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A note on the product of normal and Laplace random variables

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Abstract: The distribution of the product $| XY |$ is derived when X and Y are normal and Laplace random variables distributed independently of each other.

Key words: Laplace distribution, normal distribution, products of random variables.

1 Introduction

For given random variables X and Y, the distribution of the product $| XY |$ is of interest in problems in biological and physical sciences, econometrics, and classification. As an example in Physics, Sornette (1998) mentions:

> "... To mimic system size limitation, Takayasu, Sato, and Takayasu introduced a threshold x_c ... and found a stretched exponential truncating the power-law pdf beyond x_c. Frisch and Sornette recently developed a theory of extreme deviations generalizing the central limit theorem which, when applied to multiplication of random variables, predicts the generic presence of stretched exponential pdfs. The problem thus boils down to determining the tail of the pdf for a product of random variables ..."

The distribution of $| XY |$ has been studied by several authors especially when X and Y are independent random variables and come from the same family. For instance, see Sakamoto (1943) for uniform family, Harter (1951) and Wallgren (1980) for Student's t family, Springer and Thompson (1970) for normal family, Stuart (1962) and Podolski (1972) for gamma family, Steece (1976), Bhargava and Khatri (1981) and Tang and Gupta (1984) for beta family, Abu-Salih (1983) for power function family, and Malik and Trudel (1986) for exponential family (see also Rathie and Rohrer (1987) for a comprehensive review of known results). However, there is relatively little work of this kind when X and Y belong to different families. In the applications mentioned above, it is quite possible that X and Y could arise from different but similar distributions.

In this note, we study the distribution of $\mid XY \mid$ when X and Y are independent random variables having the normal and Laplace distributions with pdfs

$$f(x) \quad = \quad \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \tag{1.1}$$

and

$$f(y) \quad = \quad \frac{\lambda}{2} \exp\left(-\lambda \mid y \mid\right), \tag{1.2}$$

respectively, for $-\infty < x < \infty$, $-\infty < y < \infty$, $\sigma > 0$ and $\lambda > 0$. Note that these two distributions – proposed by Laplace in 1774 and 1778, respectively – are the oldest known continuous distributions in statistics.

The aim of this note is to calculate the distribution of the product $\mid XY \mid$ when X and Y are distributed according to (1.1) and (1.2), respectively. A MAPLE program for computing the associated percentage points is also provided. The calculations of this note involve the complementary error function defined by

$$\mathrm{erfc}(x) \quad = \quad \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp\left(-t^2\right) dt$$

and the hypergeometric function defined by

$$G\left(a; b, c, d; x\right) \quad = \quad \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k\,(c)_k\,(d)_k} \frac{x^k}{k!},$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. We also need the following important lemma.

Lemma 1 *(Equation (2.8.5.14), Prudnikov et al., 1986, volume 2) For* $p > 0$,

$$\begin{aligned}
& \int_0^{\infty} x^{\alpha-1} \exp\left(-p/x\right) \mathit{erfc}(cx) dx \\
= \; & p^{\alpha}\Gamma(-\alpha) - \frac{2cp^{\alpha+1}}{\sqrt{\pi}}\Gamma(-\alpha-1) G\left(\frac{1}{2}; \frac{3}{2}, \frac{3+\alpha}{2}, 1+\frac{\alpha}{2}; -\frac{c^2p^2}{4}\right) \\
& + \frac{1}{c^{\alpha}\sqrt{\pi}\alpha}\Gamma\left(\frac{\alpha+1}{2}\right) G\left(-\frac{\alpha}{2}; 1-\frac{\alpha}{2}, \frac{1}{2}, \frac{1-\alpha}{2}; -\frac{c^2p^2}{4}\right) \\
& + \frac{p}{c^{\alpha-1}\sqrt{\pi}(1-\alpha)}\Gamma\left(\frac{\alpha}{2}\right) G\left(\frac{1-\alpha}{2}; \frac{3-\alpha}{2}, 1-\frac{\alpha}{2}, \frac{3}{2}; -\frac{c^2p^2}{4}\right).
\end{aligned}$$

Further properties of the complementary error function and the hypergeometric function can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2 CDF

Theorem 2 derives an explicit expression for the cdf of $\mid XY \mid$ in terms of the hypergeometric function.

Theorem 2 *Suppose X and Y are distributed according to (1.1) and (1.2), respectively. Then, the cdf of $Z = \mid XY \mid$ can be expressed as*

$$F(z) = \frac{\lambda z}{\sqrt{2}\sigma} \left\{ \frac{3C}{\sqrt{\pi}} G \left(\frac{1}{2}; \frac{3}{2}, 1, \frac{1}{2}; -\frac{\lambda^2 z^2}{8\sigma^2} \right) + \frac{\lambda z}{\sqrt{2}\sigma} G \left(1; 2, \frac{3}{2}, \frac{3}{2}; -\frac{\lambda^2 z^2}{8\sigma^2} \right) \right\}, \tag{2.1}$$

where C denotes Euler's constant.

Proof. The cdf $F(z) = \Pr(\mid XY \mid \leq z)$ can be expressed as

$$F(z) = \frac{\lambda}{2} \int_{-\infty}^{\infty} \left\{ \Phi \left(\frac{z}{\sigma \mid y \mid} \right) - \Phi \left(-\frac{z}{\sigma \mid y \mid} \right) \right\} \exp\left(-\lambda \mid y \mid \right) dy, \tag{2.2}$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution. Using the relationship

$$\Phi(-x) = \frac{1}{2} \mathrm{erfc} \left(\frac{x}{\sqrt{2}} \right),$$

(2.2) can be rewritten as

$$\begin{aligned} F(z) &= \frac{\lambda}{2} \int_{-\infty}^{\infty} \exp\left(-\lambda y \right) \mathrm{erfc} \left(-\frac{z}{\sqrt{2}\sigma \mid y \mid} \right) dy - 1 \\ &= \lambda \int_{0}^{\infty} \exp\left(-\lambda y \right) \mathrm{erfc} \left(-\frac{z}{\sqrt{2}\sigma y} \right) dy - 1 \\ &= \lambda \int_{0}^{\infty} y^{-2} \exp\left(-\lambda / y \right) \mathrm{erfc} \left(-\frac{yz}{\sqrt{2}\sigma} \right) dy - 1. \end{aligned} \tag{2.3}$$

The integral in (2.3) can be calculated by direct application of Lemma 1. The result follows. ■

Note that the parameters in (2.1) are functions of λ/σ (ratio of scale parameters). Figure 1 illustrates possible shapes of the pdf of $\mid XY \mid$ for a range of values of λ/σ. Note that the shapes are unimodal and that the value of λ/σ largely dictates the behavior of the pdf near $z = 0$.

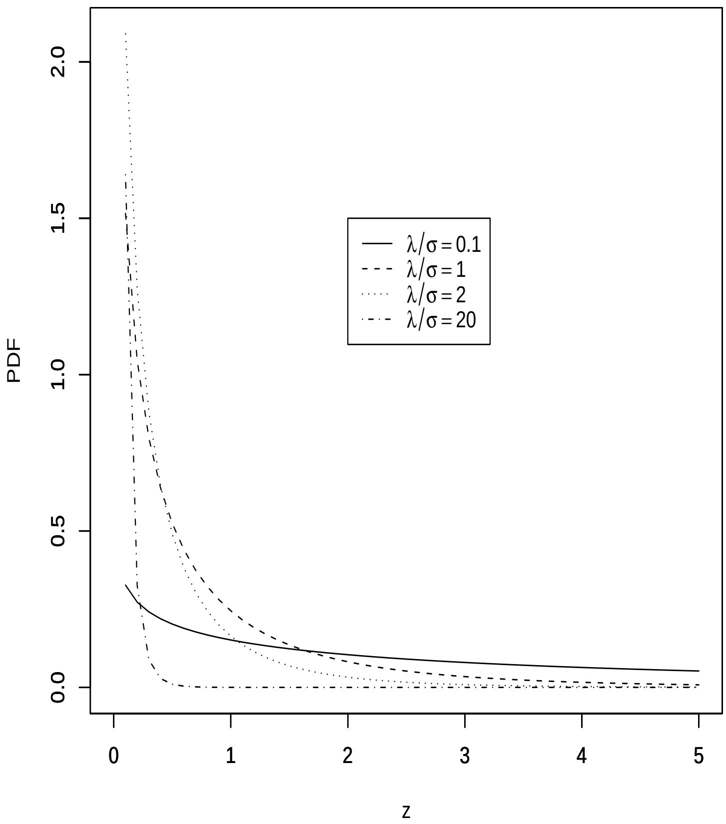

Figure 1 *Plots of the pdf of (2.1) for* $\lambda/\sigma = 0.1, 1, 2, 20$ *and* $\sigma = 1$.

Appendix

The program in MAPLE appended below can be used to generate percentage points z_p associated with the cdf (2.1). The values z_p are obtained by numerically solving the equation

$$\frac{\lambda z_p}{\sqrt{2}\sigma} \left\{ \frac{3C}{\sqrt{\pi}} G\left(\frac{1}{2}; \frac{3}{2}, 1, \frac{1}{2}; -\frac{\lambda^2 z_p^2}{8\sigma^2}\right) + \frac{\lambda z_p}{\sqrt{2}\sigma} G\left(1; 2, \frac{3}{2}, \frac{3}{2}; -\frac{\lambda^2 z_p^2}{8\sigma^2}\right) \right\} = p.$$

Evidently, this involves computation of the hypergeometric function and routines for this are widely available. We used the function hypergeom $(\cdot)$ in MAPLE.

```
lambdaoversigma:=lambda/sigma:
f1:=(3*gamma/(sqrt(2*Pi))*(u*lambdaoversigma):
f1:=f1*hypergeom([1],[3/2,1,1/2],-(1/8)*(u*lambdaoversigma)**2):
f2:=(1/2)*(u*lambdaoversigma)**2:
f2:=f2*hypergeom([1],[2,3/2,3/2],-(1/8)*(u*lambdaoversigma)**2):
ff:=f1+f2:
p1:=fsolve(ff=0.01,u=0..10000):
p2:=fsolve(ff=0.05,u=0..10000):
p3:=fsolve(ff=0.1,u=0..10000):
p4:=fsolve(ff=0.90,u=0..10000):
p5:=fsolve(ff=0.95,u=0..10000):
p6:=fsolve(ff=0.99,u=0..10000):
print(lambdaoversigma,p1,p2,p3,p4,p5,p6);
```

We hope that this program will be of use to the practitioners mentioned in Section 1.

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