010 | 0.023 | 0.150 | 0.307 | 0.415 |
| | | AR(1) | 0.748 | 0.069 | 0.020 | 0.366 | **0.029** | **0.117** | 0.865 | 0.999 | 1.000 | 1.000 |
| | 0.25 | Bloomfield | 1.000 | 1.000 | 0.999 | 0.917 | 0.071 | 0.046 | 0.858 | 0.999 | 1.000 | 1.000 |
| | | AR(1) | 0.853 | 0.935 | 0.347 | 0.101 | **0.025** | **0.156** | 0.632 | 0.983 | 1.000 | 1.000 |
| 300 | 0.50 | Bloomfield | 1.000 | 1.000 | 0.999 | 0.992 | 0.442 | 1.000 | 0.296 | 0.978 | 1.000 | 1.000 |
| | | AR(1) | 0.837 | 0.985 | 0.994 | 0.544 | **0.025** | **0.128** | 0.369 | 0.519 | 0.907 | 0.998 |
| | 0.50 | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.995 | 0.013 | 0.041 | 0.112 | 0.880 | 0.999 |
| | | AR(1) | 0.605 | 0.856 | 0.980 | 0.980 | **0.068** | **0.043** | 0.691 | 0.246 | 0.313 | 0.409 |
| | 0.95 | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.019 | 0.056 | 0.113 | 0.303 | 0.405 |

Vetterling (1986), with 10,000 replications of each case. The sample sizes are $T = 100$, 200 and 300 observations and the nominal size is in all cases 5%.[3]

Table 1 imposes $d = 1$ and AR(1) $u_t$ with the AR coefficient $\phi = 0.25$, 0.50, 0.75 and 0.95 (negative values were also considered and the results were in line with those presented here). The alternatives are in all cases fractional, testing $H_0$ (1.2) in (1.1) with $d_0 = 0$, (0.25), 2, and supposing that the disturbances are AR(1) and Bloomfield (1). The rejection frequencies correspond to the one-sided statistic given by $\hat{r}$ in (2.1). Thus, the rejection probabilities associated to $d = 1$ will indicate the size of the tests.

The first thing we observe in this table is that the sizes of the tests are too small when directed against $d > d_0$ but too large against $d < d_0$ when $\phi = 0.25$, 0.50 and 0.75. However, if $\phi = 0.95$, they behave in the opposite way, with larger sizes when directed against $d > d_0$ (e.g., 10.1% against $d > d_0$ and 3.7% against $d < d_0$ with $T = 100$ for a nominal size of 5%). Using the Bloomfield (1) model instead of the AR(1) disturbances, the sizes improve and they approximate to the nominal value when $\phi = 0.25$. However, increasing the value of the AR coefficient, the rejection frequencies substantially increase

[3]The FORTRAN code used to obtain the test statistic is available from the author upon request.

Table 2. Rejection frequencies of Robinson's (1994) tests with AR(1) disturbances. In bold: the sizes of the tests. The nominal size is 0.050.

| True model: $(1-L)^d x_t = u_t;\ u_t = \phi u_{t-1} + \varepsilon_t;\ d = 0.50$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Alternatives: $(1-L)^d x_t = u_t;\ u_t \sim$ AR(1) and Bloomfield (1) | | | | | | | | | | | | |
| | | | $H_a : d > d_0$ | | | | | $H_a : d < d_0$ | | | | |
| Size | $\phi$ | $u_t/d$ | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 |
| 100 | 0.25 | AR(1) | 0.015 | 0.038 | 0.056 | 0.040 | **0.017** | **0.221** | 0.319 | 0.470 | 0.649 | 0.802 |
| | | Bloomfield | 0.719 | 0.523 | 0.310 | 0.130 | 0.038 | 0.104 | 0.233 | 0.421 | 0.637 | 0.808 |
| | 0.50 | AR(1) | 0.104 | 0.041 | 0.014 | 0.013 | **0.011** | 0.329 | 0.364 | 0.410 | 0.492 | 0.601 |
| | | Bloomfield | 0.875 | 0.751 | 0.566 | 0.350 | 0.161 | 0.018 | 0.054 | 0.138 | 0.286 | 0.479 |
| | 0.75 | AR(1) | 0.694 | 0.432 | 0.199 | 0.070 | **0.019** | **0.189** | 0.324 | 0.400 | 0.389 | 0.361 |
| | | Bloomfield | 0.989 | 0.973 | 0.939 | 0.863 | 0.733 | 0.003 | 0.014 | 0.104 | 0.213 | 0.037 |
| | 0.95 | AR(1) | 0.833 | 0.804 | 0.657 | 0.356 | **0.101** | **0.037** | 0.157 | 0.341 | 0.454 | 0.368 |
| | | Bloomfield | 0.999 | 0.999 | 0.999 | 0.997 | 0.993 | 0.000 | 0.010 | 0.109 | 0.298 | 0.304 |
| 200 | 0.25 | AR(1) | 0.063 | 0.178 | 0.190 | 0.100 | **0.025** | **0.148** | 0.323 | 0.603 | 0.850 | 0.968 |
| | | Bloomfield | 0.996 | 0.863 | 0.624 | 0.279 | 0.056 | 0.063 | 0.240 | 0.560 | 0.848 | 0.971 |
| | 0.50 | AR(1) | 0.140 | 0.068 | 0.046 | 0.040 | **0.020** | **0.204** | 0.294 | 0.432 | 0.614 | 0.797 |
| | | Bloomfield | 0.995 | 0.973 | 0.883 | 0.662 | 0.317 | 0.003 | 0.027 | 0.126 | 0.374 | 0.694 |
| | 0.75 | AR(1) | 0.869 | 0.602 | 0.278 | 0.088 | **0.021** | **0.145** | 0.284 | 0.355 | 0.353 | 0.356 |
| | | Bloomfield | 0.999 | 0.999 | 0.999 | 0.993 | 0.961 | 0.007 | 0.020 | 0.209 | 0.320 | 0.021 |
| | 0.95 | AR(1) | 0.959 | 0.955 | 0.847 | 0.443 | **0.080** | **0.041** | 0.236 | 0.516 | 0.560 | 0.325 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.004 | 0.043 | 0.147 | 0.338 | 0.347 |
| 300 | 0.25 | AR(1) | 0.130 | 0.344 | 0.323 | 0.154 | **0.029** | **0.118** | 0.366 | 0.724 | 0.946 | 0.994 |
| | | Bloomfield | 0.997 | 0.969 | 0.811 | 0.405 | 0.071 | 0.047 | 0.278 | 0.696 | 0.949 | 0.996 |
| | 0.50 | AR(1) | 0.181 | 0.112 | 0.094 | 0.067 | **0.024** | **0.156** | 0.287 | 0.509 | 0.745 | 0.916 |
| | | Bloomfield | 0.999 | 0.998 | 0.976 | 0.837 | 0.442 | 0.001 | 0.016 | 0.142 | 0.494 | 0.849 |
| | 0.75 | AR(1) | 0.947 | 0.725 | 0.359 | 0.114 | **0.024** | **0.128** | 0.278 | 0.360 | 0.371 | 0.416 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 0.999 | 0.995 | 0.010 | 0.031 | 0.309 | 0.341 | 0.012 |
| | 0.95 | AR(1) | 0.991 | 0.990 | 0.935 | 0.519 | **0.068** | **0.043** | 0.331 | 0.649 | 0.645 | 0.303 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.012 | 0.077 | 0.302 | 0.389 | 0.334 |

when directed against $d > d_0$, and strongly reduce against $d < d_0$. Thus, for example, if $\phi = 0.95$ the test is clearly oversized when the alternative is of form $d < 1$ and undersized for $d > 1$. This bias in the size may be a consequence of the different models considered under the null and the alternative hypotheses.

If we concentrate on the rejection frequencies when $d \neq 1$ we see that they are always higher when we employ the Bloomfield model if $d < 1$. Of particular interest are the results when $\phi = 0.95$. In this case, if we test $H_0 : d = 0$ with AR(1) $u_t$, the rejection frequencies are 0.442; 0.538 and 0.605 with $T = 100$, 200 and 300 respectively. Thus, even if the true process contains a unit root (i.e., $d = 1$), testing the null of $d = 0$ with AR disturbances results in a significant probability of non-rejection of the null hypothesis of stationarity $I(0)$. Using, however, the Bloomfield model, these rejection frequencies are 1 in all cases. The relative low values obtained in the AR case may be due in large part to the fact that the AR parameters are estimated by Yule-Walker, implying roots that are automatically less than one in absolute value but that can be arbitrarily close to one. Thus, the AR estimates may be competing with the order of integration in describing the unit root component of the series. If $d > 1$, the rejection frequencies are higher with AR(1)

TABLE 3. Rejection frequencies of Robinson's (1994) tests with AR(2) disturbances. In bold: the sizes of the tests. The nominal size is 0.050.

| True model: $(1-L)^d x_t = u_t;\ u_t = \phi_1 u_{t-1} + \phi_2 u_{t-2} + \varepsilon_t;\ d = 1$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Alternatives: $(1-L)^d x_t = u_t;\ u_t \sim$ AR(2) and Bloomfield (2) | | | | | | | | | | | | |
| | | | $H_a : d > d_0$ | | | | | $H_a : d < d_0$ | | | | |
| Size | $\phi_1, \phi_2$ | $u_t/d$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 |
| 100 | 0.25, 0.25 | AR(1) | 0.677 | 0.190 | 0.099 | 0.065 | **0.018** | **0.137** | 0.567 | 0.809 | 0.905 | 0.988 |
| | | Bloomfield | 0.998 | 0.952 | 0.743 | 0.466 | 0.042 | 0.112 | 0.556 | 0.911 | 1.000 | 1.000 |
| | 0.25, 0.50 | AR(1) | 0.800 | 0.689 | 0.258 | 0.076 | **0.015** | **0.119** | 0.452 | 0.675 | 0.909 | 0.982 |
| | | Bloomfield | 0.997 | 0.991 | 0.879 | 0.764 | 0.131 | 0.011 | 0.233 | 0.681 | 0.971 | 0.991 |
| | 0.50, 0.25 | AR(1) | 0.921 | 0.889 | 0.850 | 0.311 | **0.029** | **0.119** | 0.405 | 0.362 | 0.555 | 0.847 |
| | | Bloomfield | 0.989 | 0.984 | 0.982 | 0.961 | 0.718 | 0.045 | 0.048 | 0.132 | 0.338 | 0.973 |
| | 0.50, 0.45 | AR(1) | 0.62 | 0.832 | 0.866 | 0.761 | **0.114** | **0.039** | 0.414 | 0.186 | 0.106 | 0.365 |
| | | Bloomfield | 1.000 | 1.000 | 0.995 | 0.991 | 0.953 | 0.010 | 0.012 | 0.032 | 0.065 | 0.345 |
| 200 | 0.25, 0.25 | AR(1) | 0.682 | 0.193 | 0.123 | 0.266 | **0.029** | **0.132** | 0.738 | 0.995 | 1.000 | 1.000 |
| | | Bloomfield | 1.000 | 0.997 | 0.991 | 0.760 | 0.049 | 0.068 | 0.822 | 0.991 | 1.000 | 1.000 |
| | 0.25, 0.50 | AR(1) | 0.903 | 0.890 | 0.467 | 0.087 | **0.032** | **0.212** | 0.621 | 0.903 | 0.999 | 1.000 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 0.971 | 0.201 | 0.013 | 0.312 | 0.910 | 0.999 | 0.999 |
| | 0.50, 0.25 | AR(1) | 0.942 | 0.924 | 0.971 | 0.354 | **0.030** | **0.123** | 0.332 | 0.411 | 0.735 | 0.935 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 0.991 | 0.911 | 0.003 | 0.013 | 0.233 | 0.785 | 0.993 |
| | 0.50, 0.45 | AR(1) | 0.738 | 0.801 | 0.941 | 0.913 | **0.076** | **0.044** | 0.066 | 0.289 | 0.383 | 0.413 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.997 | 0.009 | 0.017 | 0.107 | 0.282 | 0.417 |
| 300 | 0.25, 0.25 | AR(1) | 0.456 | 0.201 | 0.085 | 0.334 | **0.037** | **0.129** | 0.899 | 0.999 | 1.000 | 1.000 |
| | | Bloomfield | 1.000 | 1.000 | 0.999 | 0.954 | 0.077 | 0.049 | 0.867 | 0.999 | 1.000 | 1.000 |
| | 0.25, 0.50 | AR(1) | 0.953 | 0.921 | 0.754 | 0.202 | **0.035** | **0.165** | 0.576 | 0.993 | 1.000 | 1.000 |
| | | Bloomfield | 1.000 | 1.000 | 0.948 | 0.923 | 0.146 | 0.043 | 0.287 | 0.998 | 0.997 | 1.000 |
| | 0.50, 0.25 | AR(1) | 0.939 | 0.935 | 0.896 | 0.543 | **0.035** | **0.123** | 0.355 | 0.519 | 0.909 | 0.998 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 0.994 | 0.895 | 0.006 | 0.009 | 0.119 | 0.890 | 0.999 |
| | 0.50, 0.45 | AR(1) | 0.708 | 0.697 | 0.943 | 0.807 | **0.066** | **0.046** | 0.694 | 0.434 | 0.416 | 0.597 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.015 | 0.105 | 0.210 | 0.413 | 0.411 |

disturbances if $d = 1.25$ or 1.50, being similar to the Bloomfield model when $d \geq 1.50$. As a conclusion, we can summarize the results in this table by saying that the Bloomfield exponential spectral model can be used as an approximation to the AR(1) $u_t$ when testing the null hypothesis of a unit root in the presence of AR(1) disturbances. The rejection frequencies are higher in all cases when the alternatives are of form $d < 1$, and even if $d > 1$, they are competitive to the AR(1) model for values of d relatively far away from 1.

Table 2 extends the results of Table 1 to the case of fractionally integrated processes. Thus, the true model is now given by an $I(0.5)$ process, with the alternatives testing $H_0$ (1.2) with $d_0$-values = 0.10, (0.10), 0.90. Other fractional models were also considered and the results were completely in line with those presented in Table 2. The same conclusions as in Table 1 are obtained here. Thus, starting with the size, we see that if the AR coefficient is low ($\phi = 0.25$), the sizes of the tests improve when using the Bloomfield approximation, however, as we increase the value of the AR coefficient, the distortions also increase. If the alternatives are of form $d < 0.5$, the rejection frequencies are in all cases higher with Bloomfield disturbances, being particularly remarkable this improvement if

TABLE 4. Rejection frequencies of Robinson's (1994) tests with AR(1) disturbances. In bold: the sizes of the tests. The nominal size is 0.050.

| True model: $(1-L)^d x_t = u_t;\ u_t = \phi u_{t-1} + \varepsilon_t;\ d = 1$ ($t_3$-distribution) | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Alternatives: $(1-L)^d x_t = u_t;\ u_t \sim$ AR(1) and Bloomfield (1) | | | | | | | | | | | | |
| | | | $H_a : d > d_0$ | | | | | $H_a : d < d_0$ | | | | |
| Size | $\phi$ | $u_t/d$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 |
| 100 | 0.25 | AR(1) | 0.000 | 0.005 | 0.017 | 0.033 | **0.015** | **0.219** | 0.365 | 0.726 | 0.959 | 0.996 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 0.894 | 0.028 | 0.094 | 0.309 | 0.707 | 0.970 | 1.000 |
| | 0.50 | AR(1) | 0.000 | 0.002 | 0.010 | 0.020 | **0.009** | **0.232** | 0.296 | 0.437 | 0.800 | 0.965 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 0.959 | 0.089 | 0.022 | 0.088 | 0.354 | 0.822 | 0.986 |
| | 0.75 | AR(1) | 0.000 | 0.000 | 0.009 | 0.011 | **0.007** | **0.295** | 0.351 | 0.163 | 0.239 | 0.636 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 0.998 | 0169 | 0.000 | 0.001 | 0.108 | 0.245 | 0.650 |
| | 0.95 | AR(1) | 0.000 | 0.000 | 0.003 | 0.006 | **0.000** | **0.097** | 0.754 | 0.291 | 0.015 | 0.018 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.203 | 0.000 | 0.100 | 0.202 | 0.204 | 0.307 |
| 200 | 0.25 | AR(1) | 0.000 | 0.009 | 0.019 | 0.088 | **0.019** | **0.153** | 0.418 | 0.945 | 0.996 | 1.000 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.058 | 0.046 | 0.392 | 0.947 | 0.999 | 1.000 |
| | 0.50 | AR(1) | 0.000 | 0.005 | 0.014 | 0.031 | **0.013** | **0.173** | 0.232 | 0.626 | 0.973 | 0.997 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.117 | 0.003 | 0.064 | 0.583 | 0.983 | 0.999 |
| | 0.75 | AR(1) | 0.000 | 0.002 | 0.011 | 0.014 | **0.009** | **0.223** | 0.191 | 0.090 | 0.394 | 0.933 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.273 | 0.000 | 0.009 | 0.204 | 0.309 | 0.947 |
| | 0.95 | AR(1) | 0.000 | 0.000 | 0.001 | 0.003 | **0.000** | **0.088** | 0.812 | 0.177 | 0.000 | 0.000 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.313 | 0.000 | 0.203 | 0.306 | 0.201 | 0.105 |
| 300 | 0.25 | AR(1) | 0.000 | 0.022 | 0.088 | 0.109 | **0.024** | **0.113** | 0.513 | 0.989 | 1.000 | 1.000 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.073 | 0.034 | 0.485 | 0.989 | 1.000 | 1.000 |
| | 0.50 | AR(1) | 0.000 | 0.004 | 0.021 | 0.044 | **0.009** | **0.113** | 0.205 | 0.836 | 0.996 | 1.000 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.252 | 0.000 | 0.039 | 0.804 | 0.999 | 1.000 |
| | 0.75 | AR(1) | 0.000 | 0.000 | 0.004 | 0.010 | **0.008** | **0.128** | 0.098 | 0.052 | 0.603 | 0.987 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.395 | 0.000 | 0.039 | 0.300 | 0.517 | 0.993 |
| | 0.95 | AR(1) | 0.000 | 0.001 | 0.003 | 0.004 | **0.000** | **0.079** | 0.905 | 0.086 | 0.000 | 0.007 |
| | | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.503 | 0.009 | 0.229 | 0.300 | 0.209 | 0.105 |

$\phi = 0.25$ or 0.50 and we test lower orders of integration. If $d > 0.5$, the AR(1) model behaves better when $d$ is relatively close to the true value, the rejection probabilities being similar if $d \geq 0.70$ with low values of the AR coefficient.

Table 3 extends the analysis to the case of AR(2) disturbances. The true model is now given by:

$$(1-L)^d x_t = u_t;\quad u_t = \phi_1 u_{t-1} + \phi_2 u_{t-2} + \varepsilon_t,$$

with $d = 1$ (a unit root) and $(\phi_1, \phi_2) = (0.25, 0.25)$; (0.25, 0.50); (0.50, 0.25) and (0.50, 0.45). The alternatives are again fractional with $d_0 = 0$, (0.25), 2, and both AR(2) and Bloomfield (2) $u_t$. Similarly to Tables 1 and 2, if $(\phi_1, \phi_2) = (0.25, 0.25)$; (0.25, 0.50); (0.50, 0.25), the sizes of the tests are too small against $d > d_0$ but too large against $d < d_0$. However, as we approximate to the nonstationary case, (i.e., $(\phi_1, \phi_2) = (0.50, 0.45)$), this behaviour reverses. Once more, if $d < 1$, higher rejection frequencies are obtained with Bloomfield disturbances, and testing with $d > 1$, the rejection probabilities are competitive in both cases for values of $d \geq 1.50$.

TABLE 5. Rejection frequencies of Robinson's (1994) tests with AR(1) disturbances. In bold: the sizes of the tests. The nominal size is 0.050.

| True model: $(1-L)^d x_t = u_t;\ u_t = \phi u_{t-1} + \varepsilon_t;\ d = 0.50$ | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Alternatives: $(1-L)^d x_t = u_t;\ u_t \sim$ AR(1) and Bloomfield (1) | | | | | | | | | | | | |
| | | | $H_a : d > d_0$ | | | | | $H_a : d < d_0$ | | | | |
| Size | $\phi$ | $u_t/d$ | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.50 | 0.60 | 0.70 | 0.80 | 0.90 |
| | | AR(1) | 0.000 | 0.011 | 0.027 | 0.020 | **0.062** | **0.012** | 0.219 | 0.253 | 0.293 | 0.413 |
| | 0.25 | Bloomfield | 1.000 | 1.000 | 1.000 | 0.976 | 0.027 | 0.083 | 0.218 | 0.554 | 0.933 | 0.990 |
| | | AR(1) | 0.001 | 0.009 | 0.039 | 0.016 | **0.010** | 0.467 | 0.339 | 0.246 | 0.218 | 0.232 |
| 100 | 0.50 | Bloomfield | 1.000 | 1.000 | 1.000 | 0.993 | 0.091 | 0.019 | 0.039 | 0.193 | 0.700 | 0.966 |
| | | AR(1) | 0.000 | 0.000 | 0.021 | 0.011 | **0.005** | **0.621** | 0.503 | 0.367 | 0.259 | 0.201 |
| | 0.75 | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.253 | 0.015 | 0.041 | 0.101 | 0.251 | 0.496 |
| | | AR(1) | 0.000 | 0.000 | 0.018 | 0.008 | **0.003** | **0.048** | 0.315 | 0.693 | 0.801 | 0.630 |
| | 0.95 | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.341 | 0.011 | 0.086 | 0.194 | 0.206 | 0.301 |
| | | AR(1) | 0.000 | 0.027 | 0.101 | 0.196 | **0.027** | **0.136** | 0.188 | 0.209 | 0.326 | 0.620 |
| | 0.25 | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.061 | 0.052 | 0.219 | 0.866 | 0.999 | 1.000 |
| | | AR(1) | 0.000 | 0.011 | 0.094 | 0.023 | **0.019** | **0.289** | 0.198 | 0.122 | 0.111 | 0.189 |
| 200 | 0.50 | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.207 | 0.013 | 0.097 | 0.349 | 0.964 | 1.000 |
| | | AR(1) | 0.000 | 0.001 | 0.071 | 0.032 | **0.003** | **0.554** | 0.363 | 0.184 | 0.083 | 0.033 |
| | 0.75 | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.371 | 0.014 | 0.102 | 0.203 | 0.318 | 0.297 |
| | | AR(1) | 0.000 | 0.001 | 0.019 | 0.020 | **0.002** | **0.184** | 0.585 | 0.862 | 0.884 | 0.745 |
| | 0.95 | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.449 | 0.009 | 0.200 | 0.407 | 0.331 | 0.200 |
| | | AR(1) | 0.000 | 0.076 | 0.213 | 0.553 | **0.018** | **0.116** | 0.187 | 0.217 | 0.442 | 0.810 |
| | 0.25 | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.056 | 0.038 | 0.283 | 0.997 | 0.999 | 1.000 |
| | | AR(1) | 0.000 | 0.031 | 0.102 | 0.179 | **0.017** | **0.225** | 0.137 | 0.079 | 0.093 | 0.259 |
| 300 | 0.50 | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.253 | 0.000 | 0.097 | 0.589 | 0.996 | 1.000 |
| | | AR(1) | 0.000 | 0.000 | 0.077 | 0.034 | **0.005** | 0.449 | 0.285 | 0.086 | 0.025 | 0.012 |
| | 0.75 | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.394 | 0.010 | 0.088 | 0.100 | 0.295 | 0.990 |
| | | AR(1) | 0.000 | 0.000 | 0.005 | 0.013 | **0.002** | **0.264** | 0.719 | 0.922 | 0.946 | 0.797 |
| | 0.95 | Bloomfield | 1.000 | 1.000 | 1.000 | 1.000 | 0.457 | 0.007 | 0.197 | 0.251 | 0.207 | 0.109 |

Though not reported in the paper, we also examined the case of misspecification in the order $p$ for the AR and the Bloomfield disturbances. As expected, the rejection probabilities were relatively high in all cases, being close to 1 if $T > 100$. Other versions of LM tests for fractional integration have been recently proposed in the literature (e.g. Tanaka, 1999; Breitung and Hassler, 2002; Dolado et al., 2002), however, they are not directly comparable to Robinson (1994) in the context of Bloomfield disturbances, since they are specified in the time domain, while Bloomfield is a frequency domain non-parametric approach.

Finally, since the tests of Robinson (1994) are supposed to be robust to non-Gaussian disturbances, we also examined in this section the case of departures from Gaussianity. Tables 4 and 5 report the rejection frequencies for the cases of $t_3$ and $t_5$ disturbances respectively. In Table 4 we suppose that the true model is given by an $I(1)$ process with AR(1) $u_t$ and perform the two test statistics (AR and Bloomfield) in a similar way as in Table 1, i.e., testing $H_0$ (1.2), with $d_0 = 0, 0.25, \ldots, 1.75$ and 2. The first thing we observe in this table is that the model based on the Bloomfield approach performs much better in all cases in terms of both the size and the power properties, especially if the AR

Table 6. Testing the order of integration in the Spanish real GDP (' and in bold: Non-rejection values of the null hypothesis at the 5% significance level).

| $u_t/d$ | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 |
|---|---|---|---|---|---|---|---|---|---|
| White noise | 22.42 | 22.18 | 21.58 | 19.16 | 10.62 | 2.51 | **-1.17'** | -2.83 | -3.74 |
| AR(1) | -0.40 | 0.05 | 1.63 | 0.18 | -0.05 | 0.31 | -0.49 | -1.33 | -2.05 |
| AR(2) | -23.81 | -9.79 | 2.13 | 1.58 | -0.67 | 0.61 | 0.33 | -0.09 | -0.56 |
| Bloomfield(1) | 11.83 | 10.78 | 10.73 | 8.46 | 4.29 | **0.59'** | **-1.32'** | -2.23 | -2.75 |
| Bloomfield(2) | 12.40 | 11.48 | 11.05 | 6.78 | 2.84 | **0.04'** | **-1.53'** | -2.31 | -2.49 |

parameter is not large. Thus, for example, if we direct the tests against alternatives of form: $H_0 : d > d_0$, the rejection frequencies for $d_0 = 0$ are zero for the AR model and 1 with the Bloomfield approach. Very similar results are obtained in Table 5, where a $t_5$-distribution is considered for an I(0.5) process, and the highest improvement is obtained for departures far below the true value of $d$ ($d$ = 0.10 or 0.20).

## 4. An empirical application

The time series data analysed in this section correspond to the Spanish real GDP in 1990 prices, annually from 1900 to 1999, obtained from the International Monetary Fund's (IMF) database. Denoting the time series by $x_t$, we employ throughout model (1.1), testing $H_0$ (1.2) with $d_0 = 0$, (0.25) and 2, and white noise, AR and Bloomfield (1973) disturbances.[4]

The test statistic reported in Table 6 is the one-sided one given by $\hat{r}$ in (2.1), so that significantly positive values of this are consistent with higher orders of integration ($d > d_0$), whereas significantly negative ones are consistent with smaller values of $d$ ($d < d_0$). Starting with the case of white noise $u_t$, we observe that the values of $\hat{r}$ monotonically decrease with $d_0$. This is something to be expected given that it is a one-sided test statistic. Thus, for example, we would wish that if $d = 0.75$ is rejected against $d > 0.75$, an even more significant result in this direction would be obtained when $d = 0.50$ or 0.25 are tested. We also observe in this case that the unit root null hypothesis (i.e., $d = 1$) is strongly rejected in favour of more nonstationary alternatives ($d > 1$), and the only non-rejection value takes place when $d = 1.50$. The following two rows in Table 6 report the results with AR(1) and AR(2) disturbances respectively. Here we observe a lack of monotonic decrease in the value of the test statistic with respect to $d_0$, for small values of $d_0$. This lack of monotonicity could be explained in terms of model misspecification as is argued, for example, in Gil-Alana and Robinson (1997). However, it may also be due to the lack of power of the tests of Robinson (1994) in this context of AR disturbances, especially if the AR parameters are close to the unit root. Thus, in the last two rows of Table 6, we report values of $\hat{r}$ using the Bloomfield (1973) exponential spectral model

[4]The tests of Robinson (1994) permit us to include deterministic components with no effect on its standard null limit distribution. We try with an intercept and with an intercept and a linear trend, and the results did not substantially change from those reported in the paper.

for the disturbances described in Section 2. We see that monotonicity is now always achieved and the non-rejection values correspond to $d = 1.25$ and 1.50. Note that these values were non-rejected when AR disturbances were entertained, suggesting that the Bloomfield (1973) model can be taken as a credible alternative to the AR specification for the $I(0)$ disturbances, especially in those cases where the roots of the AR polynomial are close to the unit circle.

## 5. Concluding remarks

We have shown in this article that the Bloomfield (1973) exponential spectral model can be a credible and useful alternative when testing I(d) statistical models with weakly autocorrelated (AR) disturbances. It is well known that unit-root tests (with integer or fractional orders of integration) have in general low power in the context of autocorrelated disturbances (Diebold and Rudebusch, 1989; Hassler and Wolters, 1994). Several experiments conducted via Monte Carlo showed that the Bloomfield (1973) model approximates fairly well autoregressive models in the context of unit roots with fractional orders of integration. We have shown that this approximation is particularly relevant in the cases where the $I(0)$ disturbances associated to the fractional model are close to the unit root case. An empirical application carried out in Section 4 showed that the tests of Robinson (1994) can have problems when looking at $I(d)$ processes in the presence of AR disturbances. In that respect, the Bloomfield (1973) exponential spectral model can be adopted as a credible alternative when fractional models are combined with autoregressions. We used data of the Spanish real GDP to illustrate this point. Thus, testing the degree of integration of the series with the tests of Robinson (1994), it was observed a lack of monotonicity in the value of the one-sided statistic with respect to d if the disturbances were autoregressive. However, using the Bloomfield's (1973) model, monotonicity was achieved in all cases and the null could not be rejected for $d = 1.25$ and 1.50, implying thus nonstationarity and lack of mean reversion in its behaviour. This final result, however, should be taken with caution. Spanish data from 1900 to 1999 cover at least three different political and economic regimes, and regime shifts with structural breaks may cause artificial long memory (see, Diebold and Inoue, 2001, and Gourieroux and Jasiak, 2001). The tests of Robinson (1994) presented in this paper permit us to incorporate dummy variables to take into account the breaks, with no effect on the standard limit behaviour. In that respect, the Monte Carlo results reported here should not either be affected by the inclusion of such deterministic regressors.

Finally, it is important to note that since the tests of Robinson (1994) are based on the spectral density function of $\hat{u}_t = (1 - L)^{d_0} x_t$, or on the truncated Fourier series (if the Bloomfield model is used), then we may expect that the ill-posed estimation problem discussed in Pötscher (2002) applies. Note that Pötscher (2002) (Theorem 5.1., page 1054) makes clear that the estimation of the long memory parameter d is an ill-posed problem. Thus, in order to overcome such a problem, quite restrictive assumptions on the set of spectral densities are needed (see, e.g., Giraitis, Robinson and Samarov, 1997, 2000). However, as mentioned in other parts of the paper, the tests of Robinson (1994) employed here do not require estimation of the long memory parameter as is the case in other procedures (Sowell, 1992; Beran, 1995), and simply computes diagnostics for departures of real values of d from the null. Moreover, the goal in Robinson's (1994) procedure is not the estimation of the short run parameters of the model but testing the order of integration of the series for a sequence of values of d, and "pointwise" consistency should then be the only requirement for the short run coefficients. The Monte Carlo experiments conducted in Section 3 show that the Bloomfield model can be considered as a viable alternative in those cases where the AR structure leads to inconsistencies in the test results.

## Acknowledgements

The author gratefully acknowledges financial support from the Minsterio de Ciencia y Tecnologia (SEJ2005-07657/ECON, Spain). Comments of two anonymous referees are gratefully acknowledged. The usual disclaimers apply.

## References

Adenstedt, R. K. (1974), 'On large sample estimation for the mean of a stationary random sequence', *Annals of Statistics* **2**, 1095-1107.

Beran, J. (1993), 'Fitting long memory models by generalized linear regressions', *Biometrika* **80**, 817-822.

Beran, J. (1995), 'Maximum likelihood estimation of the differencing parameter for invertible short and long memory ARIMA models', *Journal of the Royal Statistical Society B* **57**, 695-672.

Bloomfield, P. (1973), 'An exponential model in the spectrum of a scalar time series', *Biometrika* **60**, 27-226.

Breitung, J. and Hassler, U. (2002), 'Inference on the cointegration rank in fractionally integrated processes', *Journal of Econometrics* **110**, 167-185.

Cioczek-Georges R. and Mandelbrot, B. B. (1995), 'A class of micropulses and anti-persistent fractional Brownian motion', *Stochastic Processes and Their Applications* **60**, 1-18.

Chambers, M. (1998), 'Long memory and aggregation in macroeconomic time series', *Interntional Economic Review* **39**, 1053-1072.

Dickey, D. A. and Fuller, W. A. (1979), 'Distributions of the estimators for autoregressive time series with a unit root', *Journal of the American Statistical Association* **74**, 427-431.

Diebold, F. X. and Inoue, A. (2001), 'Long memory and regime switching', *Journal of Econometrics* **105**, 131-159.

Diebold, F. X. and Rudebusch, G. D. (1989), 'Long memory and persistence in aggregate output', *Journal of Monetary Economics* **24**, 189-209.

Dolado, J. J., Gonzalo, J. and Mayoral, L. (2002), 'A fractional Dickey-Fuller test', *Econometrica* **70**, 1963-2006.

Gil-Alana, L. A. (1999), 'Testing fractional integration with monthly data', *Economic Modelling* **16**, 613-629.

Gil-Alana, L. A. (2000), 'Mean reversion in the real exchange rates', *Economics Letters* **69**, 285-288.

Gil-Alana, L. A. (2001a), 'Testing stochastic cycles in macroeconomic time series', *Journal of Time Series Analysis* **22**, 411-430.

Gil-Alana, L. A. (2001b), 'A fractionally integrated exponential model for UK unemployment', *Journal of Forecasting* **20**, 329-340.

Gil-Alana, L. A. and Robinson, P. M. (1997), 'Testing of unit roots and other nonstationary hypotheses in macroeconomic time series', *Journal of Econometrics* **80**, 241-268.

Gil-Alana, L. A. and Robinson, P. M. (2001), 'Testing seasonal fractional integration in the UK and Japanese consumption and income', *Journal of Applied Econometrics* **16**, 93-114.

Giraitis, L., Robinson, P. M. and Samarov, A. (1997), 'Rate optimal semiparametric estimation of the memory parameter, of the Gaussian time series with long range dependence', STICERD Econometrics Paper Series 323, Suntory and Toyota Internatonal Centers for Economics and Related Disciplines, LSE.

Giraitis, L., Robinson, P. M. and Samarov, A. (2000), 'Adaptive semiparametric estimation of the memory parameter', *Journal of Multivariate Analysis* **72**, 183-207.

Gourieroux C. and Jasiak, J. (2001), 'Memory and infrequent breaks', *Economics Letters* **70**, 29-41.

Granger, C. W. J. (1980), 'Long memory relationships and the aggregation of dynamic models', *Journal of Econometrics* **14**, 227-238.

Granger, C. W. J. (1980), 'Some properties of time series data and their use in econometric model specification', *Journal of Econometrics* **16**, 121-130.

Granger, C. W. J. and Joyeux, R. (1981), 'An introduction to long memory time series and fractional differencing', *Journal of Time Series Analysis* **1**, 15-29.

Hassler, U. and Wolters, J. (1994), 'On the power of unit root tests against fractional alternatives', *Economics Letters* **45**, 1-5.

Hosking, J. R. M. (1981), 'Fractional differencing', *Biometrika* **68**, 165-176.

Lippi, M. and Zaffaroni, P. (1999), 'Contemporaneous aggregation of linear dynamic models in large economies', Manuscript, Research Department, Bank of Italy.

Parke, W. R. (1999), 'What is fractional integration?', *Review of Economics and Statistics* **81**, 632-638.

Phillips, P. C. B. and Magdalinos, T. (2007), 'Limit theory for moderate deviations from a unit root', *Journal of Econometrics* **136**, 115-130.

Phillips, P. C. B. and Perron, P. (1988), 'Testing for unit roots in time series regression', *Biometrika* **75**, 335-346.

Pötscher, B. M. (2002), 'Lower risk bounds and properties of confidence sets for ill-posed estimation problems with applications to spectral density and persistence estimation, unit roots and estimation of long memory parameters', *Econometrica* **70**, 1035-1065.

Press, W. H., Flannery, B. P., Teukolsky, S. A. and Wetterling, W. T. (1986), *Numerical recipes: The art of scientific computing*, Cambridge University Press, Cambridge.

Robinson, P. M. (1978), 'Statistical inference for a random coefficient autoregressive model', *Scandinavian Journal of Statistics* **5**, 163-168.

Robinson, P. M. (1994), 'Efficient tests of nonstationary hypotheses', *Journal of the American Statistical Association* **89**, 1420-1437.

Sowell, F. (1992), 'Maximum likelihood estimation of stationary univariate time series models', *Journal of Econometrics* **53**, 165-188.

Tanaka, K. (1999), 'The nonstationary fractional unit root', *Econometric Theory* **15**, 549-582.

Taqqu, M. S. (1975), 'Weak convergence to fractional Brownian motion and to the Rosenblatt process', *Z. Wahrscheinlichkeitstheorie verw. Geb.* **31**, 287-302.

Taqqu, M. S., Willinger, W. and Sherman, R. (1997), 'Proof of a fundamental result in self-similar traffic modelling', *Computer Communication Review* **27**, 5-23.

Velasco, C. and Robinson, P. M. (2000), 'Whittle pseudo-maximum likelihood estimation of nonstationary time series', *Journal of the American Statistical Association* **95**, 1229-1243.

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