Brazilian Journal of Probability and Statistics (2005), 19, pp. 139–154.

Existence and regularity of solutions to a stochastic Burgers-type equation

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Abstract: We consider a one-dimensional Burgers-type stochastic differential equation (SDE) with an α-Laplacian in its linear part, perturbed by a white-noise term with non-Lipschitz coefficient, and with a random, bounded initial value. We approximate the equation by finite systems of SDEs and show they have strong solutions. For $\alpha > 3/2$ we prove tightness of the approximating systems in appropriate Hilbert spaces, and obtain existence and regularity properties of weak solutions to our equation.

Key words: Burgers-type equations, Fourier analysis, fractional Laplacian, Hilbert space regularity, weak and strong solutions, white noise.

1 Introduction

The one-dimensional Burgers equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + u(t,x)\frac{\partial u}{\partial x} \tag{1.1}$$

was proposed by Burgers (1948) in 1948 as a model for turbulent phenomena of viscous fluids. Since then, Burgers equation has been investigated in many fields of application, such as traffic flows and formation of large clusters in the universe. In order to model solutions of Navier-Stokes equations, several authors have studied Burgers equations with random initial conditions, including white and stable noises (see e.g. Burgers (1974), Bertoin (2001), Woyczyński (1998), and the references therein). Equation (1.1) can be solved in closed form (in terms of the initial conditions) by using the Hopf-Cole substitution, which reduces it to a heat equation.

Burgers equations involving in their linear parts fractional powers $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ of the Laplacian, $\alpha \in (0, 2]$, have been investigated in connection with certain models of hydrodynamical phenomena; see Shlesinger et al. (1995), Funaki at al. (1995) and Biler et al. (1998). In Biler et al. (1998), Biller, Funaki and Woyczynski studied existence, uniqueness, regularity and asymptotic behavior of solutions to the multidimensional fractal Burgers-type equation

$$\frac{\partial}{\partial t}u(t,x) = \nu\Delta_\alpha u(t,x) - a\nabla u^r(t,x), \tag{1.2}$$

where $x \in \mathbb{R}^d$, $d \geq 1$, $\alpha \in (0, 2]$, $r \geq 1$, and $a \in \mathbb{R}^d$. For $\alpha > 3/2$ and $d = 1$ they prove existence of a unique regular weak solution to (1.2) for initial conditions in $H^1(\mathbb{R})$.

Burgers equations in financial mathematics arise in connection with the behavior of the risk premium of the market portfolio of risky assets under Black-Scholes assumptions; see Bick (1990), He and Leland (1991), Hodges and Selby (1997) and Hodges and Carverhill (1993). Other turbulent characteristics, such as semi-heavy tails, volatility (intermittency in turbulence) or aggregational Gaussianity, have been observed in time series of financial data such as values of stocks, logarithmic stocks returns and exchange rates; see Barndorff-Nielsen and Shephard (2001) and the references therein. In Hodges and Carverhill (1993) and Hodges and Selby (1997) the authors consider a model of an economy market with a single asset price, S_t, satisfying the equation

$$\frac{dS}{S} = [r + \sigma\Xi(\cdot)]dt + \sigma dB_t,$$

where r is a constant risk-free interest rate, σ is the volatility (which is assumed to be constant), $\{B_t,\ t \geq 0\}$ is a Brownian motion and Ξ is an adapted stochastic process representing the risk price. The time horizon H is finite and no dividends are paid. Under certain additional assumptions it is shown, using Girsanov's theorem, that the risk price Ξ satisfies the Burgers equation

$$\frac{\partial \Xi}{\partial \tau} = \frac{1}{2}\sigma^2 \Delta\Xi + \sigma\Xi\frac{\partial \Xi}{\partial x},$$

where $\tau = H - t$. See Hodges and Carverhill (1993) and Hodges and Selby (1997) for a precise formulation of the assumptions, and for some financial implications of this model in the time-homogeneous setting.

Burgers equations perturbed by space-time white noise have been studied by several authors under Lipschitz conditions on the noise term coefficient, Bertini et al. (1994), Da Prato and Gatarek (1995), Biler et al. (1998), Gyongy (1998). In Kolkovska (2003) it is proved existence of a weak solution to the one-dimensional stochastic Burgers equation

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \Delta u(t,x) + \lambda\nabla u^2(t,x) + \gamma\sqrt{u(t,x)(1-u(t,x))}\,\frac{\partial^2}{\partial t\partial x}W(t,x), \\ u(t,0) &= u(t,1) = 0, \\ u(0,x) &= f(x),\ x \in [0,1], \end{aligned} \tag{1.3}$$

where λ and γ are positive constants, $\frac{\partial^2}{\partial t\partial x}W(t,x)$ is a space-time white noise and $f : [0,1] \to [0,1]$ is a continuous function. We refer to Walsh (1986) for the definition and properties of white noise.

Solutions to equation (1.3) are interpreted in the weak sense, which means that for each $\varphi \in C^2([0,1])$,

$$\int_{[0,1]} u(t,x)\varphi(x)\,dx$$

$$= \int_{[0,1]} u(0,x)\varphi(x)\,dx + \int_{[0,1]} u(t,x)\varphi''(x)\,dx - \lambda\int_0^t\int_{[0,1]} u^2(s,x)\varphi'(x)\,dx\,ds + \gamma\int_0^t\int_{[0,1]}\sqrt{u(s,x)(1-u(s,x))}\varphi(x)\,W(ds,dx).$$

The method of proof in Kolkovska (2003) consists in approximating (1.3) by finite systems of stochastic differential equations possessing a unique strong solution. Using bounds for the fundamental solution of the discrete Laplacian, it is shown tightness of the approximating systems, and, moreover, that each weak limit point solves (1.3).

In this paper we investigate existence and regularity properties of solutions of the fractal Burgers equation

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= \Delta_\alpha u(t,x) + \lambda\nabla u^2(t,x) + \gamma\sqrt{u(t,x)(1-u(t,x))}\,\frac{\partial^2}{\partial t\partial x}W(t,x),\\ u(t,0) &= u(t,1) = 0,\ x\in[0,1],\end{aligned} \tag{1.4}$$

with a random, positive, initial condition $u(0,x)$, bounded by 1.

Recall that Δ_α is the infinitesimal generator of the symmetric α-stable motion, which was proposed by Mandelbrot in 1963 to model the non-Gaussian time evolution of log-prices of certain assets.

Notice that, due to the presence of non-Lipschitz coefficients, existence and uniqueness of a weak solution to (1.4) cannot be proved by the classical approach. Following the method of proof of Kolkovska (2003), we consider a discrete version of (1.4) and obtain existence of a strong solution to the corresponding finite system of stochastic differential equations. The principal difficulty we are dealing with here, originated by the presence of fractional powers of the discrete Laplacian, consists in proving tightness of the approximating systems and regularity properties of the solution. We overcome this difficulty by using Fourier analysis methods similar to those in Blount (1996) and Blount and Kouritzin (2001). We prove that, for $\alpha > 3/2$, each limit point is a weak $L^2([0,1])$-solution to our equation whose paths are a.s. continuous. We also prove that each such solution takes values in a Sobolev space H_β with norm $|\cdot|_\beta$ (see (2.1)), and has a modification which is Hölder-continuous with respect to $|\cdot|_\beta$. In the classical case $\alpha = 2$, our results yield solutions that are more regular than those obtained in Kolkovska (2003). Uniqueness, as well as existence of strong solutions to (1.4), remain to be investigated.

2 Existence and regularity of solutions

We introduce some notations we need. Let $S = [0,1)$, and let $\mathbf{T}$ denote the quotient space obtained from $[0,1]$ by identifying 0 and 1. We define $\varphi_0(x) = 1$ for $x \in [0,1]$, and

$$\varphi_n(x) = \sqrt{2}\cos(\pi n x),\quad \psi_n(x) = \sqrt{2}\sin(\pi n x),\quad x\in[0,1],\quad n = 2,4,\ldots.$$

Let e_m be φ_m or ψ_m, $m = 0, 2, 4, \ldots$. Notice that $\{e_m,\ m = 0, 2, \ldots\}$ is the usual orthonormal basis in $L^2(S)$, and that for all m, $\Delta e_m = -\pi^2 m^2 e_m$. For any $\beta \in \mathbb{R}$ we define H_β as the Hilbert space obtained from $L^2(S)$ by completion with respect to the norm

$$|f|_\beta = \left(\sum \langle f, e_m\rangle^2 (1 + \pi^2 m^2)^\beta\right)^{1/2}, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L^2(S)$, and the sum is taken over all distinct possibilities for e_m.

Notice that $H_0 = L^2(S)$ and $H_\beta \subset H_{\beta'}$ for $\beta' < \beta$.

For any integer $N \geq 1$, let $H(N)$ denote the set of functions $f : [0,1] \to \mathbb{R}$ that are constant on $[\frac{k}{N}, \frac{k+1}{N})$ for $k = 0, 1, 2, ..., N-1$. Clearly we have $H(N) \subset L^2([0,1])$.

Let $P_N : L^2(\mathbf{T}) \to H(N)$ be the orthogonal projection of $L^2(\mathbf{T})$ onto $H(N)$, which is given by

$$P_N f(r) = N \int_{\frac{k}{N}}^{\frac{k+1}{N}} f(s)\, ds, \quad r = \frac{k}{N}, \quad k = 0, 1, 2, ..., N-1.$$

We define $\hat{e_m} = \frac{P_N e_m}{|P_N e_m|_0}$ for $0 \leq m \leq N-1$ and N odd. Then $\{\hat{e_m}\}$ is an orthonormal basis of $H(N)$ as a subspace of $L^2([0,1])$, and $\Delta_N \hat{e_m} = -\hat{\beta_m} \hat{e_m}$, where $\hat{\beta_m} \in [4m^2, \pi^2 m^2]$. Writing $|.|_0$ for the usual norm in $L^2([0,1])$, it follows that $\lim_{N\to\infty} |e_m - \hat{e_m}|_0 = 0$.

For $f \in H(N)$ and any β we define

$$|f|_{\beta,N} = \left(\sum \langle f, \hat{e_m}\rangle^2 (1 + \hat{\beta_m})^\beta\right)^{-1/2}.$$

From Blount (1996) (Lemma 3.1) it follows that $|f|_{0,N} = |f|_0$ and

$$2^{-1/2} |f|_{-\beta} \leq |f|_{-\beta,N} \leq (\pi/2)^{\beta+1} |f|_{-\beta} \tag{2.2}$$

for $f \in H(N)$ and $\beta > 0$. We define $P_n : H_\beta \to \bigcap_\gamma H_\gamma$ as the projection

$$P_n(f) = \sum_{m \leq n} \langle f, e_m \rangle e_m,$$

and put $P_n^\perp := I - P_n$, where I is the identity operator. Similarly, for $f \in H(N)$, let

$$P_{n,N}(f) = \sum_{m \leq n} \langle f, \hat{e_m} \rangle \hat{e_m},$$

and $P_{n,N}^\perp := I - P_{n,N}$.

Without loss of generality we assume that $\lambda = \gamma = 1$. Let N be a fixed positive integer. Similarly as in Kolkovska (2003), let us consider the discretized version of (1.4), namely

$$\frac{\partial}{\partial t} X^N(t,r) = \Delta_{N,\alpha} X^N(t,r) + \nabla_N X^N(t,r)^2 + \sqrt{X^N(t,r)(1 - X^N(t,r))}\, dB_N(t,r),$$
$$X^N(0,r) = X(0,r), \quad r = 0, \frac{1}{N}, ..., \frac{N-1}{N}, \quad t \geq 0, \tag{2.3}$$

where $\Delta_{N,\alpha}$ is the fractional power of the discrete Laplacian, and $\{N^{-1/2}B_N(t,r)\}_r$ is a sequence of independent Brownian motions. Now we state our results.

Theorem 2.1 *For any positive initial random condition $X^N(0)$ bounded by 1, there exists a unique strong solution $X^N(t)$ of (2.3) in $C([0,\infty), L^2([0,1])$.*

Theorem 2.2 *a) The distributions of $\{X^N\}$ are relatively compact on $C((0,\infty), H_\beta)$ if $\beta \leq 0, \alpha > \beta + 3/2$, and on $C([0,\infty), H_\beta)$ for $\alpha > \beta + 3/2, \beta < -1/2$.*
b) For any $\alpha > 3/2$, equation (1.4) has a weak solution in $C((0,\infty), L^2([0,1]))$.

Remark 2.1 Theorems 2.1 and 2.2 are consistent with results obtained in Biler et al. (1998) for the case $\gamma = 0$. In our case, we were not able to prove uniqueness of weak solutions of (1.4); this remains to be investigated.

Theorem 2.3 *The solution $X(t)$ has a modification which is Holder continuous in time: it satisfies*

$$P\left(\sup_{0<s_0\leq s<t\leq T} \frac{|X(t)-X(s)|_\beta}{|t-s|^\delta} < \infty\right) = 1$$

for each $0 < \delta < [(2\alpha - 2\beta - 3)/(2\alpha)] \wedge 1/2$, $3/2 < \alpha \leq 2$, *and* $\beta < (2\alpha - 3)/2$.

Remark 2.2 In particular, when $\alpha = 2$ and $0 \leq \beta < 1/2$, we can take $0 < \delta < \frac{1-2\beta}{4}$, and obtain

$$P(X \in C((0,\infty) : H_\beta)) = 1,$$

thus $X(t)$ is smoother than an $L^2([0,1])$ function for $t > 0$.

3 Proofs

Let us recall the discrete approximations of the first and second derivative with respect to the variable x, namely

$$\Delta_N h\left(t, \frac{k}{N}\right) = \frac{h\left(t, \frac{k+1}{N}\right) - 2h\left(t, \frac{k}{N}\right) + h\left(t, \frac{k-1}{N}\right)}{\frac{1}{N^2}},$$

$$\nabla_N\, h\left(s, \frac{k}{N}\right) \frac{h\left(s, \frac{k+1}{N}\right) - h\left(s, \frac{k}{N}\right)}{\frac{1}{N}}, \quad 1 \leq k \leq N,$$

where h is a given function. Let us write $x_r^N(t) = X^N(t,r)$. Then (2.3) can be written in the more compact form

$$\begin{aligned} dx_i^N(t) &= \left(\sum_{j=1}^N a_{ij}^N x_j^N(t) + b_{ij}^N x_j^N(t)^2\right) dt + \sqrt{N x_i^N(t)\left(1 - x_i^N(t)\right)}\, dB_i(t) \\ x_i^N(0) &= f(i/N),\ 1 \leq i, j \leq N, \end{aligned} \tag{3.1}$$

where

$$b_{ij}^N = \begin{cases} N & \text{if} \quad j = i+1, \\ -N & \text{if} \quad j = i, \\ 0 & \text{otherwise,} \end{cases}$$

and a_{ij}^N are the coefficients of $\Delta_{N,\alpha}$. In the case $\alpha = 2$,

$$a_{ij}^N = \begin{cases} N^2 & \text{if} \quad j = i+1, i-1, \\ -2N^2 & \text{if} \quad j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.1 *For any initial random condition $X^N(0) = (x_1^N, ..., x_N^N) \in [0,1]^N$, the system*

$$\begin{aligned} dx_i^N(t) &= \left(\sum_j a_{ij}^N x_j^N(t) + \sum_j b_{ij}^N x_j^N(t)^2 \right) dt + \sqrt{N x_i^N(t)(1 - x_i^N(t))}\, dB_i(t) \\ x_i^N(0) &= x_i,\ i = 1, ..., N, \end{aligned} \tag{3.2}$$

admits a unique strong solution $X^N(t) = (x_1^N(t), \ldots, x_N^N(t)) \in C([0,\infty), [0,1]^N)$.

Proof. Let us consider the re-scaled system

$$\begin{aligned} dx_i^N(t) &= \left(\sum_j a_{ij}^N x_j^N(t) + \sum_j b_{ij}^N x_j^N(t)^2 \right) dt + \sqrt{g(x_i^N(t))}\, dB_i(t) \quad (3.3) \\ x_i^N(0) &= x_i,\ i = 1, ..., N, \end{aligned}$$

where $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(x) = Nx(1-x)$ for $0 \le x \le 1$, and $g(x) = 0$ otherwise. Since the coefficients of (3.3) are continuous, by Skorohod's existence theorem (see e.g. Skorohod (1995), Ikeda and Watanabe (1989)), we conclude that on some probability space there exists a weak solution $X^N(t)$ of (3.3). We will prove that each weak solution $X^N(t) = (x_1^N(t), \ldots, x_N^N(t))$ of this system, is bounded: $x_i^N(t) \in [0,1]$ for all $i = 1, \ldots, N$ and $t \ge 0$, thus showing that $X^N(t)$ is a solution also of (3.2).

First we show that $x_i^N(t) \ge 0$ for each $i = 1 \ldots, N$. Since the system has non-Lipschitz coefficients, the solution may explode in finite time. Let $\tau_1 \le \infty$ denote the explosion time of the solution. If some of the solution coordinates are negative, then there exists a random time $0 < \tau_2 \le \infty$ such that for $0 < t \le \tau_2$ all such coordinates are between -1 and 0. This is so because there is only finite number of coordinates, and they are continuous.

In order to obtain pathwise uniqueness of weak solutions we shall use the local time techniques of Le Gall combined by the classical method of Ikeda and Watanabe (see e.g. Rogers and Williams, 2000, Chapter V, §43).

We state the following result of Le Gall (1983).

Lemma 3.2 *Let $Z \equiv \{Z(t),\ t \geq 0\}$ be a real-valued semimartingale. Suppose that there exists a function $\rho : [0,\infty) \to [0,\infty)$ such that $\int_0^\varepsilon \frac{du}{\rho(u)} = +\infty$ for all $\varepsilon > 0$, and $\int_0^t \frac{1_{\{Z_s>0\}}}{\rho(Z_s)}\, d\langle Z\rangle_s < \infty$ for all $t > 0$ a.s. Then the local time at zero of Z, $L_t^0(Z)$, is identically zero for all t a.s.*

Applying Lemma 3.2 to $x_i^N(t)$ with $\rho(u) = u$ and using Tanaka's formula (see e.g. Revuz and Yor, 1994), after summation we obtain for $x_i^N(t)_- := \max\{0, -x_i^N(t)\}$,

$$\begin{aligned}
\sum_{i=1}^N x_i^N(t)_- &= -\int_0^t \sum_{i=1}^N 1_{x_i^N(s)<0} \sum_{j=1}^N (a_{ij}^N x_j(s) + b_{ij}^N x_j(s)^2)\, ds \\
&\leq \int_0^t \sum_{i,j=1}^N 1_{x_i^N(s)<0} a_{ij}^N x_j(s)_-\, ds + N\int_0^t \sum_{i=1}^N 1_{x_i^N(s)<0} x_i(s)^2\, ds \\
&\leq \int_0^t \sum_{i,j=1}^N a_{ij}^N x_j(s)_-\, ds + N\int_0^t \sum_{i=1}^N x_i^N(s)_-\, ds \\
&= N\int_0^t \sum_{i=1}^N x_i^N(s)_-\, ds
\end{aligned}$$

since $\sum_i a_{ij}^N = 0$, which follows from self-adjointness of $\Delta_{N,\alpha}$. Applying Gronwall's lemma we obtain that $\sum_{i=1}^N x_i^N(t)_- = 0$, and, hence, the solution is non-negative for each $t \geq 0$. By a similar argument applied to $(1 - x_i^N(t))_-$ it follows that $x_i^N(t) \leq 1$ for each $1 \leq i \leq N$.

Let $X^{1,N} = (x_1^{1,N}, \ldots, x_N^{1,N})$ and $X^{2,N} = (x_1^{2,N}, \ldots, x_N^{2,N})$ be two weak solutions of (3.2) defined with the same initial conditions and the same Brownian motions.

Then

$$\begin{aligned}
&x_i^{1,N}(t) - x_i^{2,N}(t) \\
&\quad = \int_0^t \left[\sum_j a_{ij}^N \left(x_j^{1,N}(s) - x_j^{2,N}(s) \right) + b_{ij}^N \left(x_j^{1,N}(s)^2 - x_j^{2,N}(s)^2 \right) \right] ds \\
&\qquad + \int_0^t \left[\sqrt{N x_i^{1,N}(s)(1 - x_i^{1,N}(s))} - \sqrt{N x_i^{2,N}(s)(1 - x_i^{2,N})(s)} \right] dB_i(s), \\
&\qquad i = 1, \ldots, N.
\end{aligned}$$

Since

$$\langle X\rangle_t = \int_0^t \left[\sqrt{N x_i^{1,N}(s)(1 - x_i^{1,N}(s))} - \sqrt{N x_i^{2,N}(s)(1 - x_i^{2,N})(s)} \right]^2 ds$$

and

$$\int_0^t \frac{\left[\sqrt{Nx_i^{1,N}(s)(1-x_i^{1,N}(s))}-\sqrt{Nx_i^{2,N}(s)(1-x_i^{2,N})(s)}\right]^2}{x_i^{1,N}(s)-x_i^{2,N}(s)} 1_{x_i^{1,N}(s)-x_i^{2,N}(s)>0}\, ds$$
$$\leq \int_0^t 2N 1_{x_i^{1,N}(s)-x_i^{2,N}(s)>0}\, ds \ < \ 2Nt$$

(where we used that $(\sqrt{x(1-x)} - \sqrt{y(1-y)}\,)/(x-y) < 2$ for $x, y \in [0,1]$, $x > y$, which follows from L'Hospital rule), we can apply Lemma 3.2 to $Z(t) = x_i^{1,N}(t) - x_i^{2,N}(t)$ with $\rho(x) = x$. Therefore, $L_t^0\left(x_i^{1,N}(s) - x_i^{2,N}(s)\right) = 0$ for all $i \in \{1, \ldots, N\}$.

Applying Tanaka's formula again,

$$\begin{aligned}\left|x_i^{1,N}(t) - x_i^{2,N}(t)\right| &= \int_0^t \operatorname{sgn}\left(x_i^{1,N}(s) - x_i^{2,N}(s)\right) \\ &\cdot \left[\sum_j a_{ij}^N \left(x_j^{1,N}(s) - x_j^{2,N}(s)\right) + b_{ij}^N \left(x_j^{1,N}(s)^2 - x_j^{2,N}(s)^2\right)\right] ds \\ &+ \int_0^t \operatorname{sgn}\left(x_i^{1,N}(s) - x_i^{2,N}(s)\right) \\ &\cdot \left[\sqrt{Nx_i^{1,N}(s)\left(1 - x_i^{1,N}(s)\right)} - \sqrt{Nx_i^{2,N}(s)\left(1 - x_i^{2,N}(s)\right)}\right] \\ &\cdot dB_i(s), \quad i = 1, \ldots, N.\end{aligned}$$

Since a_{ij}^N and b_{ij}^N are bounded by 1, it follows that

$$\begin{aligned} &E\sum_{i=1}^N \left|x_i^{1,N}(t) - x_i^{2,N}(t)\right| \\ &\leq \int_0^t E\sum_{i=1}^N \left|\sum_j a_{ij}^N \left(x_j^{1,N}(s) - x_j^{2,N}(s)\right) + b_{ij}^N \left(x_j^{1,N}(s)^2 - x_j^{2,N}(s)^2\right)\right| ds \\ &\leq \int_0^t K(N) E\sum_{i=1}^N \left|x_i^{1,N}(s) - x_i^{2,N}(s)\right| ds,\end{aligned}$$

where $K(N)$ is a constant depending on N. From Gronwall's inequality we conclude that

$$E\sum_{i=1}^d \left|x_i^{1,N}(t) - x_i^{2,N}(t)\right| = 0$$

for all $t \geq 0$, thus proving pathwise uniqueness. By a classical theorem of Yamada and Watanabe (1971), this is sufficient for existence of a unique strong solution of (3.2). □

Since $X^N(t,\cdot)$ is defined on a discrete system of points $\{r = k/N, k = 0, 1, ..., N-1\}$, by assigning to $X^N(t,\cdot)$ the constant value $X^N(t, k/N)$ in the interval $[k/N, (k+1)/N)$, $k = 0, 1, ..., N-1$, we can view the function $X^N(t)$ as an element of the space $H(N)$. By variation of constants, we can write (2.3) in the equivalent form

$$\begin{aligned} X^N(t) &= T_{N,\alpha}(t)X^N(0) + \int_0^t T_{N,\alpha}(t-s)[\nabla_N X^N(s)^2]\, ds \\ &\quad + \int_0^t T_{N,\alpha}(t-s)\sqrt{X^N(t)(1-X^N(t))}\, dB_N(s,r) \\ &:= T_{N,\alpha}(t)X^N(0) + V_N(t) + M_N(t), \end{aligned} \tag{3.4}$$

where $T_{N,\alpha}(t)$ is the semigroup on $H(N)$ generated by $\Delta_{N,\alpha}$.

Let $Y_N(t) = \int_0^t \sqrt{X^N(s)(1-X^N(s)}\, dB_N(s)$.

Lemma 3.3 *(i) For $\beta < -1/2$, $\{Y_N\}$ is relatively compact in $C([0,\infty) : H_\beta)$.*
(ii) For any fixed n, and any β, $\{P_n X^N\}$ is relatively compact in $C([0,\infty) : H_\beta)$.

Proof. (i) For $\beta < -1/2$ and $0 \leq t \leq t+s \leq T$, we have

$$\begin{aligned} & E[|Y_N(t+s) - Y_N(t)|_\beta^2 | \sigma(X_r), r \leq t] \\ = \; & E[\sum_{m=0}^\infty \int_t^{t+s} \langle X_N(r)(1-X_N(r)), (P_N e_m)^2\rangle dr (1+\pi^2 m^2)^\beta | \sigma(X_r), r \leq t], \end{aligned}$$

hence from a well-known criterion (see e.g. Ethier and Kurtz, 1986), $\{Y_N\}$ is relatively compact in $C([0,\infty : H_\beta)$, which proves (i).

Let consider the equality

$$\begin{aligned} P_n X^N(t) &= P_n X^N(0) + \int_0^t P_n \Delta_{N,\alpha} X^N(s)\, ds + \int_0^t P_n \nabla_N X^N(s)^2\, ds \\ &+ \int_0^t P_n \sqrt{X^N(s)(1-X^N(s))}\, dW_N(s). \end{aligned}$$

For fixed n, using the fact that Δ_N is self-adjoint on H_N and $X^N(t)$ is bounded, we obtain from Ascoli's theorem and (i) that the distributions of $P_n[X^N(t) - Y_N(t)]$ are relatively compact. □

Lemma 3.4 *For any $\varepsilon > 0$ and $T > 0$,*
(i) $\lim_{n\to\infty} \sup_N P(\sup_{0\leq t\leq T} |P_{n,N}^\perp M^N(t)|_{\beta,N} \geq \varepsilon) = 0$ *for any $\beta < 1/2$.*
(ii)$\lim_{n\to\infty} \sup_N P(\sup_{s\leq t\leq T} |P_{n,N}^\perp X^N(t)|_{\beta,N} \geq \varepsilon) = 0$ *for $s > 0$ and $\alpha > \beta + 3/2$, or $s = 0$ and $\alpha > \beta + 3/2$, $\beta < -1/2$.*
(iii)$\lim_{n\to\infty} \sup_N P(\sup_{s\leq t\leq T} |P_{n,N}^\perp X^N(t)|_\beta \geq \varepsilon) = 0$ *for $s > 0$ and $\alpha > \beta + 3/2$, $\beta \leq 0$, or $s = 0$ and $\alpha > \beta + 3/2$, $\beta < -1/2$.*

Proof. From the equality

$$\langle M_N(t), \hat{e_m}\rangle = \int_0^t \exp[-\hat{\beta}_m(t-s)]\langle X^N(t)(1-X^N(t)), (\hat{e_m})^2\rangle dB(s)$$

and Blount (1996) (Lemma 1.1), we obtain

$$P\left(\sup_{t\le T}\langle M_N(t), \hat{e_m}\rangle^2 \ge a^2\right) \le \pi^2 m^2 T[\exp(Cm^2a^2)-1]^{-1}, \qquad (3.5)$$

where $C>0$ is a constant. For $\beta<1/2$, let δ be such that $0<\delta<1$, $\beta-\delta<-1/2$. Then, for given $\varepsilon>0$, there exists $n_0>0$ such that for all $n\ge n_0$ there holds $\sum_{m\ge n} m^{2(\beta-\delta)}<\varepsilon$ and

$$\begin{aligned}
&P\left(\sup_{0\le t\le T}|P^{\perp}_{n,N}M^N(t)|_{\beta,N}\ge\varepsilon\right)\\
&\le P\left(\sup_{t\le T}\sum_{m\ge n}\langle M_N(t), \hat{e_m}^2\rangle m^{2\beta}\ge\sum_{m\ge n}m^{2(\beta-\delta)}\right)\\
&\le \sum_{m\ge n}P\left(\sup_{t\le T}\langle M_N(t), \hat{e_m}^2\rangle\ge m^{-2\delta}\right)\\
&\le \sum_{m\ge n}\pi^2m^2T[\exp(Cm^{2(1-\delta)})-1]^{-1},
\end{aligned}$$

where we used (3.5) to obtain the last inequality. Letting $n\to\infty$ yields (i).

Let denote by $T_N(t)$ the semigroup generated by Δ_N. By definition we have

$$T_{N,\alpha}(t)(x) = \int_0^\infty f_{t,\alpha}(s)T_N(s)x\,ds,$$

where $f_{t,\alpha}(s) := \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{zs-tz^{\alpha/2}}\,dz$ for $s\ge 0$. Since for $m=0,2,...,N-1$, we have $T_N(s)\hat{e_m}=e^{-s\hat{\beta_m}}\hat{e_m}$, by Proposition 1, p.260 in Yosida (1980),

$$\begin{aligned}
T_{N,\alpha}(t)(\hat{e_m}) &= \int_0^\infty f_{t,\alpha}(s)e^{-s\hat{\beta_m}}\,ds\,\hat{e_m}\\
&= e^{-t\hat{\beta_m}^{\alpha/2}}\hat{e_m}.
\end{aligned}$$

Hence $|\langle T_{N,\alpha}(t)X^N(0), \hat{e_m}\rangle| < \exp(-\hat{\beta_m}t)$ and for all natural N and $\beta<-1/2$, we have

$$\sup_{0\le t\le T}|T_{N,\alpha}(t)X^N(0)|^2_{\beta,N}\le C_1\sum_{m=0}^{N-1}m^{2\beta}<\infty, \qquad (3.6)$$

and for $s > 0$ and $\alpha > \beta + 1/2$, we obtain

$$\sup_{s\le t\le T} |T_{N,\alpha}(t)X^N(0)|^2_{\beta,N} \le \sum_{m=0}^{N-1} m^{2\beta-2\alpha} < \infty. \tag{3.7}$$

Using the selfadjointness of the operators $T_{N,\alpha}(t)$ and ∇_N on $H(N)$, it follows that

$$\begin{aligned} \langle V_N(t), \hat{e_m}\rangle &= \langle \int_0^t T_{N,\alpha}(t-s)[\nabla_N X^N(s)^2]\, ds, \hat{e_m}\rangle \\ &= \langle \int_0^t T_{N,\alpha}(t-s)\hat{e_m}, \nabla_N X^N(s)^2\rangle\, ds \\ &= \int_0^t -e^{-(t-s)\hat{\beta_m}^{\alpha/2}} \langle \nabla_N \hat{e_m}, X_N^2(s)\rangle\, ds. \end{aligned} \tag{3.8}$$

Since $4m^2 \le \hat{\beta_m} \le \pi^2 m^2$ and $\sup_x |\nabla_N \hat{e_m}(x)| \le cm$ for some constant $c > 0$ independent of N (see Blount (1996)), we obtain from (3.8), for all natural N, $s \ge 0$ and $\alpha > 3/2 + \beta$,

$$\sup_{s\le t\le T} |V_N(t)|^2_{\beta,N} = \sup_{s\le t\le T} \sum_{m=0}^{N-1} \langle V_N(t), \hat{e_m}\rangle^2 (1+\pi^2 m^2)^\beta \le C_1 \sum_{m=0}^{N-1} m^{2(1-\alpha)} m^{2\beta} < \infty, \tag{3.9}$$

where $C_1 = C_1(T)$ is a constant non depending on N.

Part (ii) of the result then follows from (3.6), (3.7), (3.8) and (3.9).

Finally, (iii) follows from (ii) and (2.2). □

Proof of Theorem 2.2a). Let consider $P_{n,N}X^N = P_n X^N + (P_{n,N} - P_n)X^N$. Since for fixed n, we have $\sup_{t\le T} |(P_{n,N} - P_n)X^N(t)|_0 \to 0$ in probability as $N \to \infty$, by Lemma 3.3 (ii) we obtain relative compactness for $P_{n,N}X^N$. Now from $X^N = P_{n,N}X^N + P^\perp_{n,N}X^N$ and Lemma 3.4(iii) we obtain relative compactness for X^N. □

Proof of Theorem 2.2b).

From Theorem 2.2a) we know that there exist a process X and a subsequence X^{N_k} of X^N such that $X^{N_k} \Rightarrow X$ in $C([0,\infty), L^2([0,1]))$. We will denote X^{N_k} by X^N.

Applying Skorohod's representation theorem, we can construct a sequence $X^{N'}$ and a random element X' on some probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ such that $\{X^N\} \overset{\mathcal{D}}{=} \{X'\}$ and $X^N \to X$ in $C([0,\infty), L^2([0,1]))$ with probability 1 (hence with probability 1 $X^N(t) \to X(t)$). Let us denote

$$K_N(t) := X^N(t) - X^N(0) - \int_0^t \Delta_{N,\alpha} X^N(s)\, ds - \int_0^t \nabla_N X^N(s)^2\, ds.$$

Then by (2.3), $K_N(t)$ is an $H(N)$-valued martingale with $\langle K_N\rangle_t = \int_0^t X^N(t)(1-X^N(t))\,ds$ and it is straightforward to see that $K_N \to K$ in $L^2([0,1])$, where

$$K(t) := X(t) - X(0) - \int_0^t \Delta_\alpha X(s)\,ds - \int_0^t \nabla X(s)^2\,ds.$$

Moreover, since $K_N(t)$ is uniformly integrable ($\sup_N E(|K_N(t)|_0) < \infty$ uniformly for $t \le T$), $K(t)$ is a $L^2([0,1])$-martingale with $\langle K\rangle_t = \int_0^t X(s)(1-X(s))\,ds$. Now as in Konno and Shiga (1998) we can construct on a extended probability space a space-time white noise $W(ds,dx)$ such that $K(t) = \int_0^1 \int_0^t \sqrt{X(t)(1-X(t)}W(ds,dx)$ and hence $X(t)$ is a weak solution of (1.4). □

Proof of Theorem 2.3. Let consider the equality

$$\begin{aligned} X(t) &= T_\alpha(t)X(0) + \int_0^t T_\alpha(t-s)[\nabla X(s)^2]\,ds \\ &\quad + \int_0^t T_\alpha(t-s)\sqrt{X(t)(1-X(t))}\,dB(s) \\ &:= T_\alpha(t)X(0) + V(t) + M(t). \end{aligned} \tag{3.10}$$

As in the proof of Theorem 1.2 and Corollary 1.1 in Blount and Kouritzin (2001), we obtain

$$P\left(\sup_{0<\le s<t\le T} \frac{|M(t)-M(s)|_\beta}{|t-s|^\delta} < \infty\right) = 1$$

for each $0 < \delta < [(\alpha - 2\beta - 1)/(2\alpha)] \wedge 1/2$, $3/2 < \alpha \le 2$, and $\beta < \frac{\alpha-1}{2}$. The condition that must hold in order to give the result is

$$\sum_{m=1}^{\infty} m^{\alpha(\delta-1)}(1+m^2)^\beta < \infty.$$

Now consider the second term in (3.10) and define $V_m(t) = \langle V(t), e_m\rangle$. From (3.8) we have

$$V_m(t) = m\int_0^t e^{-m^\alpha(t-s)} h_m(s)\,ds$$

for some bounded h_m. From

$$V_m(t) - V_m(s) = (e^{-m^\alpha(t-s)} - 1)gV_m(s) + m\int_s^t (e^{-m^\alpha(t-u)}h(u)\,du,$$

we obtain for $0 \le s < t$ and a constant c,

$$|V_m(t) - V_m(s)| \le cm\frac{1-e^{-m^\alpha(t-s)}}{m^\alpha} \le cm^{\alpha(\delta-1)+1}|t-s|^\delta, \tag{3.11}$$

where in (3.11) we used

$$(1 - e^{-a|t-s|})/a \leq \min\{|t-s|, a^{\delta-1}|t-s|^{\delta}\}$$

for $a > 0$ and $0 < \delta \leq 1$.

Hence,

$$\begin{aligned}|V_m(t) - V_m(s)|_\beta^2 &= \sum_m ([V_m(t) - V_m(s)]^2 (1+m^2)^\beta \\ &\leq c \sum_{m=1}^{\infty} m^{2\alpha(\delta-1)+2+2\beta} |t-s|^{2\delta}.\end{aligned}$$

Thus for $0 < \delta < [(2\alpha - 2\beta - 3)/(2\alpha)] \wedge 1/2, 3/2 < \alpha \leq 2$ and $\beta < (2\alpha - 3)/2$ we obtain

$$P\left(\sup_{0 < \leq s < t \leq \infty} \frac{|V(t) - V(s)|_\beta}{|t-s|^\delta} < \infty\right) = 1$$

(note that the equality holds also without the probability sign since the estimates are deterministic).

Finally, for the first term in (3.10) we have

$$|(T(t) - T(s))X(0)|_\beta^2 \leq C(s_0, \beta, \alpha)|t-s|^2$$

in the same way as in the proof of Corollary 1.1 in Blount and Kouritzin (2001). The proof is complete. □

4 Conclusions

We have proved existence and regularity properties of solutions to a fractal Burgers equation with a stochastic perturbation given by white noise term. Since the equation has non-Lipschitz coefficients, the analysis of existence and properties of weak solutions has been rather complicated, and uniqueness of weak solutions remains to be proved.

Financial applications similar to those in Hodges and Carverhill (1993) , Hodges and Selby (1997), such as numerical methods and simulations, and implications regarding the behavior of the related asset stock price, have not been considered here and remain to be investigated. The presence of a fractional power of the Laplacian in our equation is related to an option-pricing model where the asset log-prices follow a Lévy stable motion. For more realistic financial applications it is important also to consider fractal Burgers equations with less restrictive stochastic perturbations, that would allow to consider other behaviors of risky asset prices.

Acknowledgements

The author is grateful to D. Blount for many valuable discussions and ideas regarding the results of this paper. She also acknowledges an anonymous referee for a careful reading of the paper and useful comments. Research supported in part by CONACyT (grant C-02-42522).

(Received June, 2004. Accepted March, 2005.)

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