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Certainty equivalents as risk measures

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Abstract: In this paper certainty equivalents are considered as risk measures. It is shown, how certainty equivalents can be characterized axiomatically, and how properties like translation invariance, positive homogeneity, convexity and subadditivity can be characterized by properties of the utility function. It turns out that these risk measures typically are not convex, but still preserve convex stochastic ordering.

Key words: Coherence, convexity, risk measure, utility function, utility indifference pricing.

1 Introduction

In recent years there has been an increasing interest in the axiomatic approach to risk measures, where we mean by a *risk measure* a functional assigning a real number to the risk of a financial position.

Seminal papers on the axiomatic approach to risk measures have been written by Artzner et al. (1999) and Föllmer and Schied (2002a), who introduced the notions of *coherent risk measure* and *convex risk measure*, respectively.

In the following, we will describe the risk of a financial position by a random variable X, where positive values of X describe gains and negative values describe losses. A risk measure ρ is a functional, assigning a real number to the risk X. According to Artzner et al. (1999) a risk measure is *coherent*, if it fulfills the following axioms:

Monotonicity: If $X \leq Y$ then $\rho(X) \geq \rho(Y)$;

Translation invariance: if $m \in \mathbb{R}$, then $\rho(X+m) = \rho(X) - m$;

Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$;

Positive homogeneity: if $\lambda > 0$, then $\rho(\lambda X) = \lambda \rho(X)$;

Such risk measures have a dual representation of the form

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \left(E_Q(-X) \right), \tag{1.1}$$

where $\mathcal{Q}$ is some set of probability measures.

Föllmer and Schied (2002a) and Frittelli and Gianin (2002) challenge the axiom of positive homogeneity and consider the weaker concept of ρ being a *convex risk measure*. They replace the axioms of subadditivity and positive homogeneity by the axiom of

Convexity: $\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y)$ for all $\lambda \in [0,1]$.

As subadditivity and positive homogeneity together imply convexity, any coherent risk measure is also a convex risk measure.

Föllmer and Schied (2002a) show that any convex risk measure has a dual representation of the form

$$\rho(X) = \sup_{Q\in\mathcal{Q}} \left(E_Q(-X) - \alpha(Q)\right), \tag{1.2}$$

where $\mathcal{Q}$ is some set of probability measures, and α is a penalty function, which can be chosen to be convex and lower semi-continuous with $\alpha(Q) \ge -\rho(0)$.

A typical example of a convex risk measure, which is not coherent is the so called entropic risk measure $e_\beta(X)$, where one chooses in (1.2)

$$\alpha(Q) = \frac{1}{\beta} H(Q|P_X),$$

with $\beta > 0$ and $H(Q|P_X)$ the relative entropy of Q with respect to P_X. This can be stated in equivalent terms as a certainty equivalent of an exponential utility function. Indeed, an expected utility maximizer with the utility function $u(x) = 1 - e^{-\beta x}$ is indifferent between the risk X and the sure position $-e_\beta(X)$, i.e.

$$u(-e_\beta(X)) = Eu(X),$$

or in other terms, $\rho(X) = e_\beta(X)$ fulfills

$$\rho(X) = -u^{-1}(Eu(X)), \tag{1.3}$$

For a proof of this result, see e.g. Föllmer and Schied (2002b). This fact is quite often used in the context of so called utility indifference pricing, see e.g. Frittelli (2000), Becherer (2003) or Mania and Schweizer (2005).

It is natural to ask the question, whether one also obtains convex or even coherent risk measures, if one chooses other utility functions. One popular utility function is the power utility function $u(x) = x^p,\ x \ge 0$ with $p \le 1$. In this case the risk measure derived from the certainty equivalent as in (1.3) simply amounts to $\rho(X) = -\|X\|_p$. Notice that for $p < 1$ this is not really a norm, since in this case in fact $\|X+Y\|_p \ge \|X\|_p + \|Y\|_p$, see e.g. Hewitt and Stromberg (1965). Therefore the risk measure obtained this way is subadditive, and it is easy to see that it is positively homogeneous, however, it is not translation invariant, and therefore it is not coherent and not a convex risk measure in the sense of Föllmer and Schied (2002a) (though it of course fulfills the axiom of convexity as a consequence of subadditivity and homogeneity). Thus we see that increasing concave utility functions do not necessarily lead to convex risk measures in the sense of Föllmer and Schied (2002a).

It is the aim of this paper to consider an arbitrary increasing and concave utility function u and the corresponding risk measure derived from a utility indifference pricing principle via

$$\rho_u(X) = -u^{-1}(Eu(X)), \tag{1.4}$$

and to characterize the axioms for risk measures mentioned above in terms of properties of the utility function. The quantity $u^{-1}(Eu(X))$ is a well known object in the theory of individual decision making under risk, as it is the *certainty equivalent* of an expected utility maximizer with utility function u. This is also known under the name of a *quasi-linear mean*, and has a long history. Indeed, this topic has been treated already by famous people like Bonferroni, Kolmogorov and de Finetti, when they founded modern probability theory, see e.g. Muliere and Parmigiani (1993) for a review of the early history of such certainty equivalents.

There is also a rich literature in actuarial science on premium calculation principles, where similar axioms are discussed as in the theory of risk measures. In that literature certainty equivalents as discussed here can sometimes be found under the name *mean value principle*, which is a little misleading. This should not be confused with the well known *zero utility principle*, where for a utility function u one defines the premium $\pi(X)$ implicitly as the solution of the equation

$$u(0) = Eu(X - \pi(X)).$$

This zero utility principle has properties quite different from the ones of certainty equivalents. In fact, the risk measure $\rho(X) = \pi(-X)$ derived from a zero utility principle is always a convex risk measure in the sense defined by Föllmer and Schied (2002a), whereas the certainty equivalent typically is not, as we will see in this paper. Only in the case of a linear or an exponential utility function, these two concepts coincide. In fact, Bühlmann et al. (1988) generalized the concepts of certainty equivalents and zero utility principle to the so called *Swiss premium principle* $\pi_z(X)$ with a parameter $z \in [0, 1]$ which is implicitly defined by the equation

$$u((1 - z)\pi_z(X)) = Eu(X - z\pi_z(X)),$$

where the extreme cases are given by the certainty equivalent ($z = 0$) and by the zero utility principle ($z = 1$). Goovaerts and Vylder (1980) give conditions on the function u to yield some properties of the Swiss premium principle, which can be considered as weakened versions of the axioms considered here. In particular, Goovaerts and Vylder (1980) consider the properties of positive subtranslativity and of subadditivity for independent random variables, which are weakened versions of translation invariance and subadditivity, respectively.

2 Main results

Let $(\Omega, \mathcal{F}, P)$ be a general probability space, and let $I = (a, b),\ -\infty \leq a < b \leq \infty$, be an arbitrary interval. We denote by $L^\infty(I)$ the set of all bounded random variables with values in I. We follow the usual convention in mathematical finance that a *risk* is described by a random variable X, where positive values of X correspond to rewards and negative values of X to losses. So, if we assume that X has only values in $I = \mathbb{R}_+ = [0, \infty)$, this means that we only consider risks with non-negative rewards. If we want to consider only potential losses (as is usually done in actuarial sciences, for instance) then we will choose $I = \mathbb{R}_- = (-\infty, 0]$, and if we want to allow rewards as well as losses, then we will consider $I = \mathbb{R}$. Throughout the paper we will assume that one of these three cases occurs, that is we assume $I \in \{\mathbb{R}, \mathbb{R}_+, \mathbb{R}_-\}$.

We denote by $I' = (-b, -a)$ the set of possible values of $-X$, if $X \in L^\infty(I)$. For any strictly increasing and convex loss function $\ell : I' \to \mathbb{R}$ we define $\rho_\ell : L^\infty(I) \to I'$ by

$$\rho_\ell(X) = \ell^{-1}(E\ell(-X)). \tag{2.1}$$

To any such loss function ℓ there corresponds an increasing concave utility function $u_\ell : I \to \mathbb{R}$, defined by $u_\ell(x) = -\ell(-x)$. In terms of the utility function the risk measure can alternatively be defined as

$$\rho_\ell(X) = -u_\ell^{-1}(Eu_\ell(X)). \tag{2.2}$$

Throughout the paper we will assume for simplicity that all loss functions and utility functions are twice differentiable. Notice that ρ_ℓ is well defined and finite, since $m \leq X \leq M$ implies $-m \geq \rho_\ell(X) \geq -M$. The functional $-\rho_\ell$ is a *generalized mean* as considered in Chapter 4 of Pecarić et al. (1992), or *quasi-linear mean*, see Muliere and Parmigiani (1993). If Ω is finite, then $-\rho_\ell$ is a *quasi-arithmetic mean*, see e.g. Bullen et al. (1977). We will use results from these references later in this section.

Any risk measure ρ_ℓ as defined in (2.1) is obviously law invariant, i.e. if X and Y have the same distribution, then $\rho_\ell(X) = \rho_\ell(Y)$. Therefore we will also write $\rho_\ell(F) := \rho_\ell(X)$, if F is the cumulative distribution function of X.

Moreover, any risk measure ρ_ℓ is monotone, and has the following constancy property:

Constancy: $\rho_\ell(m) = -m$ for all deterministic $m \in I$,

This axiom has been introduced in Frittelli and Gianin (2002). Of course this property holds for any translation invariant risk measure fulfilling $\rho(0) = 0$.

Finally, for any ℓ the functional ρ_ℓ has the following property, which is called *associativity* or *quasi-linearity*:

Quasi-linearity: for any cumulative distribution functions F, G and H it holds: $\rho_\ell(F) = \rho_\ell(G)$ implies $\rho_\ell(\alpha F + (1-\alpha)H) = \rho_\ell(\alpha G + (1-\alpha)H)$ for all $0 < \alpha < 1$.

The following well known theorem axiomatically characterizes certainty equivalents. It is known as Nagumo-Kolmogorov-de Finetti theorem. A proof is given e.g. in Hardy et al. (1934), a more general result can be found in Chew (1983).

Theorem 2.1. *A functional $\rho : L^\infty(I) \to \mathbb{R}$ can be written in the form $\rho(X) = \ell^{-1}(E\ell(-X))$ if and only if it has the following properties:*

1. *monotonicity;*
2. *law invariance;*
3. *constancy;*
4. *quasi-linearity.*

However, such a functional does not always have all the properties of a coherent or convex risk measure as defined above, even if ℓ is increasing and convex. For general ℓ the risk measure ρ_ℓ is not translation invariant. This only holds for exponential loss functions, as is well known in the literature. As an early reference, see Bemporad (1928). For completeness we give here a simple sketch of a proof.

Theorem 2.2. *ρ_ℓ is translation invariant, if and only if ℓ is either exponential or linear.*

Proof. If ρ_ℓ is translation invariant, then

$$\frac{\partial}{\partial t}\rho_\ell(X+t) = -1$$

for all X and all t. This obviously holds for ℓ linear, i.e. for $\rho_\ell(X) = EX$. Therefore assume that ℓ is non-linear with $\ell' > 0$ on some interval (a, b). For bounded X such that $X + t$ has values in (a, b) we can interchange differentiation and expectation and get

$$\begin{aligned}\frac{\partial}{\partial t}\rho_\ell(X+t) &= \frac{\partial}{\partial t}\ell^{-1}(E\ell(-X-t)) \\ &= -\frac{E\ell'(-X-t)}{\ell'(\ell^{-1}(E\ell(-X-t)))}.\end{aligned}$$

The last expression equals -1 for all bounded X and all t if and only if

$$g^{-1}(Eg(Y)) = \ell^{-1}(E\ell(Y)) \tag{2.3}$$

for all bounded Y, where we have now written $g := \ell'$. According to Bullen et al. (1988), Theorem 5, p. 221, equation (2.3) holds if and only if $g = \alpha\ell + \beta$ for some $\beta \in \mathbb{R}$ and $\alpha \neq 0$. Thus ℓ must satisfy a linear differential equation and therefore must be exponential. Moreover, it is clear that the maximal interval (a, b) with $\ell' > 0$ then must be $(a, b) = \mathbb{R}$, and the loss function must be an exponential function on the whole real line.

Theorem 2.3. *The risk measure* $\rho_\ell : L^\infty(\mathbb{R}_-) \to \mathbb{R}_+$ *is positively homogeneous, if and only if* u_ℓ *is either a power function or* $u_\ell(x) = \log x$.

Proof. This is an immediate consequence of Theorem 2.2 noticing that $\rho_\ell(X+t) = \rho_\ell(X) - t$ for all $t \in \mathbb{R}$ and all X is equivalent to $\rho_g(sY) = s\rho_g(Y)$ for all $s \geq 0$ and all $Y \geq 0$, if we define $g(x) := \ell(\log x)$, $s := e^{-t}$ and $Y := -e^{-X}$.

Notice that we assume throughout that ℓ is increasing and convex. Therefore the result of Theorem 2.3 is more restrictive than it may look at first sight. It says that for $I = \mathbb{R}$ the only positive homogeneous risk measure ρ_ℓ is $\rho_\ell(X) = -EX$, as linear functions are the only increasing convex power functions $\ell : \mathbb{R} \to \mathbb{R}$. On $I = \mathbb{R}_+$, however, there are more positive homogeneous risk measures derived from utility function. There, the risk measure derived from logarithmic utility has this property, too, and any power function $u_\ell(x) = x^\alpha$ with $\alpha \leq 1$ yields a positively homogeneous risk measure ρ_ℓ. The same holds for the power loss function $\ell(x) = x^\alpha$ with $\alpha \geq 1$ in case $I = \mathbb{R}_-$.

As a corollary of Theorem 2.2 and 2.3 we get the following result about coherence of ρ_ℓ.

Corollary 2.1. *The risk measure* ρ_ℓ *is coherent if and only if* $\rho_\ell(X) = -EX$.

Next we want to consider convexity and subadditivity of the risk measures ρ_ℓ. To the best of our knowledge this question has not been considered before in the literature on risk measures. We need the following preliminary result.

Theorem 2.4. *Let* $f : I' \times I' \to I'$ *be an arbitrary function and define* $\tilde{f} : I \times I \to I$ *by*

$$\tilde{f}(x, y) := -f(-x, -y).$$

Then

$$\rho_\ell(\tilde{f}(X, Y)) \leq f(\rho_\ell(X), \rho_\ell(Y)) \tag{2.4}$$

holds for all $X, Y \in L^\infty(I)$ *if and only if the function* $H : \ell(I') \times \ell(I') \to \mathbb{R}$ *defined by*

$$H(x, y) := \ell(f(\ell^{-1}(x), \ell^{-1}(y))) \tag{2.5}$$

is concave.

Proof. This is a special case of Theorem 4.31 in Pecarić et al. (1992). Choose there for A the expectation operator, $n = 2$, $g_1 = -X$, $g_2 = -Y$, and $\chi = \psi_1 = \psi_2 = \ell$. For the case of a finite Ω see also Bullen et al. (1988), Theorem 3 on page 249.

Of special interest are the cases $f(x, y) = x + y$ and $f(x, y) = (x + y)/2$. For these two cases we will characterize concavity of H in the next two lemmas.

We use the following notation as an abbreviation:

$$\ell^*(x) = \frac{\ell'(x)}{\ell''(x)}, \quad x \in I'.$$

Lemma 2.1. *For $f(x,y) = \tilde{f}(x,y) = x + y$ the function H defined in* (2.5) *is concave if and only if ℓ^* fulfills the following two conditions:*

(i) $\ell^*(a+b) \geq \ell^*(a)$ *for all* $a, b \in I'$*;*

(ii) $\ell^*(a+b) \geq \ell^*(a) + \ell^*(b)$ *for all* $a, b \in I'$*.*

Proof. We prove the result only in the case that H is twice differentiable. Without differentiability the idea of proof is the same, but one has to replace the derivatives by differences and this becomes quite tedious. Recall that a differentiable H is concave if and only if its second derivatives satisfy

$$H_{xx} \leq 0 \quad \text{ and } \quad H_{xx}H_{yy} - H_{xy}^2 \geq 0.$$

Here $H(x,y) = \ell(\ell^{-1}(x) + \ell^{-1}(y))$. Thus

$$H_x(x,y) = \frac{\ell'(\ell^{-1}(x) + \ell^{-1}(y))}{\ell'(\ell^{-1}(x))}$$

and

$$H_{xx}(x,y) = \frac{\ell''(\ell^{-1}(x) + \ell^{-1}(y)) - \ell'(\ell^{-1}(x) + \ell^{-1}(y))\ell''(\ell^{-1}(x))/\ell'(\ell^{-1}(x))}{\ell'((\ell^{-1}(x))^2}.$$

Writing $a = \ell^{-1}(x)$ and $b = \ell^{-1}(y)$ we see that $H_{xx} \leq 0$ if and only if

$$\frac{\ell''(a+b)}{\ell'(a+b)} \leq \frac{\ell''(a)}{\ell'(a)}$$

i.e., if and only if $\ell^*(a) \leq \ell^*(a+b)$. For the mixed derivative we get

$$H_{xy}(x,y) = \frac{\ell''(a+b)}{\ell'(a)\ell'(b)}$$

and thus

$$\begin{aligned}
&H_{xx}(x,y)H_{yy}(x,y) - H_{xy}(x,y)^2 \\
&= \frac{\left(\ell''(a+b) - \ell'(a+b)\ell''(a)/\ell'(a)\right)\left(\ell''(a+b) - \ell'(a+b)\ell''(b)/\ell'(b)\right) - \ell''(a+b)^2}{\ell'(a)^2\ell'(b)^2} \\
&\geq 0
\end{aligned}$$

if and only if

$$\begin{aligned}
&\left(1 - \frac{\ell^*(a+b)}{\ell^*(a)}\right) \cdot \left(1 - \frac{\ell^*(a+b)}{\ell^*(b)}\right) \geq 1 \\
\Leftrightarrow \quad &\frac{\ell^*(a+b)}{\ell^*(a)} + \frac{\ell^*(a+b)}{\ell^*(b)} \leq \frac{\ell^*(a+b)^2}{\ell^*(a)\ell^*(b)} \\
\Leftrightarrow \quad &\ell^*(a) + \ell^*(b) \leq \ell^*(a+b).
\end{aligned}$$

Replacing $x+y$ by $(x+y)/2$ we get the following Lemma. As the proof is very similar, we omit it.

Lemma 2.2. *For $f(x,y) = \tilde{f}(x,y) = (x+y)/2$ the function H defined in* (2.5) *is concave if and only if ℓ^* fulfills the following two conditions:*

(i) $\ell^*((a+b)/2) \geq \ell^*(a)/2$ *for all $a, b \in I'$;*

(ii) $\ell^*((a+b)/2) \geq (\ell^*(a) + \ell^*(b))/2$ *for all $a, b \in I'$.*

These lemmas can be used to characterize convexity and subadditivity for the risk measures ρ_ℓ.

Theorem 2.5. *(a) For $I = \mathbb{R}$ the risk measure ρ_ℓ is convex, if and only if ℓ is either linear or exponential.*

(b) For $I = \mathbb{R}_+$ the risk measure ρ_ℓ is convex, if and only if $\ell^* : \mathbb{R}_- \to \mathbb{R}_+$ is decreasing and concave.

(c) For $I = \mathbb{R}_-$ the risk measure ρ_ℓ is convex, if and only if $\ell^* : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and concave.

Proof. The function $\alpha \mapsto \rho_\ell(\alpha X + (1-\alpha)Y)$ is measurable, and therefore ρ_ℓ is convex if and only if it is mid-convex, i.e. if and only if

$$\rho_\ell\Big(\frac{X+Y}{2}\Big) \leq \frac{1}{2}(\rho_\ell(X) + \rho_\ell(Y)),$$

see e.g. Rockafellar (1970). Thus we can apply Theorem 2.4 with

$$f(x,y) = \tilde{f}(x,y) = \frac{x+y}{2}.$$

Lemma 2.2 therefore implies that ρ_ℓ is convex, if and only if the function $\ell^* : I' \to \mathbb{R}_+$ is concave and satisfies

$$\ell^*\Big(\frac{a+b}{2}\Big) \geq \frac{\ell^*(a)}{2} \quad \text{for all } a, b \in I'. \tag{2.6}$$

For $I = \mathbb{R}$ this can only hold if ℓ^* is constant, due to the non-negativity of ℓ^*. This proves (a). If $I = \mathbb{R}_+$, then $I' = \mathbb{R}_-$ and $\ell^* : \mathbb{R}_- \to \mathbb{R}_+$ can only be concave if it is decreasing. But then (2.6) also holds. This shows part (b), and the proof of part (c) is similar.

Remark 2.1. A related result has been proved in Ben-Tal and Teboulle (1986) in the context of nonlinear stochastic programming.

Example 2.1. An example, where the condition of Theorem 2.5 (b) is fulfilled, is given by the utility function

$$u(x) = 1 - e^{-\sqrt{x}}(1 + \sqrt{x}), \quad x \geq 0.$$

A simple calculation shows that in this case the function ℓ^* is given by

$$\ell^*(x) = 2\sqrt{-x}, \quad x \le 0,$$

which is clearly concave and decreasing. Another even simpler example is given by $u(x) = x^\alpha,\ x \ge 0,\ 0 < \alpha < 1$, which leads to $\ell^*(x) = (-x)/(1-\alpha),\ x \le 0$. An example, where the condition of Theorem 2.5 (c) is fulfilled, is discussed in detail in Example 2.2 below.

Using Lemma 2.1 we can also characterize subadditivity. We omit the proof, as it is very similar to the one of Theorem 2.5 (b).

Theorem 2.6. *(a) For $I = \mathbb{R}$ the risk measure ρ_ℓ is subadditive, if and only if ℓ is either linear or exponential.*

(b) For $I = \mathbb{R}_+$ the risk measure ρ_ℓ is subadditive, if and only if $\ell^* : \mathbb{R}_- \to \mathbb{R}_+$ is decreasing and subadditive.

(c) For $I = \mathbb{R}_-$ the risk measure ρ_ℓ is subadditive, if and only if $\ell^* : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and subadditive.

Remark 2.2. It follows from Goovaerts and Vylder (1980), Corollary 6.4, that for ρ_ℓ to be subadditive for non-positive independent random variables X and Y, it is sufficient that ℓ^* is increasing. Thus compared to our Theorem 2.6 (c) they need a weaker assumption to prove a weaker result.

Subadditivity and Convexity are properties of a risk measure, which reflect the fact that diversification pays. This should be related to well known concepts of risk aversion like stochastic dominance. In fact, several authors have studies the question, whether convex or coherent risk measures are consistent with stochastic dominance, see e.g. Dana (2005), Leitner (2005) or Bäuerle and Müller (2006).

Recall that two random variables are said to be ordered with respect to convex order (written as $X \le_{cx} Y$), if $Ef(X) \le Ef(Y)$ for all convex functions f (see Müller and Stoyan (2002) for a detailed account to this and related stochastic orders). In Bäuerle and Müller (2006) the following result is proved.

Theorem 2.7. *Assume that $(\Omega, \mathcal{A}, P)$ is either finite with P the uniform distribution or non-atomic. Then a convex risk measure is consistent with convex stochastic order in the following sense:*

$$X \le_{cx} Y \quad \textit{implies} \quad \rho(X) \le \rho(Y). \tag{2.7}$$

Dana (2005) denotes a risk measure with property (2.7) as being *Schur-convex.* Thus Theorem 2.7 says that any convex risk measure is Schur-convex. It is a natural question to ask, whether there are interesting risk measures which are Schur-convex, but not convex. As ρ_ℓ is only convex, if ℓ is exponential or linear, the following result shows that certainty equivalents form a nice class of risk measures, which are always Schur-convex, but not necessarily convex. The proof of the following theorem is obvious.

Theorem 2.8. *$X \le_{cx} Y$ implies $\rho_\ell(X) \le \rho_\ell(Y)$ for any increasing and convex function ℓ.*

Remark 2.3. The risk measure ρ_ℓ is not only Schur-convex, it is also *quasi-convex*. Recall that a function ρ on a linear space is called quasi-convex, if the sublevel sets $A_\rho(t) := \{X : \rho(X) \le t\}$ are convex for all $t \in \mathbb{R}$. It is easy to see that any monotone transform of a convex function is quasi-convex. As the functional $X \to E\ell(X)$ is convex for an increasing convex function ℓ, this obviously implies that ρ_ℓ is quasi-convex for ℓ convex. The topic of quasi-convex risk measures will be considered in more detail in a forthcoming paper.

Example 2.2. Assume that risks describe potential losses (as in a typical insurance context), i.e. $I = \mathbb{R}_-$ and therefore $I' = \mathbb{R}_+$. Consider as loss function $\ell : I' \to \mathbb{R}$ a power function $\ell(x) = x^p, \quad x \ge 0$ with $p > 1$. This yields the risk measure

$$\rho_\ell(X) = \| - X\|_p$$

and a well known representation theorem for the norm $\| \cdot \|_p$ (see e.g. Hewitt and Stromberg (1965)) yields

$$\begin{aligned}\rho_\ell(X) &= \sup\{E(-XY) : \|Y\|_q \le 1, Y \ge 0\} \\ &= \sup\{-\int X dQ : \|dQ/dP\|_q \le 1\},\end{aligned}$$

where $1/p + 1/q = 1$.

Thus we have a representation similar to the representation of a coherent risk measure as worst case expectation described in (1.1). However, here we have a set of measures, which are in general **not** probability measures, as $\|dQ/dP\|_q \le 1$ does not imply $Q(\Omega) = 1$. This reflects the fact that ρ_ℓ is not coherent and not a convex risk measure in the sense of Föllmer and Schied, as it is not translation invariant. However, it has all other properties of interest: law invariance, monotonicity, subadditivity, positive homogeneity, convexity, constancy, quasi-linearity and monotonicity with respect to convex ordering. This is a consequence of the results of this section, taking into account that here $\ell^*(x) = x/(p-1)$ is increasing, concave and subadditive.

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