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Characterization of bivariate discrete distributions based on residual life properties

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Abstract: In the bivariate continuous set up exponential, Lomax and FR distributions have been characterized by Roy and Gupta (1996) using constancy of the coefficient of variation of the residual lives. The present work examines a related characterization result for the bivariate discrete set up. In this process importance of the coefficient of factorial variation has been emphasized and bivariate generalization of Waring and negative hypergeometric distributions have been obtained. Some other associated results using the measure of memory have been studied. An application has also been added at the end to demonstrate the strength of the characterization result.

Key words: Bivariate memory, coefficient of variation, geometric distribution, negative hypergeometric distribution, residual life, Waring distribution.

1 Introduction

While theoretical framework of multivariate reliability analysis shows its inclination towards continuous life distribution models (see DeMasi, 2000), in reality product lives are often described in terms of a discrete random variable like the number of strokes, number of shots or number of rotations. As a result, development of a theoretical framework for reliability analysis in the discrete domain is desirable to meet the needs of the users.

Some discussions on discrete life models had already taken place in the literature, as may be seen from the works of Barlow and Proschan (1965), Cox (1972), Kalbfleisch and Prentice (1980), Cox and Oakes (1984), and Kaio and Osaki (1988). Important characterization and classification results are also available in Xekalaki (1983), Hitha and Nair (1989), Roy (1993, 1997), Nair and Asha (1997) and Roy and Gupta (1999).

It may be observed from the literature that many of the results obtained in the continuous set up needs the attention of the researchers for the development of their discrete counterparts. Keeping this objective in mind the present paper makes an attempt to discretize the characterization results based on constancy of coefficient of variations and study one related relationship. In Section 2 we present the characterization results and in Section 3 we indicate an application of one of those results.

2 Main results

Let us restrict our discussion to nonnegative integer valued random variables and let X and Y be two such variables jointly following a bivariate life distribution with survival function S(x, y) and frequency function f(x, y), where,

$$S(x,y) = P[X \geq x, Y \geq y] = \sum_{u \geq x} \sum_{v \geq y} f(u,v).$$

We denote the bivariate failure rate of X by $\lambda_1(x,y)$ and that of Y by $\lambda_2(x,y)$ where

$$\lambda_1(x,y) := \sum_{v \geq y} f(x,v)/S(x,y); x, y = 0, 1, \ldots$$

$$\lambda_2(x,y) := \sum_{u \geq x} f(u,y)/S(x,y); x, y = 0, 1, \ldots$$

and the bivariate mean residual life of X by $M_1(x,y)$ and that of Y by $M_2(x,y)$ in the extended domain where

$$M_1(x,y) := E(X - x | X > x, Y > y); x, y = -1, 0, 1, \ldots$$

$$M_2(x,y) := E(Y - y | X > x, Y > y); x, y = -1, 0, 1, \ldots \ .$$

Since, we propose to consider a discrete version of the constancy of the coefficient of variation for unique determination of some bivariate discrete distributions of importance, we introduce the concept of coefficient of factorial variation, which will have a greater appeal in the discrete domain. For a univariate random variable X, coefficient of factorial variation (CFV) will be defined by the ratio

$$CFV = E[(X)_2]/(E(X))_2$$

provided E(X) is neither 0 nor 1. This restriction may not have any effect on the subsequent results because we will be concerned with constancy of CFV i.e. a relationship of the following type:

$$E[(X)_2] - K(E(X))_2 = 0, \tag{2.1}$$

where K is constant. It is easy to verify that (2.1) reduces to the following condition that explains its conceptual similarity with the square of the coefficient of variation:

$$Var(X) = (K-1)(E(X))_2.$$

It is obvious that (2.1) itself cannot characterize a distribution. We propose to consider the residual life in place of the original life to induce uniqueness in this condition. Thus, for the univariate residual life we rewrite the condition (2.1) as

$$E[(X-x)_2 | X > x] - K(E(X - x | X > x))_2 = 0,$$

and for the bivariate residual lives let us generalize the condition (2.1), for all x and y and for a constant K independent of x and y, as

$$E[(X-x)_2|X>x,Y>y]-K(E(X-x|X>x,Y>y))_2=0,$$
$$E[(Y-y)_2|X>x,Y>y]-K(E(Y-y|X>x,Y>y))_2=0. \qquad (2.2)$$

This is a general definition and may also cover continuous variables. The following theorem presents an implicative relationship of the condition (2.2).

Theorem 2.1. *If (2.2) holds in the extended domain then*

$$M_1(x,y)-M_1(x+1,y)=(2-K)/K=M_2(x,y)-M_2(x,y+1).$$

Proof. Under (2.2)

$$E[(X-x)_2|X>x,Y>y]-K(E(X-x|X>x,Y>y))_2=0$$

or

$$\sum_{r=1}^{\infty}\frac{r(r-1)[S(x+r,y+1)-S(x+r+1,y+1)]}{S(x+1,y+1)}=K(M_1(x,y))_2$$

or

$$\sum_{r=1}^{\infty}r(r-1)S(x+r,y+1)-\sum_{r=1}^{\infty}r(r+1)S(x+r+1,y+1)$$
$$+\,2\sum_{r=1}^{\infty}rS(x+r+1,y+1)=KS(x+1,y+1)(M_1(x,y))_2$$

or

$$2\sum_{r=1}^{\infty}rS(x+r+1,y+1)=KS(x+1,y+1)(M_1(x,y))_2. \qquad (2.3)$$

Since (2.3) is true for all choices of x we get

$$2\sum_{r=1}^{\infty}r[S(x+r+1,y+1)-S(x+1+r+1,y+1)]$$
$$=\ KS(x+1,y+1)(M_1(x,y))_2-KS(x+2,y+1)(M_1(x+1,y))_2$$

or

$$2\sum_{r=1}^{\infty}rf(x+r+1,y+1)=KS(x+1,y+1)(M_1(x,y))_2-KS(x+2,y+1)(M_1(x+1,y))_2$$

or

$$2S(x+2,y+1)(M_1(x+1,y)) = KS(x+1,y+1)M_1(x,y)(M_1(x,y)-1)$$
$$-KS(x+2,y+1)M_1(x+1,y)(M_1(x+1,y)-1). \tag{2.4}$$

But

$$S(x+1,y+1)M_1(x,y) = \sum_{r=1}^{\infty} S(x+r,y+1)$$

or

$$S(x+1,y+1)M_1(x,y) = S(x+1,y+1) + \sum_{r=2}^{\infty} S(x+r,y+1)$$

or

$$S(x+1,y+1)M_1(x,y) = S(x+1,y+1) + S(x+2,y+1)M_1(x+1,y)$$

or

$$S(x+2,y+1)M_1(x+1,y) = S(x+1,y+1)[M_1(x,y)-1]. \tag{2.5}$$

Using (2.5) in (2.4) we obtain

$$\begin{aligned} 2S(x+1,y+1)[M_1(x,y)-1] &= KS(x+1,y+1)M_1(x,y)[M_1(x,y)-1] \\ &- KS(x+1,y+1)[M_1(x,y)-1] \\ & [M_1(x+1,y)-1] \end{aligned}$$

or

$$2 = KM_1(x,y) - K[M_1(x+1,y)-1]$$

or

$$M_1(x,y) - M_1(x+1,y) = (2-K)/K.$$

Similarly,

$$M_2(x,y) - M_2(x,y+1) = (2-K)/K.$$

■

Next theorem is an easy consequence of Theorem 2.1.

Theorem 2.2. *If the condition (2.2) holds in the extended domain then $h_1(x,y)$ is inversely and locally linear in x and $h_2(x,y)$ is inversely and locally linear in y.*

Proof. In view of Theorem 2.1 we have

$$\begin{aligned} M_1(0,y) &= (2-K)/K + M_1(1,y) = 2(2-K)/K + M_1(2,y) = \dots \\ &= x(2-K)/K + M_1(x,y) \end{aligned}$$

or

$$M_1(x,y) = M_1(0,y) - x(2-K)/K.$$

Similarly,

$$M_2(x,y) = M_2(x,0) - y(2-K)/K.$$

Now, from the relationships between bivariate failure rates and mean residual lives (see Roy, 1993) we get

$$h_1(x,y) = \frac{1 - M_1(0,y-1) + (x-1)(2-K)/K}{M_1(0,y-1) - x(2-K)/K} + 1$$

$$= \frac{1 - (2-K)/K}{M_1(0,y-1) - x(2-K)/K}.$$

Thus, $h_1(x,y)$ is inversely and locally linear in x. Similarly,

$$h_2(x,y) = \frac{1 - (2-K)/K}{M_2(x-1,0) - y(2-K)/K}$$

which is inversely and locally linear in y. ■

We are now in a position to present the main result characterizing discrete bivariate life distributions based on constancy of the coefficient of factorial variation. In this process, we obtain the bivariate geometric distribution studied in Roy (1993) (to be abbreviated as BVG), a bivariate Waring distribution and a bivariate negative hypergeometric distribution.

Theorem 2.3. *Let the bivariate factorial moments satisfy the condition (2.2) in the extended domain. Then, with symbols having their usual meanings, (i) K=2 if and only (X, Y) follows the BVG distribution with survival function given by*

$$S(x,y) = {\theta_1}^x {\theta_2}^y {\theta_3}^{xy}, 0 < \theta_1, \theta_2 < 1, (\frac{\theta_1 + \theta_2 - 1)}{(\theta_1\theta_2)} \le \theta_3 \le 1; \tag{2.6}$$

(ii) $K > 2$ if and only (X, Y) follows a bivariate Waring distribution with survival function given by

$$S(x,y) = \frac{\left(\frac{M_2(0,0)-1-e}{e}\right)_{(y)} \left(\frac{M_1(0,0)+\beta y-1-e}{e}\right)_{(x)}}{\left(\frac{M_2(0,0)}{e}\right)_{(y)} \left(\frac{M_1(0,0)+\beta y}{e}\right)_{(x)}}; \tag{2.7}$$

(iii) $K < 2$ if and only if (X, Y) follows a bivariate negative hypergeometric distribution with survival function given by

$$S(x,y) = \frac{\left(\frac{M_2(0,0)-1+d}{d}\right)_y \left(\frac{M_1(0,0)+\beta y-1+d}{d}\right)_x}{\left(\frac{M_2(0,0)}{d}\right)_y \left(\frac{M_1(0,0)+\beta y}{d}\right)_x}. \tag{2.8}$$

Proof. We have already observed in Theorem 2.2 that if (2.2) holds then

$$M_1(x,y) = M_1(0,y) - dx, M_2(x,y) = M_2(x,0) - dy \tag{2.9}$$

where d = (2 - K) / K.

Case (i) Let K = 2. Then from (2.9) we observe that

$$M_1(x,y) = M_1(0,y)$$

$$M_2(x,y) = M_2(x,0).$$

In other words, $M_1(x,y)$ is a constant in respect of x and $M_2(x,y)$ is a constant in respect of y. These imply that the bivariate failure rates are locally constants. Thus, from the relationships

$$S(0,y) = \sum_{v \geq y} [1-\lambda_2(0,0)]...[1-\lambda_2(0,v-1)]\lambda_2(0,v)]$$

and

$$S(x,y) = S(0,y)[1-\lambda_1(0,y)]...[1-\lambda_1(x-1,y)], \quad x,y = 0,1,2,\ldots \tag{2.10}$$

we get

$$S(x,y) = S(0,y)[1-\lambda_1(0,y)]^x. \tag{2.11}$$

Similarly,

$$S(x,y) = S(x,0)[1-\lambda_2(x,0)]^y. \tag{2.12}$$

Using (2.12) we get $S(0,y)$, and can simplify (2.11) as

$$S(x,y) = [1-\lambda_2(0,0)]^y[1-\lambda_1(0,y)]^x. \tag{2.13}$$

Similarly, we observe that

$$S(x,y) = [1-\lambda_2(x,0)]^y[1-\lambda_1(0,0)]^x. \tag{2.14}$$

Comparing (2.13) with (2.14) we ensure that

$$log[1-\lambda_1(0,y)] = \alpha + \beta y. \tag{2.15}$$

Simplifying (2.13) with (2.15) we get

$$S(x,y) = {\theta_1}^x {\theta_2}^y {\theta_3}^{xy}$$

which is the survival function of the BVG distribution. The converse is easy to prove.

Case (ii) Let $K > 2$. Then $d < 0$. Hence from (2.9) we can write

$$M_1(x,y) = M_1(0,y) + ex\ , \quad M_2(x,y) = M_2(x,0) + ey, \tag{2.16}$$

where $e = -d$ is a positive quantity. Thus,

$$\lambda_1(x,y) = \frac{1+e}{M_1(0,y)+ex}, \quad \lambda_2(x,y) = \frac{1+e}{M_2(x,0)+ey}. \tag{2.17}$$

Hence from Theorem 2.2 we get via (2.10) that

$$S(x,y) = S(0,y)\Pi_{s=1}^{x}\frac{M_1(0,y) - 1 - e + e(s-1)}{M_1(0,y) + e(s-1)}. \tag{2.18}$$

Similarly, we obtain

$$S(x,y) = S(x,0)\Pi_{r=1}^{y}\frac{M_2(x,0) - 1 - e + e(r-1)}{M_2(x,0) + e(r-1)}. \tag{2.19}$$

Obtaining an expression of $S(0,y)$ from (2.19) and substituting the same in (2.18) we get

$$S(x,y) = \Pi_{r=1}^{y}\frac{M_2(0,0) - 1 - e + e(r-1)}{M_2(0,0) + e(r-1)}\Pi_{s=1}^{x}\frac{M_1(0,y) - 1 - e + e(s-1)}{M_1(0,y) + e(s-1)} \tag{2.20}$$

or

$$\frac{S(x+1,y+1)S(x,y)}{S(x+1,y)S(x,y+1)} = \left[1 - \frac{1+e}{M_1(0,y+1)+ex}\right]\left[1 + \frac{1+e}{M_1(0,y) - 1 - e + ex}\right]. \tag{2.21}$$

Similarly, obtaining an expression of $S(x,0)$ from (2.18) and substituting the same in (2.19) we get

$$S(x,y) = \Pi_{r=1}^{y}\frac{M_2(x,0) - 1 - e + e(r-1)}{M_2(x,0) + e(r-1)}\Pi_{s=1}^{x}\frac{M_1(0,0) - 1 - e + e(s-1)}{M_1(0,0) + e(s-1)} \tag{2.22}$$

or

$$\frac{S(x+1,y+1)S(x,y)}{S(x+1,y)S(x,y+1)} = \left[1 - \frac{1+e}{M_2(x+1,0)+ey}\right]\left[1 + \frac{1+e}{M_2(x,0) - 1 - e + ey}\right]. \tag{2.23}$$

Thus, comparing (2.21) with (2.23) we note that $M_1(0,y)$ must be linear in y and $M_2(x,0)$ must be linear in x. Also,

$$M_1(0,y) + ex = M_2(x,0) + ey.$$

Writing, $M_1(0,y) = \alpha + \beta y$ we note that $\alpha = M_1(0,0)$, and hence

$$M_1(0,y) = M_1(0,0) + \beta y. \tag{2.24}$$

Simplifying (2.20) with (2.24) we obtain the survival function of (X,Y) as

$$S(x,y) = \prod_{r=1}^{y}\frac{M_2(0,0) - 1 - e + e(r-1)}{M_2(0,0) + e(r-1)}\prod_{s=1}^{x}\frac{M_1(0,0) + \beta y - 1 - e + e(s-1)}{M_1(0,0) + \beta y + e(s-1)} \tag{2.25}$$

or

$$S(x,y) = \prod_{r=1}^{y}\frac{\frac{M_2(0,0)-1-e}{e} + (r-1)}{\frac{M_2(0,0)}{e} + (r-1)}\prod_{s=1}^{x}\frac{\frac{M_1(0,0)+\beta y-1-e}{e} + (s-1)}{\frac{M_1(0,0)+\beta y}{e} + (s-1)}$$

or

$$S(x,y) = \frac{\left(\frac{M_2(0,0)-1-e}{e}\right)_{(y)}}{\left(\frac{M_2(0,0)}{e}\right)_{(y)}} \frac{\left(\frac{M_1(0,0)+\beta y-1-e}{e}\right)_{(x)}}{\left(\frac{M_1(0,0)+\beta y}{e}\right)_{(x)}},$$

where $(a)_{(x)} = \Gamma(a+x)/\Gamma(a)$ is the Pochhammer's symbol. Alternatively,

$$S(x,y) = \frac{\left(\frac{M_1(0,0)-1-e}{e}\right)_{(x)}}{\left(\frac{M_1(0,0)}{e}\right)_{(x)}} \frac{\left(\frac{M_2(0,0)+\delta y-1-e}{e}\right)_{(y)}}{\left(\frac{M_2(0,0)+\delta y}{e}\right)_{(y)}},$$

where δ is a constant such that

$$M_2(x,0) = M_2(0,0) + \delta x \ . \tag{2.26}$$

The converse is easy to prove.

Case (iii) The line of approach is similar to that of case (ii). However for $K < 2$ the value of d is positive. Then from (2.25) we have after necessary modifications (i.e. by replacing e by -d)

$$S(x,y) = \prod_{r=1}^{y} \frac{M_2(0,0)-1+d-d(r-1)}{M_2(0,0)-d(r-1)} \prod_{s=1}^{x} \frac{M_1(0,0)+\beta y-1+d-d(s-1)}{M_1(0,0)+\beta y-d(s-1)}$$

or

$$S(x,y) = \prod_{r=1}^{y} \frac{\frac{M_2(0,0)-1+d}{d}-(r-1)}{\frac{M_2(0,0)}{d}-(r-1)} \prod_{s=1}^{x} \frac{\frac{M_1(0,0)+\beta y-1+d}{d}-(s-1)}{\frac{M_1(0,0)+\beta y}{d}-(s-1)}$$

or

$$S(x,y) = \frac{\left(\frac{M_2(0,0)-1+d}{d}\right)_y}{\left(\frac{M_2(0,0)}{d}\right)_y} \frac{\left(\frac{M_1(0,0)+\beta y-1+d}{d}\right)_x}{\left(\frac{M_1(0,0)+\beta y}{d}\right)_x},$$

where $(a)_r$ stands for the standard notation for permutation. ■

Remark 2.1. The concept of quadrant dependence was introduced in Lehmann (1966) and studied by Shaked (1982) through unidirectional ordering between the joint survival function and the product of the marginal survival functions. Following Lehmann (1966), quadrant dependence is positive if the former is greater than the latter and negative if the former is less than the latter. Under the BVG distribution, $S(x_1,x_2) \le S_1(x_1)S_2(x_2)$, and hence the BVG distribution has negative quadrant dependence. The corresponding correlation coefficient is negative.

Theorem 2.4. *For discrete bivariate setup let* $\lambda_1(x,y)M_1(x,y-1) = c = \lambda_2(x,y)$ $M_2(x-1,y)$ *for all* $x, y = 0, 1, 2, \ldots$. *Then (i)* $c = 1$ *if and only if* (X,Y) *follows BVG with survival function given by (2.6), (ii)* $c > 1$ *if and only if* (X,Y) *follows a bivariate Waring distribution with survival given by (2.7), (iii)* $c < 1$ *if and only if* (X,Y) *follows a bivariate negative hypergeometric distribution with survival function given by (2.8).*

Proof. From the relationships between bivariate failure rates and mean residual lives we have

$$M_1(x, y-1)\lambda_1(x, y) = 1 - M_1(x-1, y-1) + M_1(x, y-1)$$

or,

$$M_1(x, y-1)\lambda_1(x, y) = 1 + \Delta M_1(x-1, y-1)$$

or,

$$\Delta M_1(x-1, y-1) = 1 - c, \quad \text{for} \quad x, y = 0, 1, 2, \dots \, .$$

Similarly,

$$\Delta M_2(x-1, y-1) = 1 - c, \quad \text{for} \quad x, y = 0, 1, 2, ...$$

Thus,

$$\begin{aligned} M_1(x, y-1) &= (1-c) + M_1(x-1, y-1) \\ &= (1-c) + (1-c) + M_1(x-2, y-1) = \dots \\ &= (1-c)x + M_1(0, y-1). \end{aligned}$$

Similarly,

$$M_2(x-1, y) = (1-c)y + M_2(x-1, 0).$$

Rest of the proof follows from that of Theorem 2.3 with $d = (2-K)/K$ replaced by $(1-c)$. ■

3 An application in stochastic modeling

Let us consider a set of sperm viability data observed over 20 weeks for twins belonging to a lower fertility group studied under recovery of testicular function. The sperm viability is measured in terms of percentage of living normal cells in the sperm specimen. We may consider sperm viability as a discrete variable as it gets generated under a counting process. Other biomarkers like sperm concentration measured in terms of sperm count per milliliter ejaculate have not been taken into consideration for this modeling problem to restrict our analysis to a bivariate set up. The sperm viability for the first twin be represented by X and that of the second twin be represented by Y. The bivariate information on (X, Y) is given by (48, 47), (53, 58), (63, 55), (61, 58), (52, 50), (50, 54), (50, 47), (48, 45), (44, 41), (45, 50), (42, 46), (45, 47), (43, 45), (44, 49), (43, 44), (46, 45), (43, 43), (41, 47), (42, 40) and (40, 44).

It is easy to note that there will be at the most 400 combinations each to be studied for residual life analysis of X and Y. Our objective is to calculate the coefficient of factorial variation for each of X and Y based on each set of residual data. The meaning of the term residual should be treated as a conditional one and CFV should be treated as a conditional measure of the bivariate distribution. For the purpose of demonstration of the calculation of CFV, we may consider the residual observations under the condition $\{X > 45, Y > 41\}$. Following is the set of residual observations satisfying this condition: (48, 47), (53, 58), (63, 55), (61, 58), (52, 50), (50, 54), (50, 47), (48, 45) and (46, 45).

Then $(X - 45)$ will have 9 values as 3, 8, 18, 16, 7, 5, 5, 3 and 1. The corresponding value of $(E(X-45|X > 45, Y > 41))_2$ will be equal to (7.333)(7.333-1)=46.440 and that of $E[(X-45)_2|X > 45, Y > 41]$ will be equal to 77.333. Thus, CFV works out as

$$CFV = E[(X - 45)_2|X > 45, Y > 41]/(E(X - 45|X > 45, Y > 41))_2 = 1.665.$$

But we need to examine CFV values for X for all choices of x and y of X and Y respectively. To be more specific, we need to examine the constancy of those CFV values. For doing so, we have calculated the variance of such CFV values using a computer program. There are all total 358 effective combinations with variance of CFV values of X as 0.176. This being a very small value we may arrive at the conclusion that CFV values for X are nearly constant. We shall treat the mean CFV value as that constant value. It comes out as 1.648. Similar such studies for Y reveals that there are all total 360 effective combinations with variance of CFV values as 0.169. This is again a very small value and hence we may conclude that CFV values for Y are almost constant. We shall treat the mean CFV value as that constant value. It comes out as 1.762. Since these two constants are quite close we may consider K value of the condition (2.2) as $(1.648 + 1.762)/2 = 1.705$, the average of these two values. Now, making appeal to Theorem 2.1 we conclude that the underlying distribution can be uniquely determined by the corresponding characterization result of bivariate negative hypergeometric distribution of Theorem 2.3. However, we have to shift the point of origin to the point (40, 40) to match with the starting point of the bivariate model at (0,0). Parameters of the fitted model can be obtained by the standard moment method. In case one considers K values, 1.648 and 1.762, as markedly different we cannot apply the Theorem 2.3. In case there were negative results that the conditions of constancy of both the CFV values do not hold true we would have concluded that neither bivariate geometric distribution nor bivariate Waring distribution nor bivariate hypergeometric distribution could describe the variations in the data set.

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References

Barlow, R. E. and Proschan, F. (1965). *Mathematical Theory of Reliability.* New York: Wiley.

Cox, D. R. (1972). Regression models with life tables (with discussions). *Journal of Royal Statistical Society, B*, **34**, 187–226.

Cox, D. R. and Oakes, D. (1984). *Analysis of Survival Data*. London: Chapman and Hall.

DeMasi, R. A. (2000). Statistical methods for multivariate failure time data and competing risk, In Handbook of Statistics, ed. Sen, P. K. and Rao, C. R., 749–781, North-Holland, Amsterdam: Elsevier.

Hitha, N. and Nair, U. N. (1989). Characterization of some discrete models by properties of residual life functions. *Calcutta Statistical Association Bulletin*, **38**, 219–223.

Kaio, N. and Osaki, S. (1988). Review of discrete and continuous distributions in replacement models. *International Journal of Systems Science*, **19**, 171–177.

Kalbfleisch, J. D. and Prentice, R. L. (1980). *The Statistical Analysis of Failure Time Data*. New York: Wiley.

Lehmann, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist.*, **37**, 1137–1153.

Nair, U. N. and Asha, G. (1997). Some classes of multivariate life distributions in discrete time. *Journal of Multivariate Analysis*, **62**, 181–189.

Roy, D. (1993). Reliability measures in the discrete bivariate set up and related characterization results for a bivariate geometric distribution. *Journal of Multivariate Analysis*, **46**, 362–373.

Roy, D. (1997). On classifications of multivariate life distributions in the discrete setup. *Microelectronics and Reliability*, **37**, 361–366.

Roy, D. and Gupta, R. P. (1996). Bivariate extension of Lomax and finite range distributions through chracterization approach. *Journal of Multivariate Analysis*, **59**, 22–33.

Roy, D. and Gupta, R. P. (1999). Chracterizations and model selections through reliability measures in the discrete case. *Statistics and Probability Letters*, **43**, 197–206.

Shaked, M. (1982). A general theory of some positive dependence, *Journal Multivariate Analysis*, **12**, 199–218.

Xekalaki, E. (1983). Hazard functions and life distributions in discrete time. *Communication in Statistics - Theory and Methods*, **12**, 2503–2509.

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