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Skewed distributions generated by the Cauchy kernel

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Abstract: Following the recent paper by A. K. Gupta *et al.* [*Random Operators and Stochastic Equations*, **10**, 2002, 133–140], we generate skew pdfs of the form $2f(u)G(\lambda u)$, where f is taken to be a Cauchy pdf while the cdf G is taken to come from one of normal, Student's t, Cauchy, Laplace, logistic or uniform distribution. The properties of the resulting distributions are studied. In particular, expressions for the characteristic functions are derived. We also provide graphical illustrations and an application to exchange rate data.

Key words: Cauchy distribution, skewed distributions.

1 Introduction

Univariate skew-symmetric models have been considered by several authors. A classical example is the skew normal distribution with the probability density function (pdf) $f(x) = 2\phi(x)\Phi(\lambda x)$ (where $\phi(\cdot)$ and $\Phi(\cdot)$, respectively, denote the pdf and the cumulative distribution function (cdf) of the standard normal distribution). This distribution was introduced by Azzalini (1985). See Gupta *et al.* (2002) for a most detailed discussion of skew-symmetric models based on the normal, Student's t, Cauchy, Laplace, logistic and uniform distributions. The main feature of these models is that a new parameter λ is introduced to control skewness and kurtosis. Thus, for example, the skew normal distribution allows for continuous variation from normality to non-normality, which is useful in many practical situations (Hill and Dixon, 1982; Arnold *et al.*, 1993). Skew-symmetric models have also been used in studying robustness and as priors in Bayesian estimation (O'Hagan and Leonard, 1976; Mukhopadhyay and Vidakovic, 1995).

Lemma 1 *Let U and V be two arbitrary absolutely continuous independent random variables symmetric about 0, with pdfs f and g and cdfs F and G, respectively. Then for any $\lambda \in \Re$, the function*

$$f_X(x) \quad = \quad 2f(x)G(\lambda x) \tag{1.1}$$

is a valid pdf of a random variable, say X.

The construction of univariate skew-symmetric models is based on the above general result due to Azzalini (1985). For example, the skew normal distribution is obtained by taking $f \equiv \phi$ and $G \equiv \Phi$ in (1.1). The models in Gupta *et al.* (2002) are obtained by taking both f and G to belong to one of normal, Student's t, Cauchy, Laplace, logistic or uniform family. See also Balakrishnan and Ambagaspitiya (1994) and Arnold and Beaver (2000a, 2000b) for similar constructions.

Mukhopadhyay and Vidakovic (1995) pointed out an extension of the above approaches by suggesting that one takes f and G in (1.1) to belong to different families. This idea was first followed up by Nadarajah (2003) and Nadarajah and Kotz (2003, 2004), where f was taken to be a normal pdf while g was taken to belong to one of normal, Student's t, Cauchy, Laplace, logistic or uniform families. In this paper, we carry out an analogous construction by taking f to be the pdf of a Cauchy distribution with scale parameter γ. Consequently, we have the following skewed models generated by a Cauchy kernel: the skew Cauchy-normal model (Section 2), the skew Cauchy-t model (Section 3), the skew Cauchy-Cauchy model (Section 4), the skew Cauchy-Laplace model (Section 5), the skew Cauchy-logistic model (Section 6) and the skew Cauchy-uniform model (Section 7). We study the characteristic function of each of these models and provide graphical illustrations. We also provide an application to exchange rate data. We assume without loss of generality that $\lambda \geq 0$ in (1.1) since the corresponding properties for $\lambda < 0$ can be obtained by using the fact $G(\lambda x) = 1 - G(-\lambda x)$. Note that just like for the Cauchy distribution the moments of X do not exist. The characteristic function for the six models to be considered involves that of the Cauchy distribution, which we shall denote by $\phi(t)$. Several closed form expressions for $\phi(t)$ have been derived in the literature (see Johnson *et al.* (1995) for a good collection).

Besides the applications mentioned above, the model (1.1) can be motivated stochastically by one of the following representations (due to Azzalini (1986)):

- $X = S_U U$, where, conditionally on $U = u$, $S_U = +1$ with probability $G(\lambda u)$ and $S_U = -1$ with probability $1 - G(\lambda u)$.
- $X = S_U \mid U \mid$, where, conditionally on $\mid U \mid = \mid u \mid$, $S_U = +1$ with probability $G(\lambda \mid u \mid)$ and $S_U = -1$ with probability $1 - G(\lambda \mid u \mid)$.

Both these representations have clear physical meanings. The model (1.1) can also be interpreted as the conditional pdf of U given $\lambda U > V$, where U and V are two absolutely continuous independent random variables symmetric about 0, with pdfs f and g.

We shall not give details of the derivations in this paper. Our calculations make use of several special functions. They are the exponential integral, the integral cosine, the integral sine, the incomplete beta function ratio, the modified Bessel functions of the first and third kind, and the Gauss hypergeometric function defined by

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{\exp(t)}{t} dt,$$

$$\begin{aligned}
\mathrm{ci}(x) &= -\int_x^\infty \frac{\cos t}{t} dt, \\
\mathrm{si}(x) &= -\int_x^\infty \frac{\sin t}{t} dt, \\
I_x(a,b) &= \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt, \\
I_a(x) &= \sum_{k=0}^{\infty} (x/2)^{2k+a} k! \Gamma(k+a+1), \\
K_a(x) &= \frac{\pi \left\{ I_{-a}(x) - I_a(x) \right\}}{2 \sin(a\pi)},
\end{aligned}$$

and

$$ {}_2F_1(\alpha, \beta; \gamma; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!}, $$

where $(c)_k = c(c+1)\cdots(c+k-1)$ denotes the ascending factorial. The properties of these special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2 Skew Cauchy-normal model

Take g to be a normal pdf with zero mean and variance σ^2. Then (1.1) yields the pdf:

$$ f_X(x) = \frac{2}{\pi\gamma} \left\{ 1 + \left(\frac{x}{\gamma} \right)^2 \right\}^{-1} \Phi\left(\frac{\lambda x}{\sigma} \right) \tag{2.1} $$

for $-\infty < x < \infty$. The characteristic function of X is

$$ E\left[\exp(itX)\right] = \gamma\phi(\gamma t) + \frac{2\gamma i}{\pi} \int_0^\infty \frac{\sin(\gamma t u)}{1+u^2} \left\{ 2\Phi\left(\frac{\lambda \gamma u}{\sigma} \right) - 1 \right\} du. $$

3 Skew Cauchy-t model

Take g to be the pdf of the Student's t distribution, i.e.

$$ g(x) = \frac{1}{\sqrt{\nu} B(\nu/2, 1/2)} \left(1 + \frac{x^2}{\nu} \right)^{-(1+\nu)/2}, \quad -\infty < x < \infty. \tag{3.1} $$

Considering the properties of the incomplete beta function ratio:

$$I_x(a,b) = \sum_{l=1}^{b} \frac{x^a (1-x)^{l-1}}{(a+l-1)B(a,l)} \quad \text{for integer } b$$

and

$$I_x\left(\frac{1}{2}, j-\frac{1}{2}\right) = \frac{2}{\pi}\arctan\sqrt{\frac{x}{1-x}} + \sum_{l=1}^{j-1} c_l,$$

where

$$c_l = \frac{x^{1/2}(1-x)^{l-1/2}}{lB\left(1/2, l+1/2\right)}.$$

it can be shown that the cdf corresponding to (3.1) is:

$$G(x) = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x}{\sqrt{\nu}}\right) + \frac{1}{2\pi}\sum_{l=1}^{(\nu-1)/2} B\left(l, \frac{1}{2}\right)\frac{(\nu)^{l-1/2}\, x}{\left(\nu+x^2\right)^{l}} \tag{3.2}$$

if ν is odd and

$$G(x) = \frac{1}{2} + \frac{1}{2\pi}\sum_{l=1}^{\nu/2} B\left(l-\frac{1}{2}, \frac{1}{2}\right)\frac{(\nu)^{l-1}\, x}{\left(\nu+x^2\right)^{l-1/2}} \tag{3.3}$$

if ν is even (Nadarajah and Kotz, 2003). Substituting (3.2) and (3.3) into (1.1), we obtain the pdf of X for the skewed Cauchy-t model. The characteristic function of X is

$$\begin{aligned} E\left[\exp(itX)\right] &= \gamma\phi(\gamma t) + \frac{4\gamma i}{\pi^2}\int_0^\infty \frac{\sin(\gamma t u)}{1+u^2}\arctan\left(\frac{\lambda\gamma u}{\sqrt{\nu}}\right) du \\ &+ \frac{2i\lambda\gamma}{\pi^2\sqrt{\nu}}\sum_{k=1}^{(\nu-1)/2} (\nu)^k\, B\left(k, \frac{1}{2}\right)\int_0^\infty \frac{u\sin(\gamma t u)}{1+u^2}\left(\nu+\lambda^2\gamma^2u^2\right)^{-k} du \end{aligned}$$

if ν is odd, and

$$\begin{aligned} E\left[\exp(itX)\right] &= \gamma\phi(\gamma t) + \frac{2i\lambda\gamma}{\pi^2\nu}\sum_{k=1}^{\nu/2} (\nu)^k\, B\left(k-\frac{1}{2}, \frac{1}{2}\right) \\ &\quad \int_0^\infty \frac{u\sin(\gamma t u)}{1+u^2}\left(\nu+\lambda^2u^2\right)^{1/2-k} du \end{aligned}$$

if ν is even.

4 Skew Cauchy-Cauchy model

If g is taken to be the Cauchy pdf

$$g(x) = \frac{1}{\pi\gamma'}\left\{1+\left(\frac{x}{\gamma'}\right)^2\right\}^{-1}, \qquad -\infty < x < \infty$$

then from (1.1) we obtain the skew Cauchy-Cauchy model for X. The pdf of X becomes

$$f_X(x) = \frac{1}{\pi\gamma}\left\{1+\left(\frac{x}{\gamma}\right)^2\right\}^{-1}\left\{1+\frac{2}{\pi}\arctan\left(\frac{\lambda x}{\gamma'}\right)\right\} \tag{4.1}$$

for $-\infty < x < \infty$. This distribution is the particular case of the skew Cauchy-t distribution for $\nu = 1$ with λ replaced by λ/γ'. Hence, the characteristic function of X becomes:

$$E\left[\exp(itX)\right] = \gamma\phi(\gamma t) + \frac{4\gamma i}{\pi^2}\int_0^\infty \frac{\sin(\gamma t u)}{1+u^2}\arctan\left(\frac{\lambda\gamma u}{\gamma'}\right)du.$$

5 Skew Cauchy-Laplace model

If g is the pdf of a Laplace distribution given by

$$g(x) = \begin{cases} 1/(2\phi)\exp\left(x/\phi\right), & \text{if } x \le 0, \\ 1/(2\phi)\exp\left(-x/\phi\right), & \text{if } x \ge 0 \end{cases}$$

then substituting into (1.1) we obtain the skew Cauchy-Laplace distribution for X. The pdf of X is:

$$f_X(x) = \frac{1}{\pi\gamma}\left\{1+\left(\frac{x}{\gamma}\right)^2\right\}^{-1}\exp\left(\frac{\lambda x}{\phi}\right) \tag{5.1}$$

if $x \le 0$, and

$$f_X(x) = \frac{1}{\pi\gamma}\left\{1+\left(\frac{x}{\gamma}\right)^2\right\}^{-1}\left\{2-\exp\left(-\frac{\lambda x}{\phi}\right)\right\} \tag{5.2}$$

if $x > 0$. Using equations (2.5.6.4)–(2.5.6.5) in volume 1 of Prudnikov *et al.* (1986) and the fact $\Gamma(1-z)\Gamma(z) = \pi/\sin(\pi z)$, the characteristic function can be calculated as

$$\begin{aligned} E\left[\exp\left(itX\right)\right] &= \gamma\sqrt{\frac{2\gamma t}{\pi}}K_{-1/2}(\gamma t) + \frac{\gamma i}{\pi}\left\{\exp(-\gamma t)\mathrm{Ei}(\gamma t) - \exp(t)\mathrm{Ei}(-\gamma t)\right\} \\ &- \frac{2\gamma i}{\pi}\int_0^\infty \frac{\sin(\gamma t u)}{1+u^2}\exp\left(-\frac{\lambda\gamma u}{\phi}\right)du. \end{aligned}$$

6 Skew Cauchy-logistic model

If g denotes the pdf of a logistic distribution

$$g(x) = \frac{1}{\beta} \frac{\exp(x/\beta)}{\{1+\exp(x/\beta)\}^2}, \qquad -\infty < x < \infty$$

then from (1.1) we obtain the skew Cauchy-logistic distribution for X. The pdf of X is given by

$$f_X(x) = \frac{2}{\pi\gamma} \left\{1 + \left(\frac{x}{\gamma}\right)^2\right\}^{-1} \left\{1 + \exp\left(-\frac{\lambda x}{\beta}\right)\right\}^{-1} \tag{6.1}$$

for $-\infty < x < \infty$. Using the Taylor series expansion for $(1+z)^{-1}$, one can obtain the following series representations for (6.1):

$$f_X(x) = \frac{2}{\pi\gamma} \left\{1 + \left(\frac{x}{\gamma}\right)^2\right\}^{-1} \sum_{k=0}^{\infty} (-1)^k \exp\left(-\frac{\lambda k x}{\beta}\right)$$

for $x > 0$, and

$$f_X(x) = \frac{2}{\pi\gamma} \left\{1 + \left(\frac{x}{\gamma}\right)^2\right\}^{-1} \sum_{k=0}^{\infty} (-1)^k \exp\left(\frac{\lambda (k+1) x}{\beta}\right)$$

for $x < 0$. Using equation (2.3.7.13) in volume 1 of Prudnikov *et al.* (1986), the characteristic function of X can be calculated as:

$$\begin{aligned} E[\exp(itX)] &= \frac{2\gamma}{\pi} \sum_{k=0}^{\infty} (-1)^k Q\left(\frac{\lambda\gamma k}{\beta} - i\gamma t\right) \\ &+ \frac{2\gamma}{\pi} \sum_{k=0}^{\infty} (-1)^k Q\left(\frac{\lambda\gamma (k+1)}{\beta} - i\gamma t\right), \end{aligned}$$

where $Q(u) = \sin(u)\mathrm{ci}(u) - \cos(u)\mathrm{si}(u)$.

7 Skew Cauchy-uniform model

Taking g to be the pdf of a uniform distribution on $[-h, h]$, we obtain the skew Cauchy-uniform model given by the pdf

$$f_X(x) = \frac{1}{\pi\gamma} \left\{1 + \left(\frac{x}{\gamma}\right)^2\right\}^{-1} \frac{\lambda x + h}{h} \tag{7.1}$$

if $-h \le \lambda x \le h$,

$$f_X(x) = \frac{2}{\pi\gamma}\left\{1 + \left(\frac{x}{\gamma}\right)^2\right\}^{-1} \tag{7.2}$$

if $\lambda x > h$, and

$$f_X(x) = 0 \tag{7.3}$$

if $\lambda x < -h$. The characteristic function of X follows by the use of equations (2.2.6.15), (2.5.6.3) and (2.5.6.4) in volume 1 of Prudnikov *et al.* (1986) and the fact $\Gamma(1-z)\Gamma(z) = \pi/\sin(\pi z)$. One obtains the expression:

$$\begin{aligned} E\left[\exp\left(itX\right)\right] &= \gamma\sqrt{\frac{2\gamma t}{\pi}}K_{-1/2}(\gamma t) + \frac{\gamma i}{\pi}\left\{\exp(-\gamma t)\mathrm{Ei}(\gamma t) - \exp(t)\mathrm{Ei}(-\gamma t)\right\} \\ &+ \frac{2\gamma i}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k(\gamma t)^{2k+1}}{(2k+1)!}\left(\frac{h}{\lambda\gamma}\right)^{2k+2} u_{2k+1} \end{aligned}$$

where

$$\begin{aligned} u_n &= \frac{1}{2+n}\,{}_2F_1\left(1+\frac{n}{2}, 1; 2+\frac{n}{2}; -\frac{h^2}{\lambda^2\gamma^2}\right) \\ &- \frac{1}{1+n}\,{}_2F_1\left(\frac{1+n}{2}, 1; \frac{3+n}{2}; -\frac{h^2}{\lambda^2\gamma^2}\right). \end{aligned}$$

8 Discussion

Figure 1 illustrates possible shapes of the six skew Cauchy distributions discussed above. It is clear that each distribution exhibits a variety of shapes. If the model (1.1) is extended to include a second location parameter – in the manner suggested by Azzalini (1986) – then a greater variety of shapes could be realized.

We now illustrate an application of the skew distributions to exchange rate (ER) data for Japanese Yen (as compared to the United States Dollar) from 1862 to 2003. The data – obtained from the web-site http://www.globalfindata.com/ – are displayed in the table below.

To obtain reasonable fits we transformed the values in the table by computing the relative change from one year to the next. We then fitted both the standard Cauchy distribution and the skew Cauchy-Cauchy distribution to the transformed data by the method of maximum likelihood. A quasi-Newton algorithm nlm in the R software package (Dennis and Schnabel, 1983; Schnabel *et al*, 1985; Ihaka and Gentleman, 1996) was used to solve the likelihood equations. The algorithm was executed several times with different starting values to make sure that the parameter solutions corresponded to the global maximum of the likelihood (this is important because local maximums can appear for skew symmetric models). The parameter estimates which corresponded to the global maximum were:

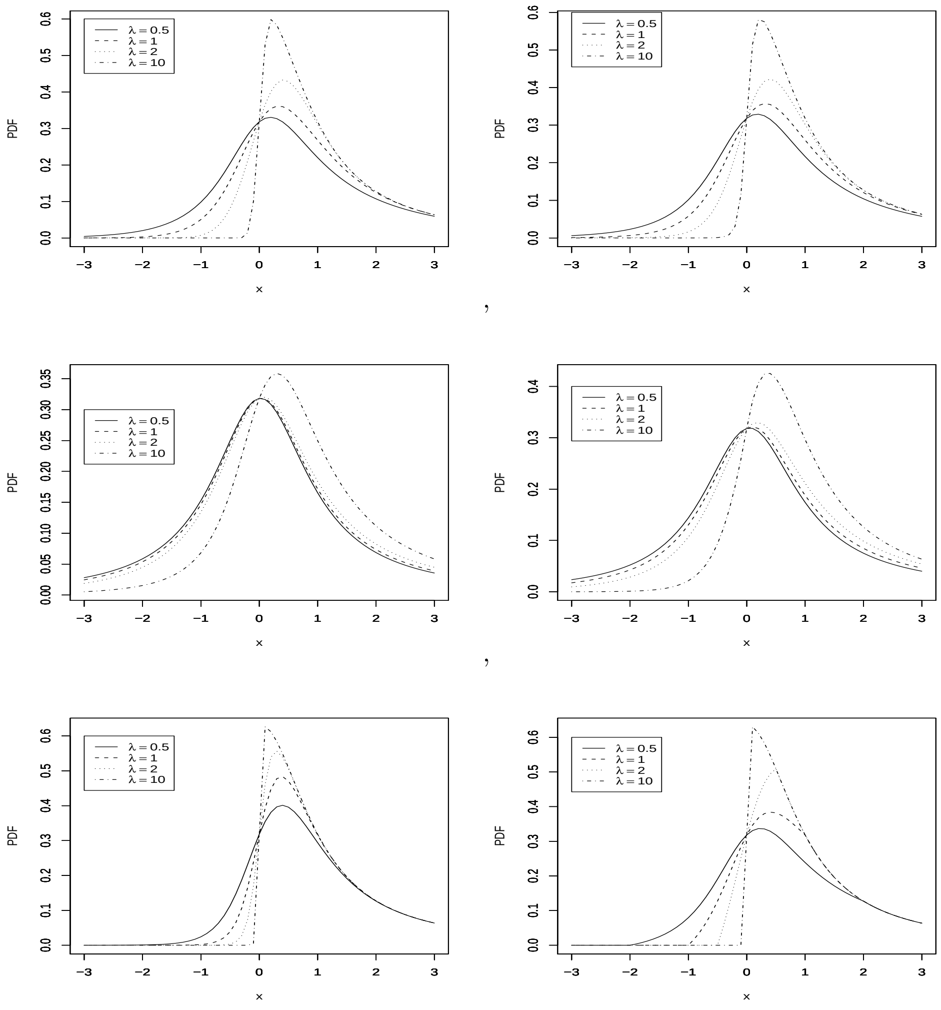

Figure 1 *Plots of the skew Cauchy pdfs. (a): the skew Cauchy-normal pdf (2.1) for* $\lambda = 0.5, 1, 2, 10$, $\gamma = 1$ *and* $\sigma = 1$*; (b): the skew Cauchy-t pdf for* $\lambda = 0.5, 1, 2, 10$, $\gamma = 1$ *and* $\nu = 5$*; (c): the skew Cauchy-Cauchy pdf (4.1) for* $\lambda = 0.5, 1, 2, 10$, $\gamma = 1$ *and* $\gamma^{'} = 8$*; (d): the skew Cauchy-Laplace pdf (5.1)–(5.2) for* $\lambda = 0.5, 1, 2, 10$, $\gamma = 1$ *and* $\phi = 5$*; (e): the skew Cauchy-logistic pdf (6.1) for* $\lambda = 0.5, 1, 2, 10$, $\gamma = 1$ *and* $\beta = 0.2$*; and, (f): the skew Cauchy-uniform pdf (7.1)–(7.3) for* $\lambda = 0.5, 1, 2, 10$, $\gamma = 1$ *and* $h = 1$.

Table 1 *Exchange rate data for Japanese Yen.*

Year	ER	Year	ER	Year	ER	Year	ER
1862	0.5982	1897	2.0566	1933	3.244	1968	360
1863	0.5415	1898	2.0253	1934	3.481	1969	360
1864	0.8996	1899	2.0228	1935	3.476	1970	360
1865	0.9205	1900	2.0305	1936	3.545	1971	315.01
1866	0.9184	1901	2.0253	1937	3.441	1972	301.66
1867	0.9064	1902	2	1938	3.7	1973	280.27
1868	0.9403	1903	2.0382	1939	4.264	1974	301.02
1869	0.9189	1904	2.0356	1940	4.309	1975	305.16
1870	0.9027	1905	2.0126	1941	4.305	1976	293.08
1871	0.9116	1906	2.0305	1942	4.3	1977	239.98
1872	0.9084	1907	2.0305	1943	4.29	1978	194.3
1873	0.9709	1908	2.0202	1944	4.29	1979	240.3
1874	0.9709	1909	2.0202	1945	15	1980	203.1
1875	1	1910	2.0279	1946	15	1981	219.8
1876	0.9804	1911	2.0279	1947	50	1982	234.7
1877	1.0417	1912	2.0177	1948	270	1983	231.7
1878	1.0811	1913	2.0279	1949	360	1984	251.6
1879	1.0929	1914	2.04	1950	360	1985	200.25
1880	1.105	1915	2.01	1951	360	1986	157.473
1881	1.1081	1916	1.985	1952	360	1987	121.012
1882	1.1332	1917	1.965	1953	360	1988	124.931
1883	1.0959	1918	1.918	1954	360	1989	143.85
1884	1.1364	1919	2.005	1955	360	1990	135.4
1885	1.2085	1920	1.99	1956	360	1991	124.8
1886	1.2659	1921	2.073	1957	360	1992	124.8
1887	1.303	1922	2.046	1958	360	1993	111.8
1888	1.3158	1923	2.162	1959	360	1994	99.7
1889	1.2699	1924	2.581	1960	360	1995	103.35
1890	1.1976	1925	2.299	1961	360	1996	115.9
1891	1.2987	1926	2.0408	1962	360	1997	130.61
1892	1.4493	1927	2.1468	1963	360	1998	113.2
1893	1.8018	1928	2.1825	1964	360	1999	102.21
1894	2.0833	1929	2.0367	1965	360	2000	114.27
1895	1.9277	1930	2.0182	1966	360	2001	131.63
1896	1.937	1931	2.852	1967	360	2002	118.74
		1932	4.878			2003	107.31

$$\hat{\gamma}' = 0.022 \text{ with } \log L = 139.6915$$

and

$$\hat{\gamma}' = 0.022, \hat{\lambda} = 0.037 \text{ with } \log L = 142.4059$$

for the two models ($\log L$ denotes the logarithm of the maximized likelihood). Thus, it follows by the standard likelihood ratio test that the skew Cauchy-Cauchy distribution is a much better model for the exchange rate data. The fitted densities for the two models are shown in Figure 2 (plotted in log scale) along with a kernel estimate of the empirical density (Silverman, 1986). Similar observations were

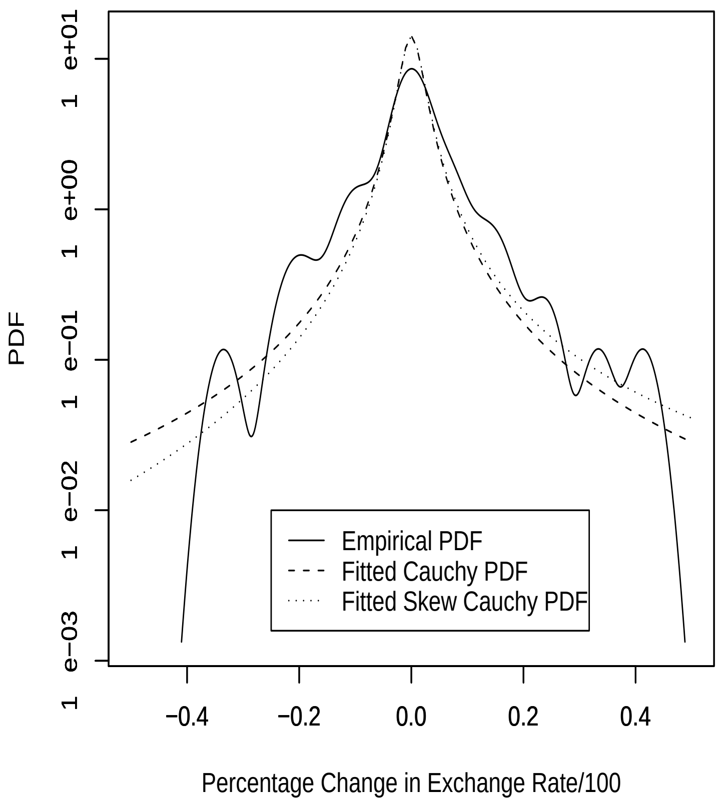

Figure 2 *Fits of the Cauchy and skew Cauchy distributions for the Japanese exchange rate data.*

noted when this exercise was repeated for exchange rate data for the United Kingdom Pound, Euro, Canadian Dollar, Australian Dollar and the Swiss Franc.

There are several ways that the work of this paper could be extended or applied to other areas of statistics. Some of these are:

- study the use of the six distributions for robustness and Bayesian estimation, along the lines suggested by Liseo and Loperfido (2003), Sahu *et al* (2003), and Kim and Mallick (2004).
- construct multivariate generalizations of the six distributions; see also Azzalini and Capitanio (2003), Capitanio *et al* (2003), Fang (2003), Gupta (2003), Gupta and Chang (2003), Liseo and Loperfido (2003), Sahu *et al* (2003) and Genton (2004).
- study the effects of alteration to the skewness of the Cauchy distribution; see Kozubowski and Panorska (2004).

We hope to address these issues in a subsequent paper.

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