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On the values of optimal stopping problems for generalized averages of discrete random variables

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Abstract: We study optimal stopping problems for generalized averages of identically distributed discrete random variables, taking values in a finite set denoted by D. We obtain a recurrence formula in the finite horizon case, which confers the value of the game in terms of associated problems of smaller horizon. We deduce analogous formulas in the infinite horizon case, and introduce a series of applications to the study of properties concerning the value as a function of the parameters. A sequence of theoretical bounds for the values is obtained, and it is proved that it converges to the true value of the game. The Bernoulli case is presented in detail.

Key words: Bernoulli variables, bounding sequence, generalized averages, optimal stopping problems, recurrence relations, value of optimal stopping games.

1 Introduction

In a space $(\Omega, \mathcal{A}, P)$ we consider a sequence $X_1, \ldots, X_n, \ldots$ of independent variables with a common law G, concentrated in a finite set D. We define the filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$ generated by our process: $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. We denote $S_n = X_1 + \cdots + X_n$.

The symbol $\mathcal{T}^N$ will denote the set of stopping times T satisfying $T \leq N$, while $\mathcal{T}$ will denote the set of finite stopping times.

We shall be dealing with wealth processes which are generalized averages of the sums S_n, those are processes defined by parameters $a \in \mathbb{R}$ and $m \in \mathbb{N}$, in the following way:

$$W_n = \frac{a + S_n}{m + n}.$$

We call a the "numerator parameter" and m the "denominator parameter". Given a and m, for each integer $N \geq 1$ we shall study the following optimal stopping values

$$V_m^N(a) = V^N(a, m) = \sup_{T\in\mathcal{T}^N} E(W_T), \quad V_m(a) = V(a, m) = \sup_{T\in\mathcal{T}} E(W_T).$$

The first one is the value of the optimal stopping game with finite horizon, the second one is the corresponding value of infinite horizon. We also define the transformed values

$$\overline{V}_m^N(a) = \overline{V}^N(a,m) = \begin{cases} V^N(a,m) - \frac{a}{m} & \text{for } m \neq 0 \\ V^N(a,0) & \text{for } m = 0, \end{cases}$$

and similarly for infinite horizon. When there is no risk of confusion, we shall write V^N for $V^N(a,m)$ and V for $V(a,m)$.

In the case $a = m = 0$, the problem is well known, and will be referred as *arithmetic means*, treated for instance by Chow et al. (1965, 1991), and also by Dalang (1996). We shall often use *GA problems* to refer to these problems.

These parametric problems have been widely studied, under different names. Chow and Robbins (1965), study the case of centered binomial variables, which is later extended by Dvoretzky (1967) to case of variables with finite second moment. The work of Shepp (1969) deserves to be mentioned in connection to this. These authors focus their attention on the problem of the existence of optimal stopping times in the infinite horizon case, which is solved in great generality by the former. He denotes

$$\mathcal{C} = \{T \in \mathcal{T} : \text{ there are } c_1, c_2, \ldots \text{ such that } T = \inf\{n : S_n \geq c_n\}\}$$

(the class of first passage times), and shows the following:

Lemma 1.1 *There exists $T \in \mathcal{C}$, which is optimal for the GA problem, and it is defined by:*

$$T = \inf\left\{n : \overline{V}_{n+m}(a + S_n) \leq 0\right\}.$$

Moreover, the set $\left\{x : \overline{V}_m(x) \leq 0\right\}$ is an interval of the form $[c, +\infty[$.

Though these results give a satisfactory solution to the infinite horizon case, they do not offer enough information about the values V^N and V, even in the simplest case of aritmetic means of binomial variables. In this case, Chow and Robbins (1965) offer an algorithm for computing the exact values in finite horizon case, and approximating from below the infinite horizon value. However, there is no upper approximation.

On the other hand, the process W is Markovian but not homogeneous, which refrains us from applying directly the classical theory of Markov homogeneous processes (see Chow et al. (1991)), giving recurrence formulas for generating V^N.

In this work, we offer an answer to the above stated questions. We shall indicate how to proceed to find the exact values in finite horizon, and then how to find upper bounds for the infinite horizon case. As a consequence, it is possible to nest the infinite horizon values in intervals of arbitrarily small length, solving in this way a problem stated in Chow et al. (1991) (Example 2 of Introduction).

To get to this, we first prove that the values of the games associated to different parameters, are related by a recurrent functional relation (Theorems 2 and 4), a fundamental result in our work. Some preliminary results on stopping times appear in the second section. Besides the approximations, we shall give a result on the existence of optimal times in infinite horizon (Theorem 5), and also on the limiting value of $V_m(a)$ when $m \to \infty$ (theorem 9).

In this paper the space $(\Omega, \mathcal{A}, P)$ will be taken as the canonical $\Omega = D^{\mathbb{N}}$, $P = \bigotimes_{i=1}^{\infty} G$, with X_n being the nth coordinate function in Ω.

2 Preliminaries

We consider the shift operator in the canonical space Ω:

$$\begin{array}{cccc} \theta & : & \Omega & \to & \Omega \\ & & (x_1, x_2, \ldots) & \mapsto & (x_2, x_3, \ldots) \end{array}$$

as well as its iterates θ^p :

$$\begin{array}{cccc} \theta^p & : & \Omega & \to & \Omega \\ & & (x_1, x_2, \ldots) & \mapsto & (x_{p+1}, x_{p+2}, \ldots) . \end{array}$$

In what follows, all stopping times will be relative to the filtration $\mathcal{F}$ defined above. An elementary property of the shift operator (specific to the filtration $\mathcal{F}$) is the following:

$$(\theta^p)^{-1}(\mathcal{F}_n) \subseteq \mathcal{F}_{n+p}, \tag{2.1}$$

in the sense of inverse images of sets.

2.1 Decomposition of stopping times

The following lemmas introduce some kind of *collage* and *splitting* operations on stopping times.

Lemma 2.1 *Let $(A_\lambda)_{\lambda \in L}$ be a partition of Ω by events of $\mathcal{F}_p$. Suppose given a family of variables $(R_\lambda)_{\lambda \in L}$, such that each R_λ is identically 0 or a finite stopping time. Then the variable T defined on each A_λ by $T = p + R_\lambda \circ \theta^p$, is a stopping time satisfying $T \geq p$.*

Proof. As $T \geq p$, it is enough to prove that $\{T = k\} \in \mathcal{F}_k$ for $k \geq p$. If $k = p$ we have

$$\{T = p\} = \bigcup_{\lambda \in L; R_\lambda = 0} A_\lambda \in \mathcal{F}_p,$$

since $A_\lambda \in \mathcal{F}_p$ for each λ. If $k = n + p$, with $n \geq 1$ it follows that

$$\{T = k\} = \bigcup_{\lambda \in L; R_\lambda \neq 0} \{R_\lambda \circ \theta^p = n\} \cap A_\lambda.$$

But $\{R_\lambda \circ \theta^p = n\} = (\theta^p)^{-1}\{R_\lambda = n\} \in \mathcal{F}_{n+p}$, by property (2.1) and the fact that $\{R_\lambda = n\} \in \mathcal{F}_n$. Therefore, $\{R_\lambda \circ \theta^p = n\} \cap A_\lambda \in \mathcal{F}_{n+p}$, which shows that $\{T = k\} \in \mathcal{F}_{n+p} = \mathcal{F}_k$. □

Lemma 2.2 *Let T be a stopping time such that $T \geq p$. Then in each atom of $\{T > p\}$ there exists a unique stopping time T' such that $T = p + T' \circ \theta^p$ in that atom. If $T \in \mathcal{T}^N$, then $T' \in \mathcal{T}^{N-p}$.*

A general version of this result (for not necessarily finite filtrations) was obtained by Courrege-Priouret, as referred by Dellacherie et Meyer (1978). For a case analogous to the present one, refer to the book of Maitra (1996).

Definition 2.1 *For $\mathcal{K} \subseteq D$ we define the class $\mathcal{T}^N(\mathcal{K})$ of stopping times by:*

$$\mathcal{T}^N(\mathcal{K}) = \left\{T \in \mathcal{T}^N : \{T > 1\} = \{X_1 \in \mathcal{K}\}\right\}.$$

We shall apply the above results to the case $p = 1$. Let $\mathcal{K} \subseteq D$, and $T \in \mathcal{T}^N(\mathcal{K})$. Let $k = card(\mathcal{K})$, and ρ a one-to-one mapping from $\mathcal{K}$ onto $\{1, \ldots, k\}$. According to Lemma 2.2, for each $d \in \mathcal{K}$ there exists a stopping time $\mathcal{T}^{N-1}$ representing T in the atom $\{X_1 = d\}$, and we denote it by $T_{\rho(d)}$. Now let us define the mapping:

$$\begin{array}{rccc} \varphi_{N,\mathcal{K}}: & \mathcal{T}^N(\mathcal{K}) & \rightarrow & \bigotimes_{i=1}^{k} \mathcal{T}^{N-1} \\ & T & \mapsto & (T_1, T_2, \ldots, T_k). \end{array}$$

Theorem 1 *The mapping $\varphi_{N,\mathcal{K}}$ is one-to-one and onto.*

Proof. During the proof we will write φ instead of $\varphi_{N,\mathcal{K}}$.

1. φ is one-to-one.

In fact, given $T, U \in \mathcal{T}^N(\mathcal{K})$ and $\varphi(T) = \varphi(U)$, it follows by definition of φ that in each atom $\{X_1 = d\}$ of $\{T > 1\}$ we have $T = 1+T_{\rho(d)}\circ\theta = 1+U_{\rho(d)}\circ\theta = U$. This shows that $T = U$ in $\{T > 1\}$, and since $\{T = 1\} = \{U = 1\}$, it follows that $T = U$ all over Ω.

2. φ is onto

Given $\overline{T}_1, \overline{T}_2, \ldots, \overline{T}_k \in \mathcal{T}^{N-1}$, we define

$$T = \begin{cases} 1 & \text{on} \quad \{X_1 \notin \mathcal{K}\} \\ 1 + \overline{T}_{\rho(d)} \circ \theta & \text{on} \quad \{X_1 = d\}, \quad \text{for } d \in K \end{cases}$$

By Lemma 2.1, T is a stopping time that satisfies $N \geq T \geq p$, and $\{T > 1\} = \{X_1 \in \mathcal{K}\}$. Due to the uniqueness part in that lemma, it is clear that $T_{\rho(d)} = \overline{T}_{\rho(d)}$ for each $d \in \mathcal{K}$. This shows that $\varphi(T) = \left(\overline{T}_1, \overline{T}_2, \ldots, \overline{T}_k\right)$. □

2.2 General properties of value functions

The following results resume some important properties of values as functions of the parameter a.

Lemma 2.3 *For m and N fixed, the functions V_m and V_m^N are convex and increasing, strictly increasing in the case of V_m^N. If $m > 0$ the functions $\overline{V}_m$ and $\overline{V}_m^N$ are convex and strictly decreasing.*

Proof. It is not difficult to verify that $a \to W_T(a, m)$ is a strictly increasing mapping for all time T, and this implies that $V_m(\cdot)$ is increasing and $V_m^N(\cdot)$ is strictly increasing. The convexity is a consequence of the fact that the values of the game are supreme of affine functions (Roberts, 1973, may be consulted). Finally, for $\varepsilon > 0$ and $T \in \mathcal{T}$ we have

$$E\left(\frac{a+\varepsilon+S_T}{m+T}\right) - \frac{a+\varepsilon}{m} \le E\left(\frac{a+S_T}{m+T}\right) - \frac{a}{m} - \frac{\varepsilon}{m(m+1)},$$

from which we obtain that both $\overline{V}_m$ $\overline{V}_m^N$ are strictly decreasing. □

For the following result, remember that a function f is contractive in an interval I, with contraction constant k, if for $x, y \in I$ we have $|f(x) - f(y)| \le k|x-y|$.

Lemma 2.4 *Consider fixed $m, N \in \mathbb{N}$. The functions V_m and V_m^N are contractive in $\mathbb{R}$, with contraction constant $1/(m+1)$.*

Proof. Just observe that for any $\varepsilon > 0$ we have

$$V_m(a+\varepsilon) = \sup_T \left(E\left(\frac{a+S_T}{m+T}\right) + E\left(\frac{\varepsilon}{m+T}\right)\right) \le V_m(a) + \frac{\varepsilon}{m+1}. \square$$

Despite its apparent simplicity, the following lemma will be extremely useful.

Lemma 2.5 *Let $d_1 = \max D$. For each $N \ge 1$ and $m \ge 1$ the sets*

$$\left\{a \in \mathbb{R} : \overline{V}_m^N(a) \le 0\right\} \quad \textit{and} \quad \left\{a \in \mathbb{R} : \overline{V}_m(a) \le 0\right\}$$

are nonempty intervals of the form $[c, +\infty[$ and contain md_1.

Proof. For each finite stopping time T we have

$$\frac{a+S_T}{m+T} - \frac{a}{m} = \frac{mS_T - aT}{(m+T)m} \le \frac{(md_1 - a)T}{m(m+T)},$$

and therefore $E(W_T(a,m)) - \frac{a}{m} \le 0$ for any $a \ge md_1$. Due to the convexity of $\overline{V}_m$ and $\overline{V}_m^N$, the result follows. □

3 Recurrence relations for V and V^N

3.1 Finite horizon case

We will denote $\mu = E(X_1)$. It is evident from the definition that

$$V^1(a,m) = \frac{a+\mu}{m+1}.$$

This fact together with next theorem, allows us to compute recursively the value of the game in the finite horizon case.

Let us consider $N > 1$, and denote $D_d = D \cap]-\infty, d]$. Notice that $\mathcal{C}^N$ is the disjoint union of the classes

$$\mathcal{C}_d^N = \mathcal{C}^N \cap \mathcal{T}^N(D_d), \quad d \in D \cup \{-\infty\},$$

and therefore

$$V^N = \max_{d \in \{-\infty\} \cup D} \left(\sup_{T \in \mathcal{C}_d^N} E(W_T) \right).$$

This shows that

$$V^N = \max_{d \in \{-\infty\} \cup D} \left(\sup_{T \in \mathcal{T}^N(D_d)} E(W_T) \right), \tag{3.1}$$

a very useful fact in the proof of the following theorem.

Let us denote by $G(x)$ the value of the distribution function of G in x. For each $d \in D$ we have

$$\int_{-\infty}^{d} \overline{V}^N(a+x,m)\, dG(x) = \sum_{d' \leq d} G(\{d'\}) \overline{V}^N(a+d',m).$$

Theorem 2 ***(Recurrence relation for*** V^N*)* *For $N > 1$ we have*

$$V^N(a,m) = \frac{a+\mu}{m+1} + \max_{d \in \{-\infty\} \cup D} \int_{-\infty}^{d} \overline{V}^{N-1}(a+x,m+1)\, dG(x).$$

Equivalently:

$$V^N(a,m) = \frac{a+\mu}{m+1} + \int \left(\overline{V}^{N-1}(a+x,m+1)\right)^+ dG(x), \tag{3.2}$$

and also

$$\overline{V}^N(a,m) = h(a,m) + \int \left(\overline{V}^{N-1}(a+x,m+1)\right)^+ dG(x), \tag{3.3}$$

where

$$h(a,m) = \begin{cases} \frac{a+\mu}{m+1} - \frac{a}{m} & \text{if } m \geq 1, \\ a+\mu & \text{if } m = 0. \end{cases}$$

Proof. If $T \in \mathcal{T}^N(D_d)$ we can express

$$\begin{aligned} E\left(W_T\right) &= E\left\{\left(\frac{a+X_1}{m+1}\right)\mathbf{1}_{\{X_1>d\}}+\left(\frac{a+S_T}{m+T}\right)\mathbf{1}_{\{X_1\le d\}}\right\} \\ &= \frac{a+\mu}{m+1}+\sum_{d'\le d} E\left\{\left(\frac{a+S_T}{m+T}-\frac{a+d'}{m+1}\right)\mathbf{1}_{\{X_1=d'\}}\right\}. \end{aligned}$$

For $d' \le d$ let $T_{d'}$ be the decomposition of T in $\{X_1 = d'\}$, given by Lemma 2.2. It follows that

$$\begin{aligned} E\left(\left(\frac{a+S_T}{m+T}\right)\mathbf{1}_{\{X_1=d'\}}\right) &= E\left\{E\left(\frac{a+d'+S_{T_{d'}\circ\theta}}{m+1+T_{d'}\circ\theta}\middle|\mathcal{F}_1\right)\mathbf{1}_{\{X_1=d'\}}\right\} \\ &= E\left(\frac{a+d'+S_{T_{d'}}}{m+1+T_{d'}}\right)G\left(\{d'\}\right) \end{aligned}$$

(the last identity by the Markov property), and consequently

$$E\left(W_T\right)=\frac{a+\mu}{m+1}+\sum_{d'\le d} E\left(\frac{a+d'+S_{T_{d'}}}{m+1+T_{d'}}-\frac{a+d'}{m+1}\right)G\left(\{d'\}\right). \tag{3.4}$$

By replacing the inequality:

$$E\left(\frac{a+d'+S_{T_{d'}}}{m+1+T_{d'}}\right)\le V^{N-1}(a+d',m+1),$$

in (3.4) we obtain

$$\begin{aligned} E\left(W_T\right) &\le \frac{a+\mu}{m+1}+\sum_{d'\le d}\left(V^{N-1}(a+d',m+1)-\frac{a+d'}{m+1}\right)G\left(\{d'\}\right) \quad (3.5)\\ &= \frac{a+\mu}{m+1}+\int_{-\infty}^{d}\overline{V}^{N-1}\left(a+x,m+1\right)dG(x). \end{aligned}$$

Being T arbitrary in $\mathcal{T}^N(D_d)$, this shows that the quantity

$$g(d)=\frac{a+\mu}{m+1}+\int_{-\infty}^{d}\overline{V}^{N-1}\left(a+x,m+1\right)dG(x),$$

bounds $\sup_{T\in\mathcal{T}^N(D_d)} E(W_T)$ from above. Next we construct $T \in \mathcal{T}^N(D_d)$ such that $E(W_T) = g(d)$. For this we consider, for each $d' \le d$, an optimal stopping time $T_{d'} \in \mathcal{T}^{N-1}$ for the game with parameters $a + d'$ and $m + 1$, and horizon $N - 1$. That is

$$E\left(W_{T_{d'}}\right)=V^{N-1}(a+d',m+1).$$

By Theorem 1 there exists $T \in \mathcal{T}^N(D_d)$ such that $\varphi_{N,D_d}(T) = \left(T_{\min(D)}, ..., T_d\right)$, and therefore for this time T we obtain from (3.4) that $E(W_T) = g(d)$.

Finally, from (3.1) we obtain

$$V^N = \max_{d \in \{-\infty\} \cup D} g(d).$$

Identities (3.2) and (3.3) are obtained directly by recalling that

$$\left\{x : \overline{V}^{N-1}(a+x, m+1) \leq 0\right\}$$

is an interval of the form $[c, +\infty[$. □

3.2 Case of infinite horizon

The problem of computing (or approximating) $V(a, m)$ can be addressed from finite horizon problems. In fact, thanks to the hypothesis on the distribution of X_n, the process W is of class $\mathcal{D}$, and by general theorems (Dalang, 1996):

$$V(a, m) = \lim_{N \to \infty} V^N(a, m).$$

Therefore we can find ε-optimal times for the infinite horizon problem: If T is an optimal stopping time for the problem of horizon N, and if $\left|V(a, m) - V^N(a, m)\right| < \varepsilon$, it follows that

$$|V(a, m) - E(W_T)| < \varepsilon.$$

We can also deduce a recurrence relation for $V(a, m)$. In fact, in the recurrence relation of Theorem 2 we can take the limit to get:

Theorem 3 *The infinite horizon problem satisfies*

$$V(a, m) = \frac{a+\mu}{m+1} + \int \left(\overline{V}(a+x, m+1)\right)^+ dG(x), \tag{3.6}$$

and equivalently

$$\overline{V}(a, m) = h(a, m) + \int \left(\overline{V}(a+x, m+1)\right)^+ dG(x). \tag{3.7}$$

In this relation, the value of the infinite horizon problem with parameters a and m, is expressed in terms of values of infinite horizon problems with different parameters.

3.3 Fundamental inequalities

Recurrence relations and Jensen's inequality allow us to deduce recurrent inequalities for the values of the game. In fact, thanks to Lemma 2.3 the functions $x \to \overline{V}_m(a+x)^+$ are convex, and from Jensen's inequality:

$$\int \left(\overline{V}_m(a+x)\right)^+ dG(x) \geq \left(\overline{V}_m(a+\mu)\right)^+.$$

This and the recurrent relation give:

Theorem 4 *For each $a \in \mathbb{R}$, $m \in \mathbb{N}$ and $N \in \mathbb{N}$ we have*

$$\overline{V}_m(a) \geq h\,(a, m) + \left(\overline{V}_{m+1}(a+\mu)\right)^+, \qquad \overline{V}_m^N(a) \geq h\,(a, m) + \left(\overline{V}_{m+1}^{N-1}(a+\mu)\right)^+.$$

Equivalently:

$$V_m(a) \geq \max\left(\frac{a+\mu}{m+1}, V_{m+1}(a+\mu)\right), \qquad V_m^N(a) \geq \max\left(\frac{a+\mu}{m+1}, V_{m+1}^{N-1}(a+\mu)\right).$$

The following result will be useful later.

Corollary 3.1 *If G is non-degenerated and $N > 1$, then $V_m^N(m\mu) > \mu$ and $V_m(m\mu) > \mu$. Equivalently: $\overline{V}_m^N(m\mu) > 0$ and $\overline{V}_m(m\mu) > 0$.*

Proof. The inequality $V_m^N(m\mu) \geq \mu$ derives directly from the previous theorem. To see that this inequality is strict, let $\alpha = \min D$. The hypothesis implies that $\alpha < \mu$, and therefore $h\,(m\mu + \alpha, m+1) > 0$. By the previous theorem we have then $\overline{V}_{m+1}^{N-1}(m\mu + \alpha) > 0$, and using the recurrence relation (3.2) we get to

$$V_m^N(m\mu) \geq \mu + \overline{V}_{m+1}^{N-1}(m\mu + \alpha)G(\{\alpha\}) > \mu.$$

Since $V_m(m\mu) \geq V_m^N(m\mu)$, we also obtain the inequality for infinite horizon. □

The formulas given in Theorem 2 allow us to establish some interesting results on the existence of optimal stopping times, and variations of values depending on the parameters. For instance, it is interesting that the lower bounds of theorem 4 are actually reached eventually, as the following theorem shows. We denote $d_0 = \min D$ and $d_1 = \max D$.

Theorem 5 *For $a \geq (m+1)\,d_1 - d_0$ we have*

$$\overline{V}_m^N(a) = \overline{V}_m(a) = h(a, m), \qquad V_m(a) = V_m^N(a) = \frac{a+\mu}{m+1}. \tag{3.8}$$

If $N = 1$ this is valid for all a. If $N > 1$, it is necessary that $a \geq m\mu$ ($a > m\mu$ if G is non-degenerate). The relation (3.8) is the same as to say that the problem of parameters a and m (finite or infinite horizon) admits the constant time $T = 1$ as optimal.

Proof. Let X be a variable with law G, and m any integer. By Lemma 2.5 we deduce that the variable $\overline{V}_{m+1}(a + X)$ is non-positive for $a \geq (m+1)\,d_1 - d_0$, and in particular $E\left(\overline{V}_{m+1}^+(a+X)\right) = 0$ for such values of a. From the recurrence relation (3.7) we deduce the relation (3.8) for $V_m(a)$. Then, as

$$\frac{a+\mu}{m+1} \leq V_m^N(a) \leq V_m(a),$$

we obtain the same for $V_m^N(a)$. The necessary condition results from Corollary 3.1. □

Remark. The lowest a for which (3.8) holds can depend on G (and also on N in finite horizon). But in any case we have $a \leq (m+1)\, d_1 - d_0$ and $a \geq m\mu$ ($a > m\mu$ if G is non-degenerate).

The contractivity of V_m and V_m^N can also be obtained from this result, given that these functions are convex and eventually afine. Moreover, from (3.8) it follows that the contraction constant $\frac{1}{m+1}$ obtained in Lemma 2.4 is optimal.

From the structure of optimal times given in Section 2, it is also possible to prove this result, without using the recurrence relation.

4 Theoretical bounds for the values

Recall the recurrence relations

$$\overline{V}_m^1(a) = h_m(a), \qquad \overline{V}_m^N(a) = h_m(a) + \int \overline{V}_{m+1}^{N-1}(a+x)^+\, dG(x), \tag{4.1}$$

where $V_m^N(a)$ is the payoff function of parameters a and m, and horizon N. For infinite horizon this gives

$$\overline{V}_m(a) = h_m(a) + \int \overline{V}_{m+1}(a+x)^+ dG(x).$$

We shall try to find upper bounds $C_m^N(a)$ and $C_m(a)$ for $\overline{V}_m^N(a)$ and $\overline{V}_m(a)$, respectively. That is, we want to establish

$$\overline{V}_m^N(a) \leq C_m^N(a), \qquad \overline{V}_m(a) \leq C_m(a).$$

These bounding functions should be expressed in terms of the law of the random walk. From Theorem 2, these bounds will be useful to approximate the optimal times too. In certain cases of Bernoulli type variables, we will also find a closed formula for these bounds, which will allow us an effective approximation of the payoff functions.

For each $a \in \mathbb{R}$, $k \in \mathbb{N}$, we set

$$\begin{aligned} C_m^N(a) &= h_m(a) + \sum_{k=1}^{N-1} E\left(h_{m+k}\,(a+S_k)^+\right) \\ C_m(a) &= h_m(a) + \sum_{k=1}^{\infty} E\left(h_{m+k}\,(a+S_k)^+\right) \end{aligned}$$

4.1 Bounds in finite horizon

We first prove that the just defined constants are bounds for the transformed values, in the finite horizon case.

Theorem 6 *For all $a \in \mathbb{R}$ and $m \in \mathbb{N}$ we have* $\overline{V}_m^N(a) \leq C_m^N(a)$.

Proof. For each $i-$ vector $(x_1, \ldots x_n)$ we denote

$$s_i = x_1 + x_2 + \ldots + x_i, \qquad d^i G(x) = dG(x_1) \ldots dG(x_i).$$

By formula (4.1) it follows that

$$\overline{V}_{m+1}^{N-1}(a+x) = h_{m+1}(a+x) + \int \overline{V}_{m+2}^{N-2}(a+x+x_2)^+ dG(x_2),$$

and hence

$$\overline{V}_{m+1}^{N-1}(a+x)^+ \leq h_{m+1}(a+x)^+ + \int \overline{V}_{m+2}^{N-2}(a+x+x_2)^+ dG(x_2).$$

By replacing in (4.1) this expression and using Fubini's theorem we get

$$\overline{V}_m^N(a) \leq h_m(a) + \int h_{m+1}(a+x_1)^+ dG(x_1) + \int \overline{V}_{m+2}^{N-2}(a+x_1+x_2)^+ d^2G(x).$$

We proceed by induction, by supposing that for some $1 \leq N' < N-1$ the following holds:

$$\begin{aligned} \overline{V}_m^N(a) &\leq h_m(a) + \sum_{k=1}^{N'-1} \int h_{m+k}(a+s_k)^+ d^k G(x) \\ &+ \int \overline{V}_{m+N'}^{N-N'}(a+s_{N'})^+ d^{N'} G(x). \end{aligned} \tag{4.2}$$

We use again the recurrence formula for $\overline{V}_{m+N'}^{N-N'}(a+s_{N'})$ and take the positive part to obtain

$$\overline{V}_{m+N'}^{N-N'}(a+s_{N'})^+ \leq h_{m+N'}(a+s_{N'})^+ + \int \overline{V}_{m+N'+1}^{N-N'-1}(a+s_{N'+1})^+ d^{N'+1}G(x),$$

and integrate it using Fubini again

$$\begin{aligned} \int \overline{V}_{m+N'}^{N-N'}(a+s_{N'})^+ d^{N'} G(x) &\leq \int h_{m+N'}(a+s_{N'})^+ d^{N'} G(x) \\ &+ \int \overline{V}_{m+N'+1}^{N-N'-1}(a+s_{N'+1})^+ d^{N'+1} G(x). \end{aligned}$$

When replaced in (4.2), this leads to the inequality for $N'+1$.

Now, for $N' = N-1$ we have

$$\overline{V}^{N-N'}_{m+N'}(a+s_{N'})^+ = \overline{V}^{1}_{m+N-1}(a+s_{N-1})^+ = h_{m+N-1}\,(a+s_{N-1})^+\,,$$

and therefore (4.2) becomes

$$\overline{V}^{N}_{m}(a) \leq h_m(a) + \sum_{k=1}^{N-1} \int h_{m+k}\,(a+s_k)^+\, d^k G\,(x)\,,$$

which is the sought inequality. It now suffices to observe that

$$\int h_{m+k}\,(a+s_k)^+\, d^k G\,(x) = E(h_{m+k}\,(a+S_k)^+),$$

and the theorem is proved. □

4.2 Bounds in infinite horizon

The bounds for $\overline{V}_m(a)$ are deduced from those of $\overline{V}^N_m(a)$ by taking limits, as $\overline{V}_m(a) = \lim\limits_{N\to\infty} \overline{V}^N_m(a)$ and $\lim\limits_{N\to\infty} C^N_m(a) = C_m(a)$. However, we have to verify that $C_m\,(a)$ is finite. For this we observe that

$$E\,|S_N - N\mu| \leq \sqrt{Var\,(S_N)} = \sigma\sqrt{N}. \tag{4.3}$$

Theorem 7 *For the infinite horizon problem we have* $\overline{V}_m(a) \leq C_m(a) < \infty$.

Proof. It remains to verify that $C_m(a)$ is finite for any a and m.

In the relations

$$\begin{aligned} E\left(h_{m+p}\,(a+S_p)^+\right) &= \frac{E\left(((m+p)\,\mu - (a+S_p))^+\right)}{(m+p)\,(m+p+1)} \\ &\leq \frac{(m\mu - a)^+}{(m+p)\,(m+p+1)} + \frac{E\,(p\mu - S_p)^+}{(m+p)\,(m+p+1)}, \end{aligned}$$

the first term on the right corresponds to the general term of a convergent series, while the second one satisfies, thanks to (4.3),

$$\frac{E\,(p\mu - S_p)^+}{(m+p)\,(m+p+1)} \leq \frac{\sigma\sqrt{p}}{(m+p)\,(m+p+1)} \sim \frac{1}{p^{\frac{3}{2}}},$$

the right member being the general term of a convergent series. □

4.3 Properties of the bounds in infinite horizon

Theorem 8 *Let the law G be non-degenerated.*

1. *For $m \geq 1$, the function $C_m(\cdot)$ is decreasing, convex and*

$$\lim_{a\to\infty} (C_m(a) - h_m(a)) = 0.$$

2. *Furthermore, $h_m(a) + E\left(C_{m+1}(a+X)^+\right) \leq C_m(a)$.*

Proof.

1. By definition, the functions $a \to h_m(a)$ are decreasing and convex, then for each elementary event $h_{m+p}(a+S_p)^+$ defines a decreasing and convex function of a. It follows that the function

$$a \mapsto \sum_{p=1}^{\infty} E\left(h_{m+p}(a+S_p)^+\right) \tag{4.4}$$

is decreasing and convex.

Let us denote by d_0 the minimum of the support of G. By definition of $C_m(a)$ it is to be proved that for each fixed m, the functional series (4.4) tends to 0 as $a \to \infty$. Let us observe that

$$\begin{aligned} E\left(h_{m+p}(a+S_p)^+\right) > 0 \quad &\Leftrightarrow \quad (m+p)\,\mu - (a+S_p) > 0 \qquad \text{somewhere} \\ &\Leftrightarrow \quad a < m\mu + p\,(\mu - d_0) \end{aligned}$$

Then, for $a \geq 0$,

$$\begin{aligned} \sum_{p=1}^{\infty} E\left(h_{m+p}(a+S_p)^+\right) &= \sum_{p:m\mu+p(\mu-d_0)>a} E\left(h_{m+p}(a+S_p)^+\right) \\ &\leq \sum_{p:m\mu+p(\mu-d_0)>a} E\left(h_{m+p}\left(S_p\right)^+\right). \end{aligned}$$

The quantity on the right tends to zero as $a \to \infty$, as it corresponds to a tail of the series defining $C_m(0)$. The last assertion because

$$\min\left\{p \in \mathbb{N} : m\mu + p\,(\mu - d_0) > a\right\} \to \infty, \quad \text{as } a \to \infty,$$

thanks to the inequality $\mu - d_0 > 0$.

2. It is enough to prove that

$$\sum_{p=1}^{\infty} E\left(h_{m+p}(a+S_p)^+\right) \geq E\left(C_{m+1}(a+X)^+\right).$$

By monotone convergence

$$\sum_{p=1}^{\infty} E\left(h_{m+p}(a+S_p)^+\right) = E\left(\sum_{p=1}^{\infty} h_{m+p}(a+S_p)^+\right).$$

Expressing for each $p \geq 1 : S_p = S_{p-1} \circ \theta + X_1$, with $S_0 = 0$, and taking the conditional expectation with respect to X_1 :

$$E\left(\sum_{p=1}^{\infty} h_{m+p}(a+S_p)^+\right) = E\left(\left(E\sum_{p=1}^{\infty} h_{m+p}\left(a+S_{p-1}\circ\theta+X_1\right)^+ \middle| X_1\right)\right). \tag{4.5}$$

But the variables $S_{p-1} \circ \theta$ being independent with respect to X_1, the conditional expectation on the right-hand side is

$$E\left(\sum_{p=0}^{\infty} h_{m+p+1}(a+S_p+x)^+\right)\Bigg|_{x=X_1} = h_{m+p+1}\left(a+x\right)^+\Big|_{x=X_1}$$
$$+E\left(\sum_{p=1}^{\infty} h_{m+p+1}(a+S_p+x)^+\right)\Bigg|_{x=X_1}$$

and this last expression is greater than

$$\left(h_{m+p+1}\left(a+X_1\right)+E\left(\sum_{p=1}^{\infty} h_{m+p+1}(a+S_p+x)^+\right)\Bigg|_{x=X_1}\right)^+$$
$$= C_{m+1}(a+x)^+\Big|_{x=X_1}.$$

Therefore, from (4.5) we deduce

$$E\left(\sum_{p=1}^{\infty} h_{m+p}\left(a+S_p+x\right)^+\right) \geq E\left(C_{m+1}(a+X)^+\right),$$

the desired inequality. □

4.4 Application: More properties of the value functions

Thanks to the results of the last section we can establish in full generality the following result concerning the limit behavior of the value functions.

Theorem 9 *For all a we have*

$$\lim_{m\to\infty} V_m(a) = \lim_{m\to\infty} \overline{V}_m(a) = \lim_{m\to\infty} C_m(a) = \mu^+.$$

Proof. First, we notice that

$$
\begin{aligned}
E\left(h_{m+k}\left(a+S_k\right)^+\right) &= \frac{E\left[\left\{(m+k)\,\mu-(a+S_k)\right\}^+\right]}{(m+k)\,(m+k+1)} \\
&\leq \frac{(m\mu-a)^+ + E\left((k\mu-S_k)^+\right)}{(m+k)\,(m+k+1)} \\
&\leq \frac{(m\mu-a)^+ + \sqrt{k}\sigma}{(m+k)\,(m+k+1)},
\end{aligned}
$$

and on the other hand

$$\sum_{k=1}^{\infty} \frac{(m\mu-a)^+}{(m+k)\,(m+k+1)} = \frac{(m\mu-a)^+}{m+1}.$$

Replacing this in the definition of $C_m(a)$ gives

$$C_m(a) \leq h_m(a) + \frac{(m\mu-a)^+}{m+1} + \sigma \sum_{k=1}^{\infty} \frac{\sqrt{k}}{(m+k)\,(m+k+1)}.$$

Since $\overline{V}_m(a) \leq C_m(a)$, the last inequality implies that

$$\overline{\lim_{m\to\infty}}\, \overline{V}_m(a) \leq \overline{\lim_{m\to\infty}} C_m(a) \leq \mu^+.$$

On the other hand, by Theorem 4 it follows that $\overline{V}_m(a) \geqslant \max\left(h\,(a,m)\,,\mu-\frac{a}{m}\right)$ and then

$$\underline{\lim_{m\to\infty}}\, \overline{V}_m(a) \geqslant \mu^+.$$

This and the above inequality imply the result. □

Next we analyze a problem in which both parameters vary. First the following result.

Lemma 4.1 *For all a and m we have*

$$\lim_{k\to\infty} E\left(C_{m+k}\left(a+S_k\right)\right) = 0.$$

Proof. By monotone convergence,

$$
\begin{aligned}
E\left(C_{m+k}\left(a+S_k\right)\right) &= E\left(h_{m+k}\left(a+S_k\right)\right) \\
&+ \left(\sum_{p=1}^{\infty} E\left(E\left(h_{m+k+p}\left(a+x+S_p\right)^+\right)\Big|_{x=S_k}\right)\right),
\end{aligned}
$$

where

$$E\left(E\left(h_{m+k+p}\left(a+x+S_p\right)^+\right)\Big|_{x=S_k}\right) = E\left(h_{m+k+p}\left(a+S_{p+k}\right)^+\right),$$

thanks to Fubini's formula. But by definition

$$E\left(h_{m+k}\left(a+S_k\right)\right)=\frac{m\mu-a}{\left(m+k\right)\left(m+k+1\right)}\to 0$$

as $k\to\infty$. On the other hand,

$$\sum_{p=1}^{\infty}E\left(h_{m+k+p}\left(a+S_{p+k}\right)^{+}\right)$$

tends to zero when $k\to\infty$, as it is the tail of order k of a convergent series. □

Corollary 4.1 *For all a and m, $\overline{V}_{m+k}\left(a+S_k\right)$ converges to zero in L^1. That is,*

$$\lim_{k\to\infty}E\left(\left|\overline{V}_{m+k}\left(a+S_k\right)\right|\right)=0.$$

Proof. Let us prove first that $\lim\limits_{k\to\infty}E\left(\overline{V}_{m+k}\left(a+S_k\right)\right)=0$. Since

$$E\left(\overline{V}_{m+k}\left(a+S_k\right)\right)\leq E\left(C_{m+k}\left(a+S_k\right)\right),$$

from Lemma 4.1 we conclude that $\overline{\lim\limits_{k\to\infty}}E\left(\overline{V}_{m+k}\left(a+S_k\right)\right)\leq 0$. On the other hand, thanks to the recurrent relation

$$\overline{V}_{m+k}\left(a+S_k\right)\geq h_{m+k}\left(a+S_k\right),$$

and then $E\left(\overline{V}_{m+k}\left(a+S_k\right)\right)\geq E\left(h_{m+k}\left(a+S_k\right)\right)$, implying that

$$\begin{aligned}\underline{\lim}_{k\to\infty}E\left(\overline{V}_{m+k}\left(a+S_k\right)\right) &\geq \underline{\lim}_{k\to\infty}E\left(h_{m+k}\left(a+S_k\right)\right)=\lim_{k\to\infty}\frac{m\mu-a}{\left(m+k\right)\left(m+k+1\right)}\\ &= 0.\end{aligned}$$

Therefore $\lim\limits_{k\to\infty}E\left(\overline{V}_{m+k}\left(a+S_k\right)\right)=0$. From the recurrent relation we get

$$E\left(\overline{V}_{m+k}\left(a+S_k\right)\right)=E\left(h_{m+k}\left(a+S_k\right)\right)+E\left(\overline{V}_{m+k+1}\left(a+S_{k+1}\right)^{+}\right).$$

But $E\left(h_{m+k}\left(a+S_k\right)\right)$ converges to 0, and applying the preceding result by taking limits in the above relation we deduce that $E\left(\overline{V}_{m+k+1}\left(a+S_{k+1}\right)^{+}\right)$ converges to 0. This completes the proof. □

The following is a remarkable property of the limit behavior of the value functions (that complements somewhat Corollary 3.1).

Theorem 10 *For all a and m,*

$$\lim_{k\to\infty}\overline{V}_{m+k}\left(a+k\mu\right)=0.$$

Proof. This is deduced from the preceding corollary and from the inequality

$$\left|\overline{V}_{m+k}\left(a+k\mu\right)\right|\leq E\left(\left|\overline{V}_{m+k}\left(a+S_k\right)\right|\right),$$

which in turn is a corollary of Jensen's inequality □

4.5 Construction of a sequence of theoretical bounds

Given the parameter values a and m, we expose a method to find a sequence of upper bounds $\left(C_m^{(n)}(a)\right)_{n\in\mathbb{N}}$ for the transformed value $\overline{V}_m(a)$, that improve the preceding $C_m(a)$ of last sections, as stated in next theorem. This solves the problem of the approximating the values $\overline{V}_m(a)$ by an iterative procedure.

Let X be a variable of law G. We define by induction:

$$C_m^{(0)}(a) = C_m(a), \qquad C_m^{(n+1)}(a) = h_m(a) + E\left(C_{m+1}^{(n)}(a+X)^+\right). \tag{4.6}$$

Theorem 11 *The sequence* $\left(C_m^{(n)}(a)\right)_{n\in N}$ *is decreasing and converges to* $\overline{V}_m(a)$.

Proof. Let us prove by induction on n that for all a and m we have

$$\overline{V}_m(a) \leq C_m^{(n+1)}(a) \leq C_m^{(n)}(a).$$

- For $n = 0$: from relation

  $$\overline{V}_m(a) = h_m(a) + E\left(\overline{V}_{m+1}(a+X)^+\right)$$

  and $C_m^{(0)}(a) \geq \overline{V}_m(a)$, it is deduced that

  $$\overline{V}_m(a) \leq h_m(a) + E\left(C_{m+1}^{(0)}(a+X)^+\right) = C_m^{(1)}(a).$$

  On the other hand, by **2** of theorem 8 we have

  $$C_m^{(1)}(a) = h_m(a) + E\left(C_{m+1}^{(0)}(a+X)^+\right) \leq C_m^{(0)}(a).$$

- Inductive procedure **:** Suppose the property holds for a given n, that is

  $$\overline{V}_m(a) \leq C_m^{(n+1)}(a) \leq C_m^{(n)}(a) \quad \text{for all } a \text{ and } m.$$

  From relation

  $$\overline{V}_m(a) = h_m(a) + E\left(\overline{V}_{m+1}(a+X)^+\right)$$

  we deduce that

  $$\begin{aligned} \overline{V}_m(a) &\leq C_{m+1}^{(n+2)}(a) \\ &= h_m(a) + E\left(C_{m+1}^{(n+1)}(a+X)^+\right) \\ &\leq h_m(a) + E\left(C_{m+1}^{(n)}(a+X)^+\right) = C_m^{(n+1)}(a), \end{aligned}$$

  giving the result for $n+1$.

Then the sequence $\left(C_m^{(n)}(a)\right)_{n\in\mathbb{N}}$ is decreasing and bounded below by $\overline{V}_m(a)$. Therefore it is convergent and its limit $\overline{C}_m(a)$ satisfies

$$\overline{C}_m(a)=\lim_{n\to\infty}C_m^{(n)}(a)\geq\overline{V}_m(a).$$

Let us prove that the above relation is actually an equality. By taking the limit when $n\to\infty$ in the recurrent definition of $C_m^{(n)}(a)$, we get :

$$\overline{C}_m(a)=h_m(a)+E\left(\overline{C}_{m+1}(a+X)^+\right).$$

By means of the recurrent relation for $\overline{V}_m(a)$ and the above one, we express $\overline{C}_m(a)-\overline{V}_m(a)$ as

$$0\leq\overline{C}_m(a)-\overline{V}_m(a)=E\left(\overline{C}_{m+1}(a+X)^+-\overline{V}_{m+1}(a+X)^+\right).$$

Given that $b^+-a^+\leq b-a$ for $a\leq b$, we have

$$0\leq\overline{C}_m(a)-\overline{V}_m(a)\leq E\left(\overline{C}_{m+1}(a+X)-\overline{V}_{m+1}(a+X)\right). \tag{4.7}$$

Using the above inequality for $m+1$ and replacing, one has

$$0\leq\overline{C}_m(a)-\overline{V}_m(a)\leq E\left(E\left(\overline{C}_{m+2}(a+x+X')-\overline{V}_{m+2}(a+x+X')\right)\big|_{x=X}\right),$$

where X' has the same law as X, and they are independent. Fubini's rule gives

$$\begin{aligned}&E\left(E\left(\overline{C}_{m+2}(a+x+X')-\overline{V}_{m+2}(a+x+X')\right)\big|_{x=X(\omega)}\right)\\&=E\left(\overline{C}_{m+2}(a+S_2)-\overline{V}_{m+2}(a+S_2)\right),\end{aligned}$$

and then :

$$0\leq\overline{C}_m(a)-\overline{V}_m(a)\leq E\left(\overline{C}_{m+2}(a+S_2)-\overline{V}_{m+2}(a+S_2)\right).$$

We can successively use relation (4.7) to get :

$$\forall\ k\in N:0\leq\overline{C}_m(a)-\overline{V}_m(a)\leq E\left(\overline{C}_{m+k}(a+S_k)-\overline{V}_{m+k}(a+S_k)\right).$$

But on the other hand

$$E\left(\overline{C}_{m+k}(a+S_k)-\overline{V}_{m+k}(a+S_k)\right)\leq E\left(\overline{C}_{m+k}(a+S_k)\right)\leq EC_{m+k}(a+S_k),$$

and therefore

$$0\leq\overline{C}_m(a)-\overline{V}_m(a)\leq\limsup_{k\to\infty}E\left(C_{m+k}(a+S_k)\right).$$

Thanks to Lemma 4.1 we conclude that $\overline{C}_m(a)=\overline{V}_m(a)$. □

5 The Bernoulli case

In this section we suppose that the sequence $X_1, \ldots$ follows a Bernoulli law:

$$P(X_n = 1) = p,\ P(X_n = -1) = 1 - p = q.$$

The general case $P(X_n = a) = p$, $P(X_n = b) = 1 - p$, with $b < a$, is reduced to this one, by using a linear transformation of variables.

For $n \in \mathbb{N}$ we denote $\gamma_n = (a + S_n) - (m + n)\mu$, and define

$$T = \inf\{n : \gamma_n > 0\}.$$

Given that the support of the law of S_n consists of the integers in $[-n, n]$ of same parity as n, the support of the law of γ_n is a finite subsequence of an arithmetic sequence of rate 2. Therefore, for each n the set $]\mu - 1, \mu + 1[\bigcap \mathrm{Supp}(\gamma_n)$ possesses at most one point. Let g_n be this point in case it exists, and 0 otherwise.

Let us note that in this case we have

$$C_m^N(a) = h_m(a) + \sum_{k=1}^{N-1} \frac{E(\gamma_k^-)}{(m+k)(m+k+1)}, \tag{5.1}$$

either for $N < \infty$ as well as for $N = \infty$.

In order to transform relation (5.1), it suffices to transform the series therein, as we will see next.

Lemma 5.1 *For all $k \in \mathbb{N}$, $a \in \mathbb{R}$ and $m \in \mathbb{N}$ we have*

$$\begin{aligned} E(\gamma_{k+1}^-) &= E(\gamma_k^-) + \left(\frac{\sigma^2}{2} - qg_k - g_k^-\right) P[\gamma_k = g_k] \\ E(\gamma_{k+1}^-; T > k) &= E(\gamma_k^-; T > k-1) + \left(\frac{\sigma^2}{2} - qg_k - g_k^-\right) P[\gamma_k = g_k; T < k] \end{aligned}$$

Proof. Let $(\mathcal{F}_n)$ be the filtration generated by $(X_n)_{n\in\mathbb{N}}$.

Since $E(E(\cdot|\mathcal{F}_n)) = E(\cdot)$, we first compute $E(\gamma_{k+1}^-|\mathcal{F}_n)$. As $\{T > k\} \in \mathcal{F}_k$ it follows that on $\{T > k\}$ we have:

$$E(\gamma_{k+1}^-|\mathcal{F}_k) = E(\gamma_{k+1}^- 1_{[T>k]}|\mathcal{F}_k).$$

But on the other hand

$$\gamma_{k+1}^- = (\mu - X_{k+1} - \gamma_k) 1_{\{\gamma_k < \mu - X_{k+1}\}},$$

and therefore on the event $\{\gamma_k = g\}$ it is achieved

$$E(\gamma_{k+1}^-|\mathcal{F}_k) = E((\mu - X - g) 1_{\{g + X - \mu < 0\}}),$$

where X is a variable of law G. Hence

$$\begin{aligned} E\left(\gamma_{k+1}^{-}|\mathcal{F}_k\right) &= p\left(\mu-1-g\right)1_{\{g<\mu-1\}}+\left(1-p\right)\left(\mu+1-g\right)1_{\{g<\mu+1\}} \\ &= \left(1-p\right)\left(\mu+1-g\right)1_{\{\mu-1\leq g<\mu+1\}}-g1_{\{g<\mu-1\}} \\ &= \left(1-p\right)\left(\mu+1-g\right)1_{\{\mu-1<g<\mu+1\}}-g1_{\{g\leq\mu-1\}}. \end{aligned}$$

Given that $g_k^{-}1_{\{g=g_k\}}$ coincides with $g_k^{-}1_{\{\mu-1<g<\mu+1\}}$ on $\{\gamma_k=g\}$, and that $\{g=g_k\}$ coincides with $\{\mu-1<g<\mu+1\}$ in there, it follows from the last relations that on $\{T>k\}$

$$\begin{aligned} E\left(\gamma_{k+1}^{-}|\mathcal{F}_k\right) &= g^{-}+\left[\left(1-p\right)\left(\mu+1-g_k\right)-g_k^{-}\right]1_{\{g=g_k\}} \\ &= g^{-}+\left[\tfrac{\sigma^2}{2}-qg_k-g_k^{-}\right]1_{\{g=g_k\}}. \end{aligned}$$

Taking expectations we get :

$$E\left(\gamma_{k+1}^{-}\right)=E\left(\gamma_k^{-}\right)+\left(\frac{\sigma^2}{2}-qg_k-g_k^{-}\right)P\left[\gamma_k=g_k\right].$$

Then on $\{\gamma_k=g\}\bigcap\{T>k\}$ we have

$$E\left(\gamma_k^{-}|F_k\right)=g^{-}1_{\{T>k\}}+\left(\frac{\sigma^2}{2}-qg_k-g_k^{-}\right)1_{[g=g_k]\cap\{T>k\}}.$$

We deduce the identity for $E\left(\gamma_{k+1}^{-};T>k\right)$ by taking the expectation of the above. □

Theorem 12 *For all $a\in\mathbb{R}$:*

$$C_m^N\left(a\right)=h_m\left(a\right)+\frac{E\left(\gamma_1^{-}\right)}{m+1}-\frac{E\left(\gamma_N^{-}\right)}{m+N}+\sum_{k=1}^{N-1}\frac{\left(\frac{\sigma^2}{2}-qg_k-g_k^{-}\right)P\left[\gamma_k=g_k\right]}{m+k+1}$$

(In case $N=\infty$,the term involving γ_N^{-} does not appear)

Proof. It is enough to apply the relation of the preceding lemma to each term of (5.1), and decompose the term

$$\begin{aligned} \frac{E\left(\gamma_k^{-}\right)}{\left(m+k\right)\left(m+k+1\right)} &= \frac{E\left(\gamma_k^{-}\right)}{\left(m+k\right)}-\frac{E\left(\gamma_{k+1}^{-}\right)-\left(\frac{\sigma^2}{2}-qg_k-g_k^{-}\right)P\left[\gamma_k=g_k\right]}{\left(m+k+1\right)} \\ &= \frac{E\left(\gamma_k^{-}\right)}{\left(m+k\right)}-\frac{E\left(\gamma_{k+1}^{-}\right)}{\left(m+k+1\right)}+\frac{\left(\frac{\sigma^2}{2}-qg_k-g_k^{-}\right)P\left[\gamma_k=g_k\right]}{\left(m+k+1\right)}. \end{aligned}$$

to obtain a telescopic sum from which the formula is deduced. □

5.1 The symmetric case

In the particular case of symmetric Bernoulli variables ($p = q = \frac{1}{2}$, $\mu = 0$, $\sigma^2 = 1$), and with the asumption $a \in \mathbb{Z}$, the previous result simplifies considerably.

Corollary 5.1 *In the symmetric Bernoulli case, with $a \in \mathbb{Z}$ we have*

$$\begin{aligned} C_m^N(a) &= h_m(a) + \frac{E((a+S_1)^-)}{m+1} - \frac{E((a+S_N)^-)}{N+m} + \tfrac{1}{2}\sum_{k=1}^{N-1} \frac{P\left[a+S_k=0\right]}{m+k+1} \\ C_m(a) &= h_m(a) + \frac{E((a+S_1)^-)}{m+1} + \tfrac{1}{2}\sum_{k=1}^{\infty} \frac{P\left[a+S_k=0\right]}{m+k+1} \end{aligned}$$

Proof. In this case $g_k = 0$ for all k, and $\gamma_n = a + S_n$, so it is enough to apply Theorem 12 to get the formula for $C_m^N(a)$. The formula for $C_m(a)$ is deduced by passage to the limit in the former case, and by observing that

$$\frac{E((a+S_N)^-)}{N+m}$$

tends to zero as $N \to \infty$. $\square$

We now seek alternative formulas for the bounds $C_m(a)$, a being an integer, from the expression found in Corollary 5.1. The main problem is to find simplified formulas for the parametric series $\mathbf{\Lambda}(a,m)$ defined by

$$\Lambda(a,m) = \sum_{k=1}^{\infty} \frac{P\left\{a+S_k=0\right\}}{m+k+1}.$$

Let us observe that for calculating $\Lambda(a,m)$, it is possible to get a formula for $P\{a+S_k=0\}$ using the law of S_k. In fact, for $k = |a| + 2n$ we have

$$P\left\{a+S_k=0\right\} = \binom{k}{n}\left(\tfrac{1}{2}\right)^k.$$

and therefore

$$\begin{aligned} \Lambda(a,m) &= \sum_{n=0}^{\infty} \frac{1}{(m+|a|+2n+1)\,2^{|a|+2n}}\binom{|a|+2n}{n} \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{|a|+2n}}\binom{|a|+2n}{n}\int_0^1 t^{m+|a|+2n}\,dt \end{aligned}$$

In the case $a = 0$, the term corresponding to $n = 0$ does not appear, and we have

$$\begin{aligned} \Lambda(0,m) &= \int_0^1 \left[\sum_{n=1}^{\infty} \frac{1}{2^{2n}}\binom{2n}{n} t^{m+2n}\right] dt \\ &= \int_0^1 t^m\left(\frac{1}{\sqrt{1-t^2}} - 1\right) dt \\ &= \int_0^{\frac{\pi}{2}} \sin^m t\,dt - \frac{1}{m+1}. \end{aligned}$$

The following lemma gives a formula for $a \neq 0$.

Lemma 5.2 *If* $a \in \mathbb{Z}^*$ *we have:*

$$\begin{aligned} \Lambda(a,m) &= \int_0^1 \frac{t^{m-|a|}\left(1-\sqrt{1-t^2}\right)^{|a|}}{\sqrt{1-t^2}}dt \\ &= \int_0^{\frac{\pi}{2}} \sin^{m-|a|}(t)\,(1-\cos(t))^{|a|}\,dt. \end{aligned}$$

Proof. For $p = 1, 2, \ldots$ we let

$$f_p(t) = \frac{\left(1-\sqrt{1-t}\right)^p}{\sqrt{1-t}}$$

and observe that

$$f_{p+1}(t) = f_p(t) - \frac{p}{2}\int_0^t f_{p-1}(s)\,ds.$$

Using this and induction, it is not hard to show that

$$f_p(t) = \frac{1}{2^p}\sum_{n=0}^{\infty}\frac{1}{2^{2n}}\binom{p+2n}{n}t^{p+n}.$$

Replacing t with t^2, and p with $|a|$ we get

$$\frac{t^{m-|a|}\left(1-\sqrt{1-t^2}\right)^p}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty}\frac{1}{2^{|a|+2n}}\binom{|a|+2n}{n}t^{m+|a|+2n},$$

and integrating from 0 to 1 we get the first identity. The second one is obtained after an obvious change of variable. □

In some cases the integral expressions of Lemma 5.2 may be evaluated by means of reduction formulas. We then consider formulas for these expressions, written in the trigonometrical form :

$$I(m,a) = \int_0^{\frac{\pi}{2}} \sin^{m-|a|}(t)\,(1-\cos(t))^{|a|}\,dt.$$

In case $a = 0$ the quantities $I(m,0)$ are expressed in terms of the known Wallis formula

$$I(m,0) = \frac{(m-1)(m-3)\ldots 1}{m(m-2)\ldots 2}\frac{\pi}{2}, \quad \text{if } m \text{ is even,} \tag{5.2}$$

$$I(m,0) = \frac{(m-1)(m-3)\ldots 2}{m(m-2)\ldots 3}, \quad \text{if } m \text{ is odd} \tag{5.3}$$

In the general case, consider $a > 0$ and apply the integration by parts rule to obtain

$$I(m,a) = I(m-1,a-1) - \int_0^{\frac{\pi}{2}} \sin^{m-a}(t)\cos(t)\left(1-\cos(t)\right)^{a-1} dt,$$

and a second application of the formula of integration by parts in the second term on the right expression gives finally

$$I(m,a) = I(m-1,a-1) - \frac{1}{m-a+1} + (a-1)I(m,a-2), \tag{5.4}$$

where the expression $I(m,a-2)$ is taken as zero if $a = 1$. Relations 5.2 and 5.4 allow us to calculate recurrently the quantities $I(m,a)$, as explained in the next example.

Example 5.1 *A numerical example*

Let us explain the preceding method by the known example of the optimal stopping problem of arithmetic means (the problem of parameters $a = m = 0$), to approximate $\overline{V}(0,0) = V(0,0)$. We obtain sequentially the approximations $C_0^n(0)$, beginning with $C_0^0(0)$. Since $h(0,0) = 0$, the relation (4.6) gives in particular

$$C_{n+1}(0,0) = E\left(C_n(X,1)^+\right) = \tfrac{1}{2}\left(C_n(-1,1)^+ + C_n(1,1)^+\right),$$

where we have written $C_n(a,m)$ instead of $C_m^{(n)}(a)$.

1. **Calculation of** $C_0(0,0)$:

$$C_0(0,0) = h(0,0) + E\left(X^-\right) + \tfrac{1}{2}\Lambda(0,0) = \tfrac{1}{2} + \tfrac{1}{2}\int_0^{\pi/2}(1-\cos t)\,dt = \tfrac{\pi}{4}.$$

2. **Calculation of** $C_1(0,0)$:

$$C_1(0,0) = h(0,0) + \frac{1}{2}\left(C_0(1,1)^+ + C_0(-1,1)^+\right).$$

   Then we compute $C_0(1,1), C_0(-1,1)$, as follows.

   (a) $C_0(1,1) = h(1,1) + \frac{1}{2}\Lambda(1,1)$. Since $h(1,1) = -\frac{1}{2}$ and

$$\Lambda(1,1) = \int_0^{\frac{\pi}{2}}(1-\cos t)\,dt = \tfrac{1}{2}\pi - 1,$$

   we get $C(1,1) = -\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\pi - 1\right) = \frac{\pi}{4} - 1 < 0.$

(b) $C_0(-1,1) = h(-1,1) + \frac{E\left((X-1)^-\right)}{2} + \frac{1}{2}\Lambda(-1,1)$. We compute

$$\Lambda(-1,1) = \int_0^{\frac{\pi}{2}} (1-\cos(t))\, dt = \frac{\pi}{2} - 1,$$

and get

$$C_0(-1,1) = 1 + \frac{1}{2}\left(\frac{\pi}{2} - 1\right) = \frac{\pi+2}{4}.$$

Then $C_1(0,0) = \frac{1}{2}C_0(-1,1) = \frac{1}{2}\frac{\pi+2}{4} = \frac{\pi+2}{8} \sim 0,626...$

3. **Calculation of** $C_2(0,0)$:

$$C_2(0,0) = \frac{1}{2}\left(C_1(1,1)^+ + C_1(-1,1)^+\right).$$

We calculate $C_1(1,1)$ and $C_1(-1,1)$:

(a) $C_1(1,1) = h(1,1) + \frac{1}{2}\left(C_0(2,2)^+ + C_0(0,2)^+\right)$. We calculate $C_0(2,2)$ and $C_0(0,2)$:

i. $C_0(2,2) = h(2,2) + \frac{E(X+2)^-}{3} + \frac{1}{2}\Lambda(2,2) = -\frac{1}{3} + \frac{1}{2}\Lambda(2,2) = \frac{3\pi}{8} - \frac{4}{3} < 0.$

ii. $C_0(0,2) = h(0,2) + \frac{E(X)^-}{3} + \frac{1}{2}\Lambda(0,2) = \frac{1}{6} + \frac{1}{2}\Lambda(0,2) = \frac{\pi}{8} > 0.$ Then

$$C_1(1,1) = -\frac{1}{2} + \frac{1}{2}C(0,2) = -\frac{1}{2} + \frac{1}{2}\left(\frac{\pi}{8}\right) = \frac{\pi}{16} - \frac{1}{2} < 0.$$

(b) $C_1(-1,1) = h(-1,1) + \frac{1}{2}\left(C(0,2)^+ + C(-2,2)^+\right)$. We calculate $C(-2,2)$:

$C(-2,2) = h(-2,2) + \left(\frac{E(X-2)^-}{3}\right) + \frac{1}{2}\Lambda(-2,2) = \frac{1}{3} + \frac{2}{3} + \frac{1}{2}\Lambda(-2,2) = \frac{3\pi}{8} > 0$

Then $C_1(-1,1) = \frac{1}{2} + \frac{1}{2}(C(0,2) + C(-2,2)) = \frac{1}{2}\left(1 + 1 - \frac{\pi}{4} + \frac{3\pi}{8}\right) = 1 + \frac{\pi}{16} > 0$, and therefore

$C_2(0,0) = \frac{1}{2}C_1(-1,1) = \frac{1}{2} + \frac{\pi}{32} \sim 0,5981...$

This last upper bound, together with the lower bound obtained by Chow and Robbins (1965), give a good nest for $V(0,0)$. In fact we get

$$0,5850 < V(0,0) < 0,5982.$$

Of course one can obtain better approximations by programming the algorithm obtained here.

Finally, we point out that the problem we studied here can be used to investigate the optimal time defined by Lemma 1.1, taking advantage of recent researches on first passage times to convex barriers, as in Hammarlid (2005).

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