Brazilian Journal of Probability and Statistics (2006), 20, pp. 133–140.

An improved u chart for attributes

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Abstract: We propose in this paper an improved u chart for monitoring λ, the average number of defects per inspection unit, which can achieve large improvement over the usual u chart for attributes. This chart is correct to order $O(n^{-3/2})$, where n is the sample of inspection units, and is also a better improvement over the modified u chart proposed by Chen and Cheng (1998) which is correct only to order $O(n^{-1})$. We compare both modified u charts with the usual u chart.

Key words: Cornish-Fisher expansion, normal approximation, Poisson distribution, u chart.

1 Introduction

A u-chart is an attributes control chart used with data collected in subgroups of varying sizes. U-charts show how the process, measured by the number of nonconformities per item or group of items, changes over time. Nonconformities are defects or occurrences found in the sampled subgroup. They can be described as any characteristic that is present but should not be, or any characteristic that is not present but should be. For example, a scratch, dent, bubble, blemish, missing button, and a tear are all nonconformities. U-charts are used to determine if the process is stable and predictable, as well as to monitor the effects of process improvement theories. A u-chart is particularly useful when the item is too complex to be ruled as simply conforming or nonconforming. For example, an automobile could have hundreds of possible defects, yet still not be considered defective. Finally, use u-charts for standardization. This means you should continue collecting and analyzing data throughout the process operation. If you made changes to the system and stopped collecting data, you would have only perception and opinion to tell you whether the changes actually improved the system. Without a control chart, there is no way to know if the process has changed or to identify sources of process variability.

Denote by Y the number of nonconformities in an inspection unit of product from a process. Suppose that defects or nonconformities occur in this inspection unit according to the Poisson distribution with mean equal to λ. Therefore, a control chart for nonconformities with 3-sigma limits would be given by

$$\begin{aligned} UCL &= \lambda + 3\sqrt{\lambda} \\ CL &= \lambda \\ LCL &= \lambda - 3\sqrt{\lambda}, \end{aligned} \tag{1.1}$$

where the adequacy of the normal approximation to the Poisson distribution is assumed, in adition to the appropriateness of the Poisson distribution itself, and assuming that a standard value for λ is available.

The limits (1.1) require that λ should be at least 5. Although this requirement will often be met in practice, control limits (1.1) should not automatically be used when it is met. In particular, when $5 \leq \lambda < 9$, zero would be used for the lower limit since LCL would be negative. For λ of at least moderate size, say $\lambda \geq 20$, by the Central Limit Theorem the tail areas beyond the control limits will be very close to 0.0027. However, the false alarm probabilities of falling under LCL or over UCL can also be very different from the assumed nominal normal value 0.00135, even when LCL is positive and λ is not small. For example, when $\lambda = 10$, the lower limit is approximately $LCL = 0.5$, but $P(Y = 0) = 0.000045$, which is one-thirtieth of the assumed area. If the objective was to come as close as possible to 0.00135, the lower limit would be set at 2, and the tail area would be 0.0005. Should the calculations in (1.1) yield a negative value for the LCL, set $LCL = 0$.

If no standard is given, then λ may be estimated as the observed average number of nonconformities in a preliminary sample of inspection units, say $u = Y/n$, where Y denotes the total number of defects found in a sample of n inspection units. The control chart for nonconformities is also sometimes called the u chart. The parameters of the u chart are

$$\begin{aligned} UCL &= \bar{y} + 3\sqrt{\tfrac{\bar{y}}{n}} \\ CL &= \bar{y} \\ LCL &= \bar{y} - 3\sqrt{\tfrac{\bar{y}}{n}}, \end{aligned} \tag{1.2}$$

where $\bar{y}$ represents the observed average number of nonconformities per unit in a preliminary set of data. The limits (1.2) are almost universally used in conjunction with control charts.

We know that for small to moderate λ, the LCL of the u chart can become ineffective and therefore fail to be a possible indicator of a reduction in λ. To overcome this, some alternative charts have been considered in the literature, including the charts based on variance stabilizing transformations, the charts based on discrete probability integral transformations (the Q charts, see Quesenberry, 1991), and the charts with probability control limits, i.e., the exact charts in the sense that their false alarm probabilities are as close to 0.00135 as the discreteness of the Poisson distribution allows. Chen and Cheng (1998) review these charts and propose a modified u chart obtained from adjusting the control limits of the usual u chart by terms of order n^{-1}, where n is the size of the sample of inspection units.

In this paper, we generalize the results by Chen and Cheng (1998) by including terms of order $n^{-3/2}$ in an improved u chart. We also show that with the further adjustment to the control limits of the Chen and Cheng's (1998) modified u chart, one can achieve better improvement which is almost equivalent to using the exact Poisson distribution to design a chart.

On using the well established Cornish-Fisher expansion of quantiles to obtain better normal approximation for the Poisson distribution, we define in Section 2 an improved u chart by including terms of order $O(n^{-3/2})$. Some comparisons of this chart with the modified u chart proposed by Chen and Cheng (1998) are made in Section 3 with an example. Finally, in Section 4, we compare both modified u charts and the usual u chart through simulation studies. Based on these simulations, our proposed u chart is proved to be more sensitive to detecting changes is the process than Chen and Cheng's modified u chart.

2 An improved u chart

Let $U = Y/n$ denote the average number of defects found in a sample of n inspection units. We obtain the following central moments of U $\mu_2 = E\{(U-\lambda)^2\} = \lambda/n$, $\mu_3 = E\{(U-\lambda)^3\} = \lambda/n^2$ and $\mu_4 = E\{(U-\lambda)^4\} = \lambda(3n\lambda+1)/n^3$, and let z_α denote the αth quantile of the standard normal distribution. Then, the αth quantile of U, denoted by u_α, is obtained from the Cornish-Fisher expansion up to order $1/n$

$$\frac{u_\alpha - \lambda}{\sqrt{\lambda/n}} = z_\alpha + \frac{1}{6\sqrt{\lambda n}}\left(z_\alpha^2 - 1\right) - \frac{1}{36\lambda n}\left(2z_\alpha^3 - 5z_\alpha\right) + \frac{1}{24\lambda n}\left(z_\alpha^3 - 3z_\alpha\right).$$

Then, it follows immediately that

$$u_\alpha = \lambda + z_\alpha\sqrt{\frac{\lambda}{n}} + \frac{1}{6n}\left(z_\alpha^2 - 1\right) + \frac{z_\alpha\left(1 - z_\alpha^2\right)}{72n\sqrt{\lambda n}}. \tag{2.1}$$

Equation (2.1) generalizes formula (2) given by Chen and Cheng (1998) which includes only the term of order n^{-1}. We call the chart based on equation (2.1) "the proposed u chart". Chen and Cheng's chart, which is based on the first three terms of equation (2.1), is called here "the modified u chart". When λ is unknown, we have to estimate λ for determining the probability control limits for the average number of defects in the ith sample $u_i = y_i/n_i$ by $\hat{\lambda}_{i-1} = \sum_{j=1}^{i-1} y_j / \sum_{j=1}^{i-1} n_j$, where y_i denotes the number of defects in the ith sample of size n_i. We then have the following charts:

(i) The modified u chart when λ is known

For $i = 1, 2, ...$ plot u_i on a chart with variable control limits at $UCL = \lambda + 3\sqrt{\frac{\lambda}{n_i}} + \frac{4}{3n_i}$, $CL = \lambda$, $LCL = \lambda - 3\sqrt{\frac{\lambda}{n_i}} + \frac{4}{3n_i}$.

(ii) The proposed u chart when λ is known

For $i = 1, 2, ...$ plot u_i on a chart with variable control limits at $UCL = \lambda + 3\sqrt{\frac{\lambda}{n_i}} + \frac{4}{3n_i} - \frac{1}{3n_i\sqrt{\lambda n_i}}$, $CL = \lambda$, $LCL = \lambda - 3\sqrt{\frac{\lambda}{n_i}} + \frac{4}{3n_i} - \frac{1}{3n_i\sqrt{\lambda n_i}}$.

(iii) The modified u chart when λ is unknown

For $i = 2, 3, ...$ plot u_i on a chart with variable control limits at $UCL = \hat{\lambda}_{i-1} + 3\sqrt{\frac{\hat{\lambda}_{i-1}}{n_i}} + \frac{4}{3n_i}$, $CL = \hat{\lambda}_{i-1}$, $LCL = \hat{\lambda}_{i-1} - 3\sqrt{\frac{\hat{\lambda}_{i-1}}{n_i}} + \frac{4}{3n_i}$.

(iv) The proposed u chart when λ is unknown

For $i = 2, 3, ...$ plot u_i on a chart with variable control limits at $UCL = \hat{\lambda}_{i-1} + 3\sqrt{\frac{\hat{\lambda}_{i-1}}{n_i}} + \frac{4}{3n_i} - \frac{1}{3n_i\sqrt{\hat{\lambda}_{i-1}n_i}}$, $CL = \hat{\lambda}_{i-1}$, $LCL = \hat{\lambda}_{i-1} - 3\sqrt{\frac{\hat{\lambda}_{i-1}}{n_i}} + \frac{4}{3n_i} - \frac{1}{3n_i\sqrt{\hat{\lambda}_{i-1}n_i}}$.

3 Performance of the proposed and modified u charts

We have conducted an example to study the performance of the proposed and modified u charts. Consider a process by which moonroofs are installed in automobiles to detect several types of critical defects: (1) wind noise, (2) water leaks, (3) binding during retraction, and (4) squeaks and rattles. Table 1 gives 34 samples of data (De Vor et al., 1992) for this process. Figure 1 displays the absolute value of the difference (delta) between the modified and proposed charts versus n_i, where λ is taken as the average number 1.4 of defects per moonroof. As expected, the difference is larger for small sample sizes. For samples of size less than 5, the improvement of the proposed chart over the modified chart can be more substantial.

4 Simulation results

We study the performance of the modified and proposed u charts to detect changes in λ. For doing this, we suppose that a control chart is set up when $\lambda = \lambda_1$, and after c observations are taken, the underlying λ changes to $\lambda = \lambda_2$. If k is the number of observations taken immediately after the change in λ occurs, let $A_{c,\lambda_1,k,\lambda_2}$ ($B_{c,\lambda_1,k,\lambda_2}$) denote the event that there is at least one U_i under LCL (over UCL) when the k observations are plotted. Thus, the values of the probabilities $P(A_{c,\lambda_1,k,\lambda_2})$ and $P(B_{c,\lambda_1,k,\lambda_2})$ indicate how quickly the charts can detect a decrease and an increase in λ, respectively. These probabilities are estimated through simulation.

Table 1 *Defect data for the moonroof installation example.*

i	y_i	n_i	u_i	i	y_i	n_i	u_i
1	23	16	1.43	18	24	22	1.09
2	30	20	1.50	19	22	14	1.57
3	35	26	1.35	20	17	16	1.06
4	12	8	1.50	21	33	22	1.50
5	29	22	1.32	22	21	16	1.31
6	35	29	1.21	23	18	14	1.29
7	50	31	1.61	24	9	5	1.80
8	15	13	1.15	25	18	13	1.38
9	36	28	1.29	26	26	19	1.37
10	38	23	1.65	27	12	10	1.20
11	24	19	1.26	28	8	10	0.80
12	32	23	1.39	29	14	14	1.00
13	24	14	1.71	30	8	11	0.73
14	34	29	1.17	31	14	29	0.48
15	38	27	1.41	32	7	19	0.37
16	25	15	1.67	33	12	19	0.63
17	26	22	1.18	34	25	45	0.56

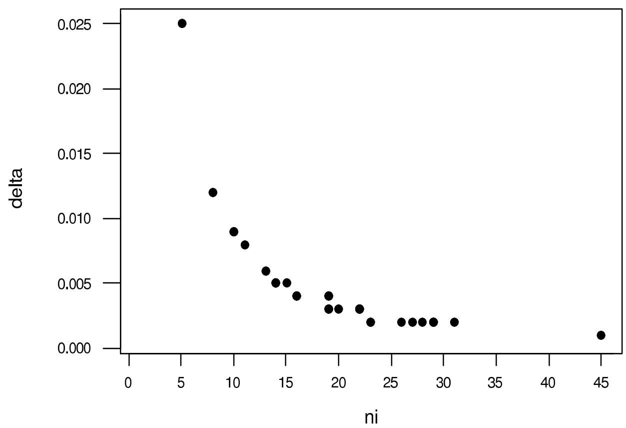

Figure 1 *Difference (delta) between the modified and proposed u charts versus n_i.*

We simulate c observations from $f(y; n_i\lambda_1)$ and k observations from $f(y; n_i\lambda_2)$, where $f(y; \lambda)$ represents the probability function of a Poisson distribution with mean λ. This process is repeated 10,000 times, and $P(A_{c,\lambda_1,k,\lambda_2})$ and $P(B_{c,\lambda_1,k,\lambda_2})$ are estimated as the proportion of times a decrease and an increase in λ is signaled on the points indexed by $c+1, c+2, ..., c+k$ for the first time, respectively. We fix $k = 20$ and $n_i = 1$. Table 2 shows these probabilities for $\lambda_1 = 10, \lambda_2 = 6$ and $c = 1, 5, 10, 20, 30$ and 50. Tables 3-6 do the same for $\lambda_1 = 10, \lambda_2 = 8, \lambda_1 = \lambda_2 = 10, \lambda_1 = 10, \lambda_2 = 12$ and $\lambda_1 = 10, \lambda_2 = 14$, respectively. All simulations were done using the MATLAB software. We see from the figures in Tables 2-6 that our proposed u chart is more sensitive in signaling decreases and increases in λ than the modified u chart. In fact, the estimated false alarm probabilities for the proposed u chart are greater than the corresponding estimated probabilities for the modified u chart. Overall, our proposed u chart seems more effective to detect changes in λ. Finally, we want to mention that the Cornish-Fisher expansion (2.1) used in the proposed u chart was to order $n^{-3/2}$. Further research should be directed to investigate if the inclusion of higher order terms would result an u chart even more sensitive.

Table 2 *A comparison of* $P\left(A_{c,\lambda_1,k,\lambda_2}\right)$ *and* $P\left(B_{c,\lambda_1,k,\lambda_2}\right)$.

		$P\left(A_{c,\lambda_1,k,\lambda_2}\right)$			$P\left(B_{c,\lambda_1,k,\lambda_2}\right)$		
λ	c	usual	mod.	prop.	usual	mod.	prop.
$\lambda_1 = 10$	1	0.0341	0.1308	0.1502	0.0814	0.0278	0.0313
$\lambda_2 = 6$	5	0.0155	0.1514	0.1750	0.0153	0.0037	0.0044
	10	0.0153	0.1920	0.2264	0.0065	0.0016	0.0019
	20	0.0210	0.2440	0.2850	0.0040	0.0010	0.0010
	30	0.0330	0.3070	0.3390	0.0020	0.0010	0.0010
	50	0.0373	0.3257	0.3580	0.0004	0	0

Table 3 *A comparison of* $P\left(A_{c,\lambda_1,k,\lambda_2}\right)$ *and* $P\left(B_{c,\lambda_1,k,\lambda_2}\right)$.

		$P\left(A_{c,\lambda_1,k,\lambda_2}\right)$			$P\left(B_{c,\lambda_1,k,\lambda_2}\right)$		
λ	c	usual	mod.	prop.	usual	mod.	prop.
$\lambda_1 = 10$	1	0.0120	0.0674	0.0800	0.1213	0.0492	0.0539
$\lambda_2 = 8$	5	0.0050	0.0659	0.0770	0.0495	0.0173	0.0189
	10	0.0040	0.0708	0.0823	0.0291	0.0100	0.0109
	20	0.0050	0.0760	0.0900	0.0160	0.0050	0.0070
	30	0.0110	0.0720	0.0900	0.0110	0.0010	0.0010
	50	0.0058	0.0785	0.0960	0.0092	0.0030	0.0031

Table 4 *A comparison of* $P\left(A_{c,\lambda_1,k,\lambda_2}\right)$ *and* $P\left(B_{c,\lambda_1,k,\lambda_2}\right)$.

		$P\left(A_{c,\lambda_1,k,\lambda_2}\right)$			$P\left(B_{c,\lambda_1,k,\lambda_2}\right)$		
λ	c	usual	mod.	prop.	usual	mod.	prop.
$\lambda_1 = 10$	1	0.0069	0.0428	0.0497	0.1744	0.0836	0.0911
$\lambda_2 = 10$	5	0.0023	0.0304	0.0352	0.1017	0.0409	0.0440
	10	0.0017	0.0283	0.0334	0.0954	0.0348	0.0381
	20	0.0010	0.0220	0.0280	0.0930	0.0150	0.0200
	30	0.0030	0.0200	0.0230	0.0660	0.0270	0.0300
	50	0.0008	0.0199	0.0285	0.0756	0.0254	0.0283

Table 5 *A comparison of* $P\left(A_{c,\lambda_1,k,\lambda_2}\right)$ *and* $P\left(B_{c,\lambda_1,k,\lambda_2}\right)$.

		$P\left(A_{c,\lambda_1,k,\lambda_2}\right)$			$P\left(B_{c,\lambda_1,k,\lambda_2}\right)$		
λ	c	usual	mod.	prop.	usual	mod.	prop.
$\lambda_1 = 10$	1	0.0049	0.0286	0.0328	0.2484	0.1351	0.1432
$\lambda_2 = 12$	5	0.0015	0.0144	0.0169	0.2169	0.1060	0.1121
	10	0.0008	0.0109	0.0132	0.2321	0.1092	0.1152
	20	0.0010	0.0070	0.0090	0.2710	0.1300	0.1390
	30	0	0.0060	0.0060	0.3260	0.1630	0.1680
	50	0.0002	0.0061	0.0079	0.2993	0.1473	0.1572

Table 6 *A comparison of* $P\left(A_{c,\lambda_1,k,\lambda_2}\right)$ *and* $P\left(B_{c,\lambda_1,k,\lambda_2}\right)$.

		$P\left(A_{c,\lambda_1,k,\lambda_2}\right)$			$P\left(B_{c,\lambda_1,k,\lambda_2}\right)$		
λ	c	usual	mod.	prop.	usual	mod.	prop.
$\lambda_1 = 10$	1	0.0041	0.0234	0.0256	0.3439	0.2247	0.2346
$\lambda_2 = 14$	5	0.0010	0.0093	0.0110	0.3856	0.2252	0.2372
	10	0.0004	0.0039	0.0050	0.4646	0.2733	0.2862
	20	0.0010	0.0050	0.0060	0.5800	0.3490	0.3640
	30	0.0010	0.0020	0.0020	0.6090	0.3650	0.3860
	50	0.0003	0.0011	0.0014	0.6750	0.4339	0.4513

5 Conclusions

We proposed a modified u chart for monitoring the average number of defects found in a sample of n inspection units which is corrected to order $O(n^{-3/2})$. It is not possible to determine for which sample sizes there is a substantial improvement using the modified chart. However, the modified chart offers always some improvement over the usual charts. The modified chart also does not detect well an increase in λ although it could be useful in this situation. We have not investigated alternative Bayesian methods for this problem. This should be done in future research. Some simulations showed the usefulness of the proposed u chart.

Acknowledgements

We would like to thank two referees for a careful reading and useful reports. We would also like to thank the Editor and Executive Editor for helpful suggestions.

(Received September, 2004. Accepted June, 2006.)

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