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Noncentral, nonsingular matrix variate beta distribution

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Abstract: In this paper, we determine the density of a nonsingular noncentral matrix variate beta type I and II distributions under different definitions.

Key words: Matrix variate beta, noncentral distribution, random matrices.

1 Introduction

Central and noncentral matrix variate beta type I and II distributions have been studied by different authors utilising diverse approaches, see Olkin and Rubin (1964), Khatri (1970), Muirhead (1982), Cadet (1996), Gupta and Nagar (2000), Díaz-García and Gutiérrez-Jáimez (2001), among many others. These distributions play a very important role in various problems for proving hypotheses in the context of multivariate analysis, including canonical correlation analysis, the general linear hypothesis in MANOVA and the multiple matrix variate correlation analysis, see Muirhead (1982), Rao (1973), Srivastava (1968) and Kshirsagar (1961). Similarly, beta noncentral distributions are to be found in the context of shape theory, see Goodall and Mardia (1993).

In all these applications, the use of beta-type distributions had not been developed as expected and wished for, due particularly to the fact that such distributions depend on hypergeometric functions with a matrix argument or on zonal polynomials, which until very recently were quite complicated to evaluate. The literature has recently included descriptions of algorithms that are very efficient for calculating both zonal polynomials and hypergeometric functions with a matrix argument; these can be used more widely and more efficiently in noncentral distributions in general, see Gutiérrez *et al.* (2000), Sáez (2004), Demmel and Koev (2006), Koev (2004), Koev and Demmel (2006) and Dimitriu *et al.* (2005).

As well as the classification of the beta distribution, as beta type I and type II (see Gupta and Nagar (2000) and Srivastava and Khatri (1979)), two definitions have been proposed for each one of these. Let us focus initially on the beta type I distribution; if A and B have a central Wishart distribution, i.e. $A \sim \mathcal{W}_m(r, I)$ and $B \sim \mathcal{W}_m(s, I)$ independently, then the beta matrix U can be defined as

$$U = \begin{cases} (A+B)^{-1/2}A((A+B)^{-1/2})', & \text{Definition 1 or,} \\ A^{1/2}(A+B)^{-1}(A^{1/2})', & \text{Definition 2,} \end{cases} \tag{1.1}$$

where $C^{1/2}(C^{1/2})' = C$ is a reasonable nonsingular factorization of C, see Gupta and Nagar (2000), Srivastava and Khatri (1979) and Muirhead (1982). Is easy to see that under definition 1 and 2 its density function is given by

$$f_U(U) = \frac{1}{\beta_m[r/2, s/2]}|U|^{(r-m-1)/2}|I_m - U|^{(s-m-1)/2}, \quad 0 < U < I_m, \tag{1.2}$$

denoting as $U \sim \mathcal{B}I_m(r/2, s/2)$, with $r \geq m$ and $s \geq m$; where $\beta_m[r/2, s/2]$ denotes the multivariate beta function defined by

$$\beta_m[b, a] = \int_{0<S<I_m} |S|^{a-(m+1)/2}|I_m - S|^{b-(m+1)/2}(dS) = \frac{\Gamma_m[a]\Gamma_m[b]}{\Gamma_m[a+b]},$$

where $\Gamma_m[a]$ denotes the multivariate gamma function and is defined as

$$\Gamma_m[a] = \int_{R>0} \operatorname{etr}(-R)|R|^{a-(m+1)/2}(dR),$$

$\operatorname{Re}(a) > (m-1)/2$ and $\operatorname{etr}(\cdot) \equiv \exp(\operatorname{tr}(\cdot))$.

Alternatively, a third version of the beta type I matrix has been proposed, see Srivastava and Khatri (1979, pp. 94-95), Srivastava (1968), Muirhead (1982, pp. 451-452) and Gupta and Nagar (2000). We assumed above that $B \sim \mathcal{W}_m(s, I)$ and we wrote $Y \sim \mathcal{N}_{r\times m}(0, I_r \otimes I_m)$, $m > r$, independently of B. Then $U = Y(Y'Y + B)^{-1}Y' = Y(A + B)^{-1}Y'$, and moreover $U \sim \mathcal{B}I_r(m/2, (s + r - m)/2)$. However, note that in the central and non-central cases, the density, properties and associated distributions can be obtained from the definitions in (1) by replacing m by r, r by m and s by $s + r - m$, i.e., by making the substitutions

$$m \to r, \quad r \to m, \quad s \to s + r - m, \tag{1.3}$$

see Srivastava and Khatri (1979, p. 96) or Muirhead (1982, eq. (7), p. 455). For this reason, we focus on the definitions given in (1.1). On extending these definitions to the noncentral case, i.e. when B has a noncentral Wishart distribution, $B \sim \mathcal{W}_m(s, I, \Omega)$, a further classification is given in the literature, in which the beta matrix is defined as follows, see Greenacre (1973) and Gupta and Nagar (2000):

$$U = \begin{cases} (A+B)^{-1/2}A((A+B)^{-1/2})', & \text{denoting as} \quad \mathcal{B}I_1(A)_m(r/2, s/2, \Omega) \\ (A+B)^{-1/2}B((A+B)^{-1/2})', & \text{denoting as} \quad \mathcal{B}I_1(B)_m(s/2, r/2, \Omega) \end{cases} \tag{1.4}$$

under Definition 1; or

$$U = \begin{cases} A^{1/2}(A+B)^{-1}(A^{1/2})', & \text{denoting as} \quad \mathcal{B}I_2(A)_m(r/2, s/2, \Omega) \\ B^{1/2}(A+B)^{-1}(B^{1/2})', & \text{denoting as} \quad \mathcal{B}I_2(B)_m(s/2, r/2, \Omega) \end{cases} \tag{1.5}$$

under Definition 2. Both distributions, types A and B, play a fundamental role in various areas of statistics, for example in the W and U criteria proposed by Wilks (1932).

The density $\mathcal{B}I_1(A)_m(\cdot,\cdot,\cdot)$, when the range of Ω is one, the linear case, has been obtained by Kshirsagar (1961). In the general case, the distributions $\mathcal{B}I_1(A)_m(\cdot,\cdot,\cdot)$ and $\mathcal{B}I_1(B)_m(\cdot,\cdot,\cdot)$ are found by Gupta and Nagar (2000, pp. 188-189)[1], but both expressions depend on an integral of the following type (see also Greenacre (1973) or Roux (1975))

$$\int_{C>0} |C|^{a+b-(m+1)/2} \operatorname{etr}\left(-\tfrac{1}{2}R^{-1}C\right) {}_0F_1\left(b; \tfrac{1}{4}SC^{1/2}X(C^{1/2})'\right)(dC), \tag{1.6}$$

where $0 < X < I_m$ and ${}_aF_b$ is the matrix argument hypergeometric function, see Muirhead (1982, p. 258). The problem of evaluating this integral was proposed earlier by Constantine (1963), Khatri (1970) and reconsidered in Farrell (1985, p. 191).

This problem of finding a closed form for the beta distributions was addressed by Greenacre (1973), who proposed the symmetrised multivariate density of a positive definite matrix, defined as

$$f_S(W) = \int_{\mathcal{O}(m)} f(HWH')(dH), \tag{1.7}$$

where $W: m\times m > 0$ has the density function $f(W)$, $\mathcal{O}(m) = \{H \in \Re^{m\times m} | HH' = H'H = I_m\}$ and (dH) denotes the normalised invariant measure on $\mathcal{O}(m)$ (Muirhead, 1982, p. 72), obtaining the symmetrised density of $\mathcal{B}I_1(A)_m(\cdot,\cdot,\cdot)$ and $\mathcal{B}II_2(B)_m(\cdot,\cdot,\cdot)$, see also Roux (1975).

Under Definition 2, only the distribution $\mathcal{B}I_2(A)_m(\cdot,\cdot,\cdot)$ presents the same problem, i.e. its density depends on an integral of the type (1.6). On the other hand, Díaz-García and Gutiérrez-Jáimez (2001) found an explicit expression for the density of the distribution $\mathcal{B}I_2(B)_m(\cdot,\cdot,\cdot)$ and applied it to calculating the expected value of a zonal polynomial, see also Srivastava (1968). This same distribution was given as an extension of the univariate beta density by Asoo (1969) (cited by Gupta and Nagar (2000)) and proposed as a definition of the noncentral matrix variate beta type I density, see Gupta and Nagar (2000, Definition 5.5.1, p. 190).

A similar situation arises with the beta type II distribution, with which we have the following three definitions:

$$V = \begin{cases} B^{-1/2}A(B^{-1/2})', & \text{Definition 1,} \\ A^{1/2}B^{-1}(A^{1/2})', & \text{Definition 2,} \\ Y^{1/2}B^{-1}Y', & \text{Definition 3.} \end{cases} \tag{1.8}$$

with the distribution being denoted as $V \sim \mathcal{B}II_m(r/2, s/2)$. Similarly to the case of the beta type I distribution, the results under Definition 3 can be found from the results obtained with Definition 2, applying the transforms (1.3), see James (1964) and Muirhead (1982, pp. 451-455).

[1] In both final expressions there is a small error, that is: in the second argument of the hypergeometric function ${}_0F_1$ in both densities is necessary to interchange Σ^{-1} and Θ, in their notation.

On extending these definitions to the noncentral case, we obtain definitions that are parallel to those given in (1.4) and (1.5),

$$V = \begin{cases} B^{-1/2}A(B^{-1/2})', & \text{denoting as} \quad \mathcal{B}II_1(A)_m(r/2, s/2, \Omega) \\ A^{-1/2}B(A^{-1/2})', & \text{denoting as} \quad \mathcal{B}II_1(B)_m(s/2, r/2, \Omega) \end{cases} \tag{1.9}$$

under Definition 1; or

$$V = \begin{cases} A^{1/2}B^{-1}(A^{1/2})', & \text{denoting as} \quad \mathcal{B}II_2(A)_m(r/2, s/2, \Omega) \\ B^{1/2}A^{-1}(B^{1/2})', & \text{denoting as} \quad \mathcal{B}II_2(B)_m(s/2, r/2, \Omega) \end{cases} \tag{1.10}$$

under Definition 2.

In this case, the distributions $\mathcal{B}II_1(A)_m(\cdot,\cdot,\cdot)$ and $\mathcal{B}II_2(B)_m(\cdot,\cdot,\cdot)$ have been studied by James (1964), Muirhead (1982, Section 10.4) and Gupta and Nagar (2000, Section 5.5). Once again, Asoo (1969) (cited by Gupta and Nagar (2000)) proposed $\mathcal{B}II_2(B)_m(\cdot,\cdot,\cdot)$ as a definition of the noncentral matrix variate beta type II density, see Gupta and Nagar (2000, Definition 5.5.12, p. 190).

Note that, to a certain extent, the fact that under the type I definitions, both for beta type I and type II, their corresponding densities cannot be found in an explicit form, which is why the type 2 definitions were proposed, thus avoiding the difficulty in evaluating the type of integrals found in (1.6).

In the present paper, we propose a very simple means of evaluating this integral (1.6), see Section 2. In Section 3 we describe all the densities of the type I distributions that are obtained from Definitions (1.4) and (1.5), observing that the corresponding non-central densities coincide under Definitions 1 and 2. These results are presented in Section 4 for the case of the beta type II distribution. Finally, we propose definitions for the beta type I and II distributions under their different definitions.

2 Preliminar results

From Greenacre (1973), denote

$$f(X) = \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-1}XX'\right),$$

from which the symmetrised function $f_s(X)$ is given by, see Muirhead (1982, Theorem 7.3.3, p. 260),

$$\begin{aligned} f_s(X) &= \int_{\mathcal{O}(m)} \operatorname{etr}(-\tfrac{1}{2}\Sigma^{-1}HXH'HX'H)(dH) \\ &= {}_0F_0^{(m)}\left(-\tfrac{1}{2}\Sigma^{-1}, XX'\right), \quad H \in \mathcal{O}(m), \end{aligned}$$

where ${}_aF_b^{(m)}$ is the hypergeometric function with two matrix arguments, see Muirhead (1982, p. 260).

Our approach is also to apply this idea from Greenacre (1973), in an inverse way, i.e. in the knowledge that

$$f_s(X) = {}_0F_0^{(m)}\left(-\tfrac{1}{2}\Sigma^{-1}, XX'\right) = \int_{\mathcal{O}(m)} f(HXH')(dH), \tag{2.1}$$

we wish to identify the function $f(X)$. Of course, this procedure can be applied for any function $f(X)$, and so it is easy to evaluate the integral (1.6) in an explicit way, as shown below:

Theorem 2.1. *Denote the integral (1.6) by $g(X)$. Then*

i) $g_s(X) = \Gamma[(a+b)]|2R|^{(a+b)}{}_1F_1^{(m)}\left(a+b; b; \frac{1}{2}SR, X\right)$,

ii) $g(X) = \Gamma[(a+b)]|2R|^{(a+b)}{}_1F_1\left(a+b; b; \frac{1}{2}SRX\right)$.

Proof.

$$g(X) = \int_{C>0} |C|^{a+b-(m+1)/2} \operatorname{etr}\left(-\tfrac{1}{2}R^{-1}C\right) {}_0F_1\left(b; \tfrac{1}{4}SC^{1/2}X(C^{1/2})'\right)(dC),$$

then the symmetrised function g is given by

$$g_s(X) = \int_{C>0} |C|^{a+b-(m+1)/2} \operatorname{etr}\left(-\tfrac{1}{2}R^{-1}C\right) \int_{\mathcal{O}(m)} {}_0F_1\left(b; \tfrac{1}{4}SC^{1/2}HXH'(C^{1/2})'\right)(dH)(dC),$$

from Muirhead (1982, theorem 7.3.3, p. 260) we have

$$g_s(X) = \int_{C>0} |C|^{a+b-(m+1)/2} \operatorname{etr}\left(-\tfrac{1}{2}R^{-1}C\right) {}_0F_1^{(m)}\left(b; \tfrac{1}{4}SC, X\right)(dC),$$

therefore, from Muirhead (1982, theorem 7.3.4, p. 260)

$$g_s(X) = \Gamma[(a+b)]|2R|^{(a+b)}{}_1F_1^{(m)}\left(a+b; b; \tfrac{1}{2}SR, X\right).$$

Now, by applying the inverse procedure (2.1)

$$\begin{aligned} g_s(X) &= \Gamma[(a+b)]|2R|^{(a+b)}{}_1F_1^{(m)}\left(a+b; b; \tfrac{1}{2}SR, X\right), \\ &= \int_{\mathcal{O}(m)} g(HXH')(dH) \\ &= \Gamma[(a+b)]|2R|^{(a+b)} \int_{\mathcal{O}(m)} {}_1F_1\left(a+b; b; \tfrac{1}{2}SRHXH'\right)(dH), \end{aligned}$$

from which

$$g(X) = \Gamma[(a+b)]|2R|^{(a+b)}{}_1F_1\left(a+b; b; \tfrac{1}{2}SRX\right).$$

□

Henceforth, the density function of X is denoted by $f_X(X)$ and its corresponding symmetrised density function by $f_s(X)$. Moreover, the density function obtained by applying the idea behind Theorem 2.1 will be termed the nonsymmetrised density function in order to distinguish it from the integral form of its density in the corresponding cases. Nevertheless, we should always bear in mind that the fundamental goal of this study is, in fact, to propose the nonsymmetrised density as the real density.

3 Noncentral beta type I distribution

Let us denote the central beta type I density (1.2) as $\mathcal{B}I_m(U; r/2, s/2)$, thus,

Theorem 3.1. *Let* $W \sim \mathcal{B}I_1(A)(r/2, s/2, \Omega)$ *then*

1. *Its density function is*

$$f_W(W) = \frac{\operatorname{etr}(-\frac{1}{2}\Omega)}{2^{m(r+s)/2}\Gamma_m[r/2]\Gamma_m[s/2]}|W|^{(r-m-1)/2}|I-W|^{(s-m-1)/2}$$

$$\times \int_{C>0} |C|^{(r+s-m-1)/2} \operatorname{etr}\left(-\tfrac{1}{2}C\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega C^{1/2}(I-W)(C^{1/2})'\right)(dC).$$

2. *Its symmetrised density function is*

$$f_s(W) = \mathcal{B}I_m(W; r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}\Omega\right) {}_1F_1^{(m)}\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega, (I-W)\right)$$

3. *Its nonsymmetrised density function is*

$$f_W(W) = \mathcal{B}I_m(W; r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}\Omega\right) {}_1F_1\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega(I-W)\right)$$

Proof. (1) is given in Gupta and Nagar (2000, Theorem 5.5.1, p. 188). (2) and (3) follow from (1) by the application of Theorem 2.1. □

Theorem 3.2. *Let* $U \sim \mathcal{B}I_1(B)(s/2, r/2, \Omega)$ *then*

1. *Its density function is*

$$f_U(U) = \frac{\operatorname{etr}(-\frac{1}{2}\Omega)}{2^{m(r+s)/2}\Gamma_m[r/2]\Gamma_m[s/2]}|U|^{(s-m-1)/2}|I-U|^{(r-m-1)/2}$$

$$\times \int_{C>0} |C|^{(r+s-m-1)/2} \operatorname{etr}\left(-\tfrac{1}{2}C\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega C^{1/2}U(C^{1/2})'\right)(dC).$$

2. *Its symmetrised density function is*

$$f_s(U) = \mathcal{B}I_m(U; s/2, r/2) \operatorname{etr}\left(-\tfrac{1}{2}\Omega\right) {}_1F_1^{(m)}\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega, U\right)$$

3. *Its nonsymmetrised density function is*

$$f_U(U) = \mathcal{B}I_m(U; s/2, r/2)\,\mathrm{etr}\left(-\tfrac{1}{2}\Omega\right) {}_1F_1\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega U\right)$$

Proof. (1) is considered in Gupta and Nagar (2000, p. 189) and Roux (1975). (2) is proposed by Greenacre (1973) and Roux (1975) and (3) follows from (1) or (2) follows from (1) or (2) by the application of Theorem 2.1. □

Similarly, from Definition 2 we have:

Theorem 3.3. *Let* $W \sim \mathcal{B}I_2(A)(r/2, s/2, \Omega)$ *then*

1. *Its density function is*

$$f_W(W) = \frac{\mathrm{etr}(-\frac{1}{2}\Omega)}{2^{m(r+s)/2}\Gamma_m[r/2]\Gamma_m[s/2]} |W|^{(r-m-1)/2}|I-W|^{(s-m-1)/2}$$

$$\times \int_{A>0} |A|^{(r+s-m-1)/2}\,\mathrm{etr}\left(-\tfrac{1}{2}AW^{-1}\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega A^{1/2}(I-W)W^{-1}(A^{1/2})'\right)(dA).$$

2. *Its symmetrised density function is the same as in Theorem 3.1(2).*

3. *Its nonsymmetrised density function is the same as in Theorem 3.1(3).*

Proof. (1) is obtained in a similar way to the result for Theorem 3.1(1). (2) and (3) follow from (1) by the application of Theorem 2.1. □

Theorem 3.4. *Let* $U \sim \mathcal{B}I_2(B)(s/2, r/2, \Omega)$ *then*

1. *Its density function and nonsymmetrised density agree and are the same that in Theorem 3.2(3).*

2. *Its symmetrised density function is the same as in Theorem 3.2(2).*

Proof. (1) is obtained by Díaz-García and Gutiérrez-Jáimez (2001), see also Srivastava (1968). And (2) follows from (1) by the application of Theorem 7.3.3 in Muirhead (1982, p.260). □

4 Noncentral beta type II distribution

Let us denote the central beta type II density as

$$\mathcal{B}II_m(V; r/2, s/2) = \frac{1}{\beta[r/2, s/2]}|V|^{(r-m-1)/2}|I+V|^{-(r+s)/2}, \quad V > 0.$$

Then from Definition 1, we have

Theorem 4.1. *Let* $V \sim \mathcal{B}II_1(A)(r/2, s/2, \Omega)$ *then*

1. *Its density function and non-symmetrised density agree, and moreover are given by*

$$f_V(V) = \mathcal{B}II_m(V; r/2, s/2)\,\mathrm{etr}\left(-\tfrac{1}{2}\Omega\right) {}_1F_1\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega(I+V)^{-1}\right)$$

2. *Its symmetrised density function is*

$$f_s(V) = \mathcal{B}II_m(V; r/2, s/2)\,\mathrm{etr}\left(-\tfrac{1}{2}\Omega\right) {}_1F_1\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega, (I+V)^{-1}\right)$$

Proof. (1) is addressed in Gupta and Nagar (2000, Theorem 5.5.3, p. 190). And (2) is obtained from (1) by the application of Theorem 7.3.3 in Muirhead (1982, p.260), see also Greenacre (1973). □

Theorem 4.2. *Let* $F \sim \mathcal{B}II_1(B)(s/2, r/2, \Omega)$ *then*

1. *Its density function is*

$$f_F(F) = \frac{\mathrm{etr}(-\frac{1}{2}\Omega)}{2^{m(r+s)/2}\Gamma_m[r/2]\Gamma_m[s/2]}|F|^{(s-m-1)/2}$$

$$\times \int_{B>0} |B|^{(r+s-m-1)/2}\,\mathrm{etr}\left(-\tfrac{1}{2}(I+F)B\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega B^{1/2}F(B^{1/2})'\right)(dB).$$

2. *Its symmetrised density function is*

$$f_s(F) = \mathcal{B}II_m(F; s/2, r/2)\,\mathrm{etr}\left(-\tfrac{1}{2}\Omega\right) {}_1F_1^{(m)}\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega, (I+F)^{-1}F\right)$$

3. *Its nonsymmetrised density function is*

$$f_F(F) = \mathcal{B}II_m(F; s/2, r/2)\,\mathrm{etr}\left(-\tfrac{1}{2}\Omega\right) {}_1F_1\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega(I+F)^{-1}F\right)$$

Proof. (1) is obtained in a similar way to the result in Theorem 3.1(1). (2) and (3) are obtained from (1) by the application of Theorem 2.1. □

Similarly, under Definition 2, we can state:

Theorem 4.3. *Let* $V \sim \mathcal{B}II_2(A)(r/2, s/2, \Omega)$ *then*

1. *Its density function is*

$$f_V(V) = \frac{\mathrm{etr}(-\frac{1}{2}\Omega)}{2^{m(r+s)/2}\Gamma_m[r/2]\Gamma_m[s/2]}|V|^{-(s+m+1)/2}$$

$$\times \int_{A>0} |A|^{(r+s-m-1)/2}\,\mathrm{etr}\left(-\tfrac{1}{2}A(I+V^{-1})\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega A^{1/2}V^{-1}(A^{1/2})'\right)(dA).$$

2. *Its symmetrised density function is the same as in Theorem 4.1(1).*

3. *Its nonsymmetrised density function is the same as in Theorem 4.1(2).*

Proof. (1) is obtained in a similar way to the result in Theorem 3.3(1). (2) and (3) follow from (1) by the application of Theorem 2.1. □

Theorem 4.4. *Let* $F \sim \mathcal{B}II_2(B)(s/2, r/2, \Omega)$ *then*

1. *Its density function and non-symmetrised density agree and are the same as in Theorem 4.2(3).*

2. *Its symmetrised density function is the same as in Theorem 4.2(2).*

Proof. (1) is obtained from Muirhead (1982, Theorem 10.4.1, p.449), see also James (1964). And (2) is obtained from (1) by the application of Theorem 7.3.3 in Muirhead (1982, p.260). □

5 Conclusions

It is immediately apparent that from nonsymmetrised densities we can obtain the same distributions of the eigenvalues of the beta type I and II matrices obtained by Constantine (1963) in the case of the beta type I distribution and by James (1964) and Muirhead (1982, pp. 450-451) for the case of the beta type II distribution. What is important, as established in Theorems 3.1- 4.4, is the fact that these nonsymmetrised densities are invariant under definitions type 1 and 2 for the beta type I and II distributions. Note, too, that there are various transformations to relate the beta type I distributions in their different versions with the beta type II distributions (also for their different versions), both in the central case and in the noncentral one. Thus it is possible in a very simple way, for example, when we know the beta type I(A) density, to determine the beta type I(B) density, see Srivastava and Khatri (1979, problem 3.24, p. 102) and Gupta and Nagar (2000, Section 5.5).

Now, let us observe that, for Theorems 3.4 and 4.4, the beta type I(B) and II(B) distributions are specified by Definitions 5.5.1 and 5.5.2 in Gupta and Nagar (2000, pp. 190 and 192), respectively, irrespective of whether the type I or type II definition is employed to define them. Similarly, for Theorems 3.3 and 4.3, we have the following definitions for the case of the beta type I(A) and II(A) distributions, respectively:

Definition 5.1 (Noncentral matrix variate beta type I(A))**.** *A symmetric positive definite random matrix* $W : m \times m$ *is said to have non central matrix variate beta type I(A) distributions with parameters* a, b *and* $\Theta : m \times m$, *if its density function is given by*

$$f_W(W) = \mathcal{B}I_m(W; a, b) \operatorname{etr}(-\Theta)\, {}_1F_1\left((a+b); b; \Theta(I-W)\right), \quad 0 < W < I.$$

where $a > (m-1)/2$ *and* $b > (m-1)/2$.

and

Definition 5.2 (Noncentral matrix variate beta type II(A)). *A symmetric positive definite random matrix $V : m \times m$ is said to have non central matrix variate beta type II(A) distributions with parameters a, b and $\Theta : m \times m$, if its density function is given by*

$$f_s(V) = \mathcal{B}II_m(V; a, b) \operatorname{etr}(-\Theta) \, {}_1F_1\left((a+b); b; \Theta(I+V)^{-1}\right), \quad V > 0$$

where $a > (m-1)/2$ and $b > (m-1)/2$.

Finally, observe that the joint cumulative distribution functions of the Beta type I and II matrices can be obtained from the corresponding nonsymmetrised densities and Theorem 7.2.10 in Muirhead (1982, p. 254).

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