Brazilian Journal of Probability and Statistics (2007), 21, pp. 13–25.

Moments of the discounted dividends in a threshold-type Markovian risk process

Andrei Badescu[1] and **David Landriault**[2]
[1] *University of Toronto*
[2] *University of Waterloo*

Abstract: In this paper, we derive an explicit expression for the n-th moment of the discounted dividend payments prior to ruin, generalizing the results on the first moment in Badescu et al. (2007). Based on the connection between an insurer's surplus process and its corresponding fluid flow process, we propose a recursive algorithm to compute the higher moments of the discounted dividend payments in a fairly general class of risk processes governed by Markovian claim arrivals. Finally, numerical examples are considered to illustrate our main results.

Key words: Discounted dividends, fluid flow processes, Markovian arrival process, phase-type distribution, threshold level.

1 Introduction

In this paper, we consider an insurer's surplus process with surplus-dependent premium rates of a threshold type. Namely, we consider a structure with a unique threshold at level b ($b > 0$), the so-called *threshold level*. We assume that a net positive premium of c (c') is received by the insurer whenever the surplus level at a given time is below (above) the threshold level b. For insurance applications, we generally choose c to be greater than c' due to a possible dividend rate paid to the shareholders whenever the insurer's surplus is at a relatively high level (here when the surplus is greater than b), lowering the net premium received by the insurer above level b.

We define the surplus process of interest in this paper, namely $\left\{R^b\left(t\right), t \geq 0\right\}$, as

$$dR^b(t) = \begin{cases} cdt - d\left(\sum\limits_{n=1}^{N(t)} U_n\right), & R^b(t) < b, \\ c'dt - d\left(\sum\limits_{n=1}^{N(t)} U_n\right), & R^b(t) \geq b, \end{cases} \tag{1.1}$$

with $R^b\left(0\right) = u$ being the initial surplus level. Note that, in the risk model (1.1), $\left\{N\left(t\right), t \geq 0\right\}$ is the claim counting process where $N\left(t\right)$ represents the total number of claims by time t. In this paper, we assume that the claim number process follows a Markovian arrival process (MAP) which includes both the classical compound Poisson risk model and the renewal risk model with phase-type interclaim times. Also, the claim size r.v.'s $\{U_n\}_{n=1}^{\infty}$ in (1.1) are assumed to form a sequence of identically distributed r.v.'s having a phase-type representation. We recall that risk processes with interarrival times correlated to claim sizes can also be modelled under the MAP framework (see Ahn and Badescu (2007) and references therein).

A MAP with representation $MAP(\alpha, D_0, D_1)$ of order m is a two-dimensional Markov process on the state space $\mathbb{N}_0 \times \{1, \ldots, m\}$. For this process, an underlying continuous-time Markov chain (CTMC) on the state space $E = \{1, \ldots, m\}$ (referred to as the environmental states) evolves such that the instantaneous rate of transition from state i to state $j \neq i$ in E without an accompanying claim is given by the (i, j)-th element of D_0, namely $D_0(i, j) \geq 0$. Similarly, the instantaneous rate of transition from state i to state j (possibly $j = i$) in E with an accompanying claim is given by the quantity $D_1(i, j) \geq 0$. The diagonal elements of D_0 are assumed to be negative and such that the sum of the elements on each row of the matrix $D_0 + D_1$ are all zero. We denote by α the initial probability vector of the underlying CTMC. For a detailed treatment of MAPs, we refer the reader to Latouche and Ramaswami (1993) and Neuts (1981).

Pertaining to the surplus process (1.1) is the event of ruin where ruin is defined to occur if and when the insurer has a negative surplus. Let us define the time to ruin $\tau^b(u)$ as $\tau^b(u) = \inf\left\{t : R^b(t) < 0\right\}$ with $\tau^b(u) = \infty$ if $R^b(t) \geq 0$, $\forall t \geq 0$ (ruin does not occur). Note that the time to ruin is a random variable of a crucial importance in ruin theory in the analysis of surplus processes. In Badescu et al. (2007), an analysis of various ruin related quantities, including the Laplace transform of the time to ruin $\tau^b(u)$, as well as the triple Laplace transform of the time to ruin $\tau^b(u)$, the surplus immediately prior to ruin $U(\tau^b(u)^-)$ and the deficit at ruin $\left|U\left(\tau^b(u)\right)\right|$, has been performed for the surplus process (1.1).

In this paper, we shift our attention to the discounted sum of dividend payments before ruin. We assume that $c' < c$ and let their difference $d = c - c'$ be the dividend rate received by the shareholders whenever the insurer's surplus is above the threshold level b. Note that no dividend is paid when the surplus level is below the barrier level b. The main focus of the paper is the calculation of the moments of the discounted dividends prior to ruin. In the actuarial literature, the expected discounted dividend payments have been analyzed by several authors in various risk models (see Badescu et al. (2007) and references therein). Recently, Albrecher et al. (2005a, 2005b) considered the calculation of higher order moments in the classical compound Poisson risk model and in the Sparre Andersen risk model with generalized Erlang-n interclaim times. The present work generalizes the semi-Markovian structure proposed by Albrecher et al. (2005a) by considering a MAP process for the claim arrivals, at the expense of phase-type distributed claim assumptions. Using first passage times analysis, our approach differs from the one of Albrecher et al. (2005c) by exploiting the existing connection between risk and fluid processes (see Ahn et al. (2007) and Asmussen (2000) for a detailed description). The mathematical tools employed under our approach are more probabilistic in nature and heavily depend on the matrix-representation of the claim size distribution.

The paper is structured as follows. Section 2 consists in a short review of the fluid flow quantities used in the subsequent analysis. In Section 3, the main results pertaining to the higher order moments of the discounted dividends are presented. Finally, numerical examples are considered in Section 4 to illustrate some applications of the main results derived in this paper.

2 Mathematical background

Underlying the fluid flow processes and risk models in this paper is an irreducible CTMC $\mathcal{J} = \{J(t), t \geq 0\}$ defining an environmental process which governs the interclaim intervals and the claim sizes. The states of this process are referred to as "phases". We assume that the CTMC $\mathcal{J}$ has state space $S = S_1 \cup S_2$ where the set S_1 contains the phases when the fluid flow increases (the interclaim intervals in the associated risk model) and the set S_2 contains the phases when the fluid flow decreases (the claim sizes in the associated risk model). The infinitesimal generator associated with $\mathcal{J}$ is partitioned as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}. \tag{2.1}$$

Several risk processes, including the classical compound Poisson risk model and most Sparre Andersen risk models, can be recovered by a proper selection of the infinitesimal generator T (see Remark 2.1 (a) and (b) in Ahn and Badescu (2007)). The paths of the risk process before ruin can be obtained from segments of the fluid process before the fluid process becomes empty and this is achieved by replacing downward linear paths in the fluid model by downward jumps of appropriate sizes. This artifice requires a change in clock time to properly embed the fluid model into the risk model (see Remark 2.1 (c) in Ahn and Badescu (2007)). Leading to the analysis of the surplus process (1.1) is the corresponding fluid flow process

$$(\mathcal{F}^b, \mathcal{J}) = \{(F^b(t), J(t)),\ t \geq 0\}, \tag{2.2}$$

which is defined such that the rate of increase/decrease is c (c') whenever the fluid level is below (above) the threshold level b, i.e.

$$dF^b(t) = \begin{cases} c, & 0 < F^b(t) < b,\ J(t) \in S_1 \\ -c, & 0 < F^b(t) < b,\ J(t) \in S_2 \\ c', & F^b(t) \geq b,\ J(t) \in S_1 \\ -c', & F^b(t) \geq b,\ J(t) \in S_2 \end{cases} \tag{2.3}$$

From (2.3), one observes that the rates of increase/decrease of the fluid flow process $\mathcal{F}^b$ are independent of the state of the Markovian environment. To analyze (2.2), we consider the infinite buffer (barrier-free) fluid model

$$(\mathcal{F}_c, \mathcal{J}) = \{(F_c(t), J(t)), t \geq 0\}, \tag{2.4}$$

defined as

$$dF_c(t) = \begin{cases} c, & F_c(t) > 0,\ J(t) \in S_1 \\ -c, & F_c(t) > 0,\ J(t) \in S_2 \end{cases}$$

and its reflected version

$$(\mathcal{F}_c^r, \mathcal{J}) = \{(F_c^r(t), J(t)), t \geq 0\}, \tag{2.5}$$

with

$$dF_c^r(t) = \begin{cases} -c, & F_c^r(t) > 0,\ J(t) \in S_1 \\ c, & F_c^r(t) > 0,\ J(t) \in S_2 \end{cases}$$

(see Ramaswami (2006) for further details). Note that the subscript c ($c > 0$) stands for the rate of increase/decrease of the fluid flow processes $\mathcal{F}_c$ and $\mathcal{F}_c^r$ (independent of the fluid level). We recall that, in the reflected fluid model, the roles of the up and down phases are simply reversed. The crucial mathematical tools of our analysis are the Laplace transforms (LTs) of the busy periods associated to the fluid processes defined by (2.4) and (2.5) respectively. These quantities, denoted by $\Psi_c(\delta)$ and $\Psi_c^r(\delta)$, were first introduced by Ahn and Ramaswami (2004) and their evaluation can be performed via the algorithm developed by Ahn and Ramaswami (2005). Similarly, we define ${}^b\Psi_c(\delta)$ (${}^b\Psi_c^r(\delta)$) as the LT of a busy period in the (reflected) finite buffer fluid model without visiting level b en route (see Ahn et al. (2007) for a more formal definition as well as their calculation).

3 Main results

In this section, we propose a two-step procedure to compute the moments of the discounted dividend payments before ruin in the surplus process (1.1). As a first step, we derive a general expression for the n-th moment of the discounted dividends in the barrier-free risk model

$$R_{c'}(t) = u + c't - \sum_{n=1}^{N(t)} U_n, \tag{3.1}$$

assuming that a dividend rate d is paid continuously from time 0 to the time of ruin $\tau_{c'}(u) = \inf\{t : R_{c'}(t) < 0\}$ (with $\tau_{c'}(u) = \infty$ if ruin does not occur). Note that the ruin process (3.1) can be linked to the "original" surplus process (1.1) by assuming that level b in the latter corresponds to the new origin (level 0) in the former (i.e. considering only the top surplus-layer of the surplus process (1.1)). Using results pertaining to the barrier-free surplus process (3.1), we then propose a recursive scheme to compute the n-th moment of the discounted dividend payments in the surplus process of interest in this paper.

To do so, let $D_{c'}(u,\delta)$ be the present value (at a force of interest δ) of the dividend payments prior to ruin in the barrier-free risk model (3.1). Also, let $\overrightarrow{V}_{n,c'}(u,\delta)$ be a column vector whose i-th element $\left[\overrightarrow{V}_{n,c'}(u,\delta)\right]_i$ represents the n-th moment of the r.v. $D_{c'}(u,\delta)$ given that $J(0) = i$ ($i \in S_1$), i.e.

$$\left[\overrightarrow{V}_{n,c'}(u,\delta)\right]_i = E_i\left[(D_{c'}(u,\delta))^n\right], \tag{3.2}$$

where $E_i[\cdot]$ stands for the conditional expectation of $\cdot$ given that the initial state of the CTMC $\mathcal{J}$ is i. In order to state the following result, we define $\overrightarrow{1}_{|S_j|}$ as a column vector of 1 of size $|S_j|$ for $j = 1, 2$ and $I_{|S_j|}$ as the identity matrix of the same dimension.

Proposition 3.1. *For the barrier-free surplus process (3.1), the n-th conditional moments of the discounted dividend payments prior to ruin ($n = 1, 2, ...$) are given by*

$$\overrightarrow{V}_{n,c'}(u,\delta) = \left(\frac{d}{\delta}\right)^n \left(\overrightarrow{1}_{|S_1|} + \sum_{j=1}^{n} \binom{n}{j} (-1)^j \overrightarrow{\rho}_{j\delta,c'}(u)\right), \tag{3.3}$$

where

$$\overrightarrow{\rho}_{\delta,c'}(u) = e^{\frac{u\delta}{2c'}} \Psi_{c'}\left(\frac{\delta}{2}\right) e^{H_{c'}\left(\frac{\delta}{2}\right)u} \overrightarrow{1}_{|S_2|}, \tag{3.4}$$

and

$$H_{c'}(\delta) = \left(T_{22} - \delta I_{|S_2|}\right) + T_{21}\Psi_{c'}(\delta).$$

Proof. The proof of this proposition uses a similar line of logic as the one used in Dickson and Waters (2004) for the classical compound Poisson risk model. From (3.2), we know

$$\left[\overrightarrow{V}_{n,c'}(u,\delta)\right]_i = E_i\left[\left(d\overline{a}_{\overline{\tau_{c'}(u)|}}\right)^n\right] = \left(\frac{d}{\delta}\right)^n E_i\left[\left(1 - e^{-\delta\tau_{c'}(u)}\right)^n\right], \tag{3.5}$$

where $\overline{a}_{\overline{t|}}$ stands for the present value of a continuous annuity that pays at a rate of 1 over the next t periods. From (3.5), a binomial expansion of $\left(1 - e^{-\delta\tau_{c'}(u)}\right)^n$ directly leads to (3.3) where $\overrightarrow{\rho}_{\delta,c'}(u)$ is a column vector (of size $|S_1|$) of the conditional Laplace transform of the time to ruin $\tau_{c'}(u)$ starting with an initial capital of u, i.e.

$$\left[\overrightarrow{\rho}_{\delta,c'}(u)\right]_i = E_i\left[e^{-\delta\tau_{c'}(u)}\right].$$

The reader is referred to Badescu et al. (2005) for a proof of the representation (3.4) for the Laplace transform $\overrightarrow{\rho}_{\delta,c'}(u)$. □

The unconditional moments of $D_{c'}(u,\delta)$ can be obtained by pre-multiplying $\overrightarrow{V}_{n,c'}(u,\delta)$ by the initial probability vector α.

Using Proposition 3.1, we now consider the calculation of the moments of the discounted dividend payments in the surplus process (1.1). This class of risk models was considered previously by Badescu et al. (2007) where an expression for the expected discounted dividend payments was derived. We let $D^b(u,\delta)$ be the discounted dividends associated to the surplus process (1.1) and denote by $\overrightarrow{V}_n^b(u,\delta)$ the column vector of its n-th conditional moment ($n = 1, 2, ...$). To find a general expression for $\overrightarrow{V}_n^b(u,\delta)$, we first derive an expression for $\overrightarrow{V}_n^b(b,\delta)$, i.e. the n-th moment of the discounted dividends with an initial surplus $u = b$.

Proposition 3.2. *For the surplus process (1.1) with an initial surplus $u = b$, the n-th conditional moment of the discounted dividend payments prior to ruin ($n = 1, 2, ...$) is given by*

$$\overrightarrow{V}_n^b(b,\delta) = \left(I_{|S_1|} - \Psi_{c'}\left(\frac{n\delta}{2}\right)\ {}^b\Psi_c^r\left(\frac{n\delta}{2}\right)\right)^{-1} \cdot \overrightarrow{V}_{n,c'}(0,\delta)$$
$$+ \left(I_{|S_1|} - \Psi_{c'}\left(\frac{n\delta}{2}\right)\ {}^b\Psi_c^r\left(\frac{n\delta}{2}\right)\right)^{-1} \cdot \sum_{j=1}^{n-1}\binom{n}{j}\left(\frac{d}{\delta}\right)^j$$
$$\sum_{k=0}^{j}\binom{j}{k}(-1)^k\,\Psi_{c'}\left(\frac{(n+k-j)\,\delta}{2}\right)\ {}^b\Psi_c^r\left(\frac{(n-j)\,\delta}{2}\right)\overrightarrow{V}_{n-j}^b(b,\delta), \quad (3.6)$$

where the starting point of this recursive scheme is given by

$$\overrightarrow{V}_1^b(b,\delta) = \left(I_{|S_1|} - \Psi_{c'}\left(\frac{\delta}{2}\right)\ {}^b\Psi_c^r\left(\frac{\delta}{2}\right)\right)^{-1}\overrightarrow{V}_{1,c'}(0,\delta). \quad (3.7)$$

Proof. To obtain the explicit expression (3.6) for the n-th moment of $D^b(b,\delta)$, we first group the dividend payments prior to ruin with respect to their timing :

(a) dividends from time 0 to the first return of the surplus process $\left\{R^b(t), t \geq 0\right\}$ to the threshold level b in S_2 (namely $\tau_{c'}(0)$);

(b) dividends paid after the first return of the surplus process $\left\{R^b(t), t \geq 0\right\}$ to the threshold level b in S_1 (namely $\tau_{c'}(0) + {}^b\tau_c^r(0)$, where ${}^b\tau_c^r(0)$ corresponds to the busy period of a reflected finite buffer (of level b) risk process operating at premium rate c, without visiting level b en route)

Using the proposed decomposition of $D^b(b,\delta)$, it follows that

$$D^b(b,\delta) = d\overline{a}_{\overline{\tau_{c'}(0)|}} + e^{-\delta\left(\tau_{c'}(0)+\ {}^b\tau_c^r(0)\right)}D_*^b(b,\delta), \quad (3.8)$$

where $D_*^b(b,\delta)$ corresponds to the present value (at time $\tau_{c'}(0) + {}^b\tau_c^r(0)$) of the future dividend payments. From (3.8), one finds

$$\left[\overrightarrow{V}_n^b(b,\delta)\right]_i = E_i\left[\left(d\overline{a}_{\overline{\tau_{c'}(0)|}} + e^{-\delta\left(\tau_{c'}(0)+\ {}^b\tau_c^r(0)\right)}D_*^b(b,\delta)\right)^n\right]. \quad (3.9)$$

Using a binomial expansion, (3.9) becomes

$$
\begin{aligned}
\left[\overrightarrow{V}^{b}_{n}(b,\delta)\right]_i &= E_i\left[\left(d\overline{a}_{\overline{\tau_{c'}(0)}|}\right)^n\right] \\
&+ \sum_{j=1}^{n-1}\binom{n}{j}E_i\left[\left(d\overline{a}_{\overline{\tau_{c'}(0)}|}\right)^j e^{-(n-j)\delta\left(\tau_{c'}(0)+\ {}^{b}\tau^{r}_{c}(0)\right)}\left(D^{b}_{*}(b,\delta)\right)^{n-j}\right] \\
&+ E_i\left[e^{-n\delta\left(\tau_{c'}(0)+\ {}^{b}\tau^{r}_{c}(0)\right)}\left(D^{b}_{*}(b,\delta)\right)^{n}\right] \\
&= e_i\overrightarrow{V}_{n,c'}(0,\delta) \\
&+ \sum_{j=1}^{n-1}\binom{n}{j}E_i\left[\left(d\overline{a}_{\overline{\tau_{c'}(0)}|}\right)^j e^{-(n-j)\delta\tau_{c'}(0)}e^{-(n-j)\delta\ {}^{b}\tau^{r}_{c}(0)}\left(D^{b}_{*}(b,\delta)\right)^{n-j}\right] \\
&+ E_i\left[e^{-n\delta\left(\tau_{c'}(0)+\ {}^{b}\tau^{r}_{c}(0)\right)}\left(D^{b}_{*}(b,\delta)\right)^{n}\right]. \qquad (3.10)
\end{aligned}
$$

where e_i is a row vector (of size $|S_1|$) with a "1" at the i-th position and 0 elsewhere. Note that, for $n > 0$, the conditional distribution of $D^{b}_{*}(b,\delta)$ given that $\tau_{c'}(0)+\ {}^{b}\tau^{r}_{c}(0) < \infty$ (i.e. that there is at least one return to level b in an increasing phase without ruin en route) is equal to the distribution of $D^{b}(b,\delta)$. Using this fact and a binomial expansion of the term $(\overline{a}_{\overline{\tau_{c'}(0)}|})^j$, (3.10) becomes

$$
\begin{aligned}
\left[\overrightarrow{V}^{b}_{n}(b,\delta)\right]_i &= e_i\overrightarrow{V}_{n,c'}(0,\delta) + \sum_{j=1}^{n-1}\binom{n}{j}\left(\frac{d}{\delta}\right)^j \\
&\quad \sum_{k=0}^{j}\binom{j}{k}(-1)^k E_i\left[e^{-(n+k-j)\delta\tau_{c'}(0)}e^{-(n-j)\delta\ {}^{b}\tau^{r}_{c}(0)}\left(D^{b}_{*}(b,\delta)\right)^{n-j}\right] \\
&+ e_i\Psi_{c'}\left(\frac{n\delta}{2}\right)\ {}^{b}\Psi^{r}_{c}\left(\frac{n\delta}{2}\right)\overrightarrow{V}^{b}_{n}(b,\delta) \\
&= e_i\overrightarrow{V}_{n,c'}(0,\delta) + \sum_{j=1}^{n-1}\binom{n}{j}\left(\frac{d}{\delta}\right)^j \\
&\quad \sum_{k=0}^{j}\binom{j}{k}(-1)^k e_i\ \Psi_{c'}\left(\frac{(n+k-j)\,\delta}{2}\right)\ {}^{b}\Psi^{r}_{c}\left(\frac{(n-j)\,\delta}{2}\right)\overrightarrow{V}^{b}_{n-j}(b,\delta) \\
&+ e_i\Psi_{c'}\left(\frac{n\delta}{2}\right)\ {}^{b}\Psi^{r}_{c}\left(\frac{n\delta}{2}\right)\overrightarrow{V}^{b}_{n}(b,\delta),
\end{aligned}
$$

from which one deduces

$$
\begin{aligned}
\overrightarrow{V}^{b}_{n}(b,\delta) &= \left(I_1 - \Psi_{c'}\left(\frac{n\delta}{2}\right)\ {}^{b}\Psi^{r}_{c}\left(\frac{n\delta}{2}\right)\right)^{-1} \cdot \overrightarrow{V}_{n,c'}(0,\delta) \\
&+ \left(I_1 - \Psi_{c'}\left(\frac{n\delta}{2}\right)\ {}^{b}\Psi^{r}_{c}\left(\frac{n\delta}{2}\right)\right)^{-1} \cdot \sum_{j=1}^{n-1}\binom{n}{j}\left(\frac{d}{\delta}\right)^{j} \\
&\quad \sum_{k=0}^{j}\binom{j}{k}(-1)^{k}\ \Psi_{c'}\left(\frac{(n+k-j)\,\delta}{2}\right)\ {}^{b}\Psi^{r}_{c}\left(\frac{(n-j)\,\delta}{2}\right)\overrightarrow{V}^{b}_{n-j}(b,\delta).
\end{aligned}
$$

Finally, the starting point (3.7) was derived in Badescu et al. (2007) and can easily be obtained from (3.6) with $n = 1$. □

From Proposition 3.2, one concludes that the calculation of the moments of the r.v. $D_{c'}(b,\delta)$ will have to be performed recursively with respect to the order of the moments (starting with the first moment and incrementing its order n at each iteration). In the next theorem, an expression for the higher order moments of the discounted dividends is given for an arbitrary initial capital $u \geq 0$.

Theorem 3.1. *For the surplus process (1.1), the n-th conditional moment of the discounted dividend payments prior to ruin ($n = 1, 2, \ldots$) is given by*

$$
\overrightarrow{V}^{b}_{n}(u,\delta) = \begin{cases}
e^{-\frac{n\delta}{2}\frac{b-u}{c}}\ {}_{0}\widehat{f}_{11,c}\left(u,b,n\frac{\delta}{2}\right)\overrightarrow{V}^{b}_{n}(b,\delta), & u<b \\
\overrightarrow{V}_{n,c'}(u-b,\delta) & \\
+\sum_{j=0}^{n-1}\binom{n}{j}\left(\frac{d}{\delta}\right)^{j}\sum_{k=0}^{j}\binom{j}{k}(-1)^{k}\, e^{\frac{(n+k-j)\delta}{2}\left(\frac{u-b}{c'}\right)} & \\
\Psi_{c'}\left(\frac{\delta(n+k-j)}{2}\right) & \\
e^{H_{c'}\left(\frac{\delta(n+k-j)}{2}(u-b)\right)}\ {}^{b}\Psi^{r}_{c}\left(\frac{(n-j)\delta}{2}\right)\overrightarrow{V}^{b}_{n-j}(b,\delta), & u\geq b
\end{cases} \tag{3.11}
$$

where ${}_{0}\widehat{f}_{11,c}(x,y,\delta)$ represents the Laplace transform of the first passage of the fluid flow process $\mathcal{F}_c$ from level x in S_1 to level y in S_1 avoiding level 0 en route and $\overrightarrow{V}^{b}_{n}(b,\delta)$ ($n = 1, 2, \ldots$) are obtained from Proposition 3.2.

Proof. For $u < b$, dividends will be paid only for those sample paths reaching level b without a visit to level 0 en route. The Laplace transform of this first passage time in the risk process is $e^{-\frac{n\delta}{2}\frac{b-u}{c}}\ {}_{0}\widehat{f}_{11,c}\left(u,b,n\frac{\delta}{2}\right)$ (see Ramaswami (2006)). Now at level b, $\overrightarrow{V}^{b}_{n}(b,\delta)$ corresponds to the n-th moment of the future discounted dividends. For $u \geq b$, a decomposition of the discounted dividend payments $D^{b}(u,\delta)$ similar to (3.8) followed by its ensuing analysis leads to (3.11). We omit the details. □

4 Illustrations

We first consider a modified version of Example 4.1 in Albrecher et al. (2005a). The assumptions are repeated here for purposes of completeness. In the context of the Sparre Andersen risk model, we assume that the interclaim times and the claim sizes are both Erlang-$(2, 2)$ distributed. The discount rate δ is chosen at 3%. While Example 4.1 of Albrecher et al. (2005a) assumes a barrier-type model at level b, a modified version is considered here by assuming a threshold-type model at level b. As in Example 4.1 of Albrecher et al. (2005a), we set the net premium rate for surplus levels below b at $c = 1.1$. However, for surplus levels greater than b, Albrecher et al. (2005a) assumes that the total premium rate is paid as a dividend rate (i.e. $c^{'} = 0$) while, in our setup, a dividend rate of $d = 1.09$ is paid whenever the surplus level is greater than b leading to a small net premium rate of $c^{'} = 0.01$ above the threshold b). Figures 1 and 2 contain the numerical values of the first moment and the standard deviation of the discounted dividend payments prior to ruin. Note that Figures 1 and 2 consider cases where $u \leq b$ and $u > b$ as opposed to Albrecher et al. (2005a) where similar graphs are obtained for $u \leq b$ only due to the barrier-type dividend structure.

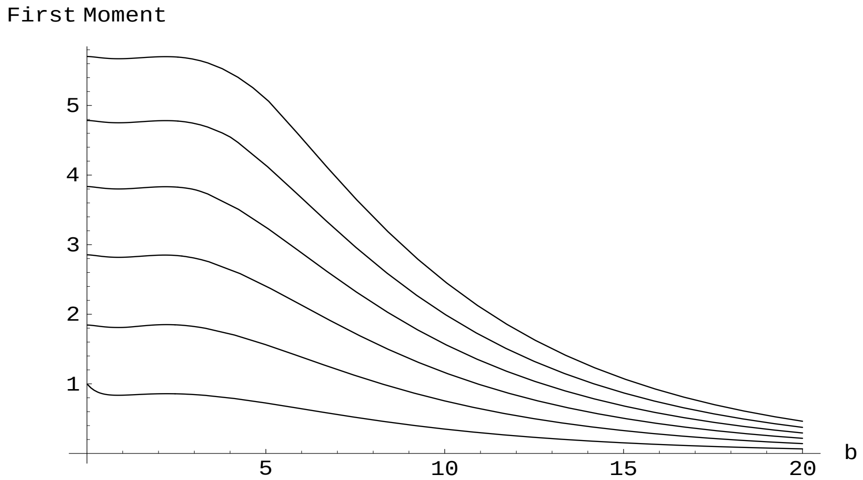

Figure 1 *The first moment of the dividends prior to ruin ($u = 0, 1, ..., 5$), from bottom to top*

A comparison of Figures 1 and 2 to those obtained by Albrecher et al. (2005a) shows that a similar behavior is observed to these two ruin related quantities for initial surplus levels below b. This can be explained by the fact that the net premium rate retained by the insurer whenever the surplus level is greater than b is relatively small ($c^{'} = 0.01$). However, an important observation in the threshold type risk model is that the derivative of the moments of the discounted dividends does not exist at $u = b$. A similar conclusion has been drawn by e.g. Lin and Sendova (2006) in the study of the Gerber-Shiu discounted penalty function in the context of the classical risk model with a threshold dividend structure.

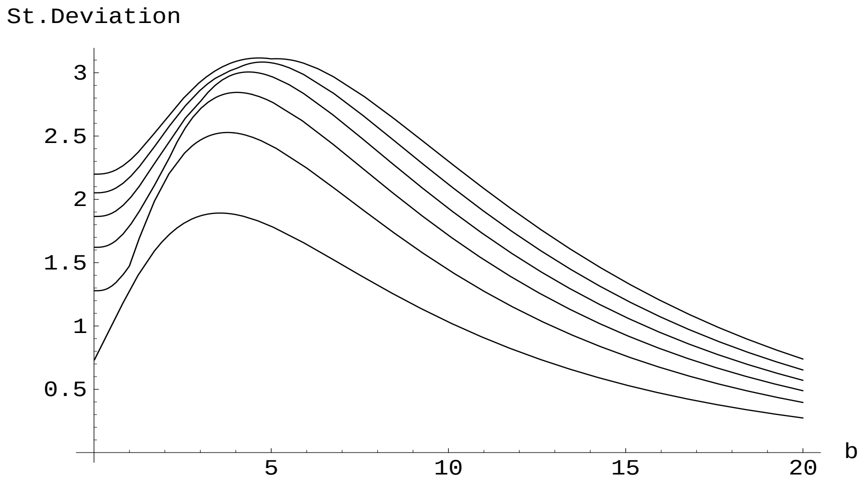

Figure 2 *The standard deviation of the dividends prior to ruin ($u = 0, 1, ..., 5$), from bottom to top*

In the second example, we consider a more complicated scenario, namely the "contagion" example first introduced in Badescu et al. (2005). This example moves away from the renewal model assumptions assuming claim amounts that are correlated with the interclaim times. In this MAP risk model, we assume the existence of a claim arrival process in which *standard* claims occur according to a Poisson process at rate $\lambda_1 = 1$ and that, during periods of contagion, *infectious* claims can also occur at Poisson rate $\lambda_2 = 10$. Standard claim sizes are exponentially distributed with mean $1/\mu_1 = 1/5$, whereas infectious claim sizes are exponentially distributed with mean $1/\mu_2 = 15/\mu_1 = 3$. The rate at which the process enters the infectious environment is $\alpha_I = 0.02$ and the return rate to the standard environment is $\alpha_S = 1$, so that in the long run, standard claims will occur with probability $\pi_S = 50/51$ (for more details see Badescu et al. (2005)).

In Figure 3, we present the first moment of the discounted dividend payments (at a discount rate $\delta = 3\%$) as a function of the threshold level b. The premium rates are $c = 1.5$ and $c' = 1$. We choose 4 different values of the initial capital, namely $u = 0,\ 10,\ 20,\ 50$.

One of the classical problems in a threshold-type risk model is to find the optimal threshold level b under different sets of constraints. In ruin theory, a standard criterion is to set b such that the expected dividend payments prior to ruin is maximized. From Figure 3, one would choose a relatively small b under such a criterion. However, a closer look at Figure 4 indicates that the standard deviation is quite significant for small values of b (when compared to larger values of b). Thus, the choice of a small threshold level b would maximize the expected discounted dividends at the cost of increasing the uncertainty (variability) in the actual sum of discounted dividends paid. This provides an example against the consideration of only the first moment as a sufficient optimization criterion. Thus, higher order moments need to be carefully investigated in order to understand the risks carried by the choice of the threshold level on the overall risk management strategy of an insurance company.

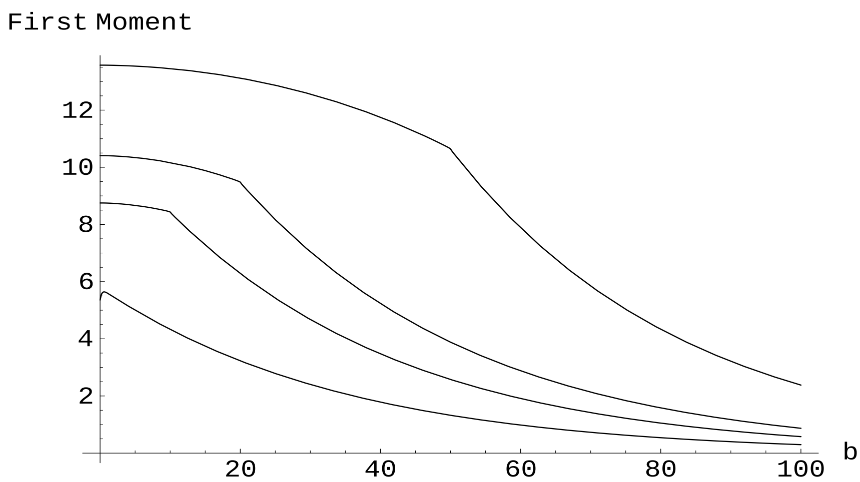

Figure 3 *The first moment of the dividends prior to ruin (*$u = 0, 10, 20, 50$*), from bottom to top*

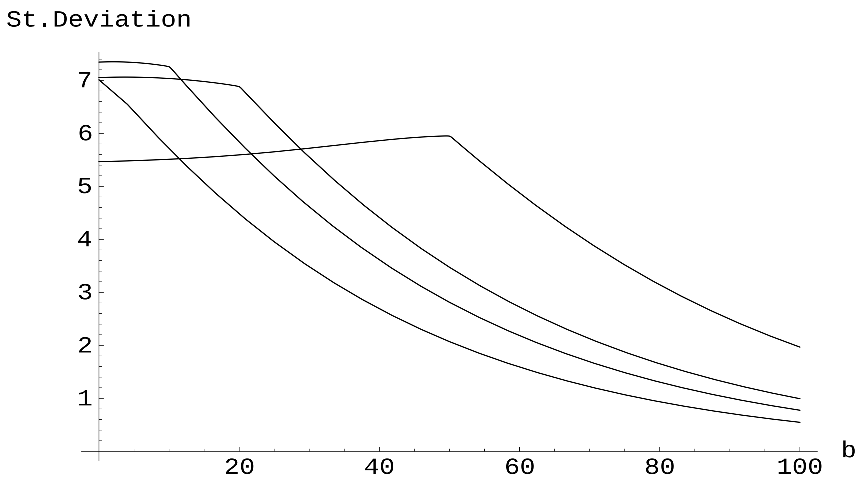

Figure 4 *The standard deviation of the dividends prior to ruin (*$u = 0, 10, 20, 50$*), from bottom to top on the right-hand side*

Acknowledgements

Both authors want to gratefully acknowledge financial support received from the Natural Sciences and Engineering Research Council of Canada (NSERC).

(Received January, 2007. Accepted April, 2007.)

References

Ahn, S., and Badescu, A. (2007). On the analysis of the Gerber-Shiu discounted penalty function for risk processes with Markovian arrivals. *Insurance: Mathematics and Economics.* In press.

Ahn, S., Badescu, A., and Ramaswami, V. (2007). Time dependent analysis of finite buffer fluid flows and risk models with a dividend barrier. *Queueing Systems*, **55**, 207–222.

Ahn, S. and Ramaswami, V. (2004). Transient analysis of fluid flow models via stochastic coupling to a queue. *Stochastic Models,* **20**, 71-101.

Ahn, S. and Ramaswami, V. (2005). Efficient algorithms for transient analysis of stochastic fluid flow models. *Journal of Applied Probability*, **42**, 531-549.

Albrecher, H., Claramunt, M., and Marmol, M. (2005a). On the distribution of dividend payments in a Sparre Andersen model with generalized Erlang(n) interclaim times. *Insurance: Mathematics and Economics*, **37**, 324-334.

Albrecher, H., Hartinger, J., and Tichy, R. (2005b). On the distribution of dividend payments and the discounted penalty function in a risk model with linear dividend barrier. *Scandinavian Actuarial Journal*, **2**, 103-126.

Albrecher, H. and Boxma, O. J. (2005c). On the discounted penalty function in a Markov-dependent risk model. *Insurance: Mathematics and Economics*, **35**, 245-254.

Asmussen, S. (2000). *Ruin Probabilities.* Singapore: World Scientific.

Badescu, A., Breuer, L., Da Silva Soares, A., Latouche, G., Remiche, M-A. and Stanford, D. (2005). Risk processes analyzed as fluid queues. *Scandinavian Actuarial Journal*, **2**, 127-141.

Badescu, A., Drekic, S., and Landriault, D. (2007). Analysis of a threshold dividend strategy for a MAP risk model. *Scandinavian Actuarial Journal.* In press.

Dickson, D. and Waters, H. (2004). Some optimal dividend problems. *Astin Bull.*, **34**, 49-74.

Latouche, G. and Ramaswami, V. (1993). A logarithmic reduction algorithm for Quasi-Birth-and-Death processes, *J. Appl. Prob.,* **30**, 650-674.

Lin, X. S. and Pavlova, K. (2006). The compound Poisson risk model with a threshold dividend strategy. *Insurance: Mathematics and Economics*, **38**, 57-80.

Neuts, M. F. (1981). *Matrix-Geometric Solutions in Stochastic Models – An Algorithmic Approach.* The Baltimore, MD: Johns Hopkins University Press.

Ramaswasmi, R. (2006). Passage times in fluid models with application to risk processes. *Methodology and Computing in Applied Probability*, **8**, 497–515.

Andrei Badescu
Department of Statistics
University of Toronto
100 St. George St., Toronto, Canada
E-mail: badescu@utstat.toronto.edu

David Landriault
Department of Statistics and Actuarial Science
University of Waterloo
200 University Avenue West, Waterloo, Canada
E-mail: dlandria@math.uwaterloo.ca