The non-unique Universe

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Abstract

The purpose of this paper is to elucidate, by means of concepts and theorems drawn from mathematical logic, the conditions under which the existence of a multiverse is a logical necessity in mathematical physics, and the implications of Gödel's incompleteness theorem for theories of everything.

1 Introduction

In modern cosmology, a *multiverse* is defined to be a collection of possible physical universes.[1] Such multiverses may be considered to be possible or actual, and may or may not be postulated to be the result of physical processes.

The logical existence of these multiverses is a consequence of the fact that modern mathematical physics represents the physical world by means of mathematical structures, i.e., structured sets. These are sets equipped with properties, relationships, operations and distinguished elements. A species of structure is defined by a set of conditions, called the axioms of the structure, which the properties, relationships, operations and distinguished elements are collectively required to satisfy. For example, the axioms for a vector space define a species of structure, and each particular vector space is a member of that species, a structured set. In a sense which will be precisely defined below, each member of the species is a 'model' of the axiomatic theory which defines the species.

If our physical universe is conceived to possess a mathematical structure, then one can define a multiverse consisting of all the models of that species of structure. In addition, the species of structures are arranged into tree-like hierarchies (McCabe 2007, p9), hence one can generalise from the structure of our own universe to find parent structures in the hierarchy of mathematics, from which one can postulate the existence of ever more general multiverses.

Let us consider a couple of examples. In general relativity, a universe is represented by a 4-dimensional differential manifold $\mathcal{M}$ equipped with a metric

[1] This paper will refrain from using the phrase 'ensemble of universes', given that an *ensemble* is typically considered to be a space which possesses a probability measure. It is debatable whether the universe collections postulated by mathematical physicists and cosmologists possess a well-defined probability measure.

tensor field g and a set of matter fields and gauge force fields $\{\phi_i\}$ which generate an energy-stress-momentum tensor T that satisfies the Einstein field equations

$$T = 1/(8\pi G)(\mathrm{Ric} - 1/2\,\mathrm{S}\,g)\,.$$

Ric denotes the Ricci tensor field determined by g, and S denotes the curvature scalar field. The matter fields have distinctive equations of state, and include fluids, scalar fields, tensor fields, and spinor fields. Gauge force fields, such as electromagnetism, are described by n-form fields. Hence, one can define a general relativistic multiverse to be the class of all models of such n-tuples $\{\mathcal{M}, g, \phi_1, ...\}$, interpreted in this sense.

Alternatively, quantum field theory represents a universe to be a Lorentzian manifold $(\mathcal{M}, g)$ which is equipped with a Hilbert space $\mathscr{H}$, a density operator ρ on $\mathscr{H}$, and a collection of operator-valued distributions $\{\hat{\phi}_i\}$ on $\mathcal{M}$ which take their values as bounded self-adjoint operators on $\mathscr{H}$ (Wallace 2001). (Given that there is no mathematically well-defined interacting quantum field theory, we shall draw a veil over the issue of the law-like equations which the operator-valued distributions are required to satisfy). A quantum field theory multiverse is the class of all models of such n-tuples $\{\mathcal{M}, g, \mathscr{H}, \rho, \hat{\phi}_1, ...\ \}$, interpreted in this sense.

Such universe collections are considered to exist merely by virtue of the absence of contradiction in their definition, and Max Tegmark (1998, 2008) is the most renowned proponent of the hypothesis that these universe collections physically exist. Tegmark considers the proposal that "some subset of all mathematical structures... is endowed with... physical existence," (1998, p1) as inadequate because it fails to explain why some particular collection of mathematical structures is endowed with physical existence rather than another. This is what philosophers would refer to as a problem of *contingency*, where a contingent fact is something which happens to be true, but which isn't true as a matter of necessity. Tegmark's response to this problem of contingency was to suggest that *all* mathematical structures have physical existence. More recently, however, Tegmark (2008) has incorporated the implications of Gödel incompleteness and Church-Turing uncomputability, by considering the possibility that only computable structures, or finite computable structures, physically exist (2008, p22). (Tegmark defines a computable structure to be one whose relations can be obtained by computations which are guaranteed to halt after a finite number of steps (2008, p20)). In the case of all Tegmark's proposals, the most general postulated multiverse is a timeless multiverse of disjoint, non-interacting universes.

Other universe collections consist of universes created over time by the operation of some postulated physical process. The primary examples here are the universe-domains in Linde's chaotic inflation theory (1983a and 1983b), and the universes created inside black holes in Smolin's theory of cosmological natural selection (2006). Such universe collections, however, are not the main focus of interest in this paper.

2 Multiverses, parameters, theories and models

Multiverses are often introduced by varying the so-called ‘parameters of physics’. These are parameters in the standard model of particle physics[2], and parameters which specify the initial conditions in general relativistic cosmology, whose values cannot be theoretically derived, and need to be determined by experiment and observation.

Philosopher of science Jesus Mosterin (2004) points out that “the set of all possible worlds is not at all defined with independence from our conceptual schemes and models. If we keep a certain model (with its underlying theories and mathematics) fixed, the set of the combinations of admissible values for its free parameters gives us the set of all possible worlds (relative to that model). It changes every time we introduce a new cosmological model (and we are introducing them all the time). Of course, one could propose considering the set of all possible worlds relative to all possible models formulated in all possible languages on the basis of all possible mathematics and all possible underlying theories, but such consideration would produce more dizziness than enlightenment.”

Mosterin’s point here is aimed at the anthropic principle, and the suggestion that there are multiverses which realise all possible combinations of values for the parameters of physics. At face value, this might seem to be a different type of multiverse than that obtained by varying mathematical structures and models. However, the values chosen for the free parameters of a theory do actually correspond to a choice of model. For example, consider the free parameters of the standard model of particle physics, which include: the coupling constants of the strong and electromagnetic forces; two parameters which determine the Higgs field potential; the Weinberg angle; the masses of the elementary quarks and leptons; and the values of four parameters in the Kobayashi-Maskawa matrix which specifies the ‘mixing’ of the $\{d, s, b\}$ quark flavours in weak force interactions. In terms of a choice of model, the value chosen for the coupling constant of a gauge field with gauge group G corresponds to a choice of metric in the lie algebra $\mathfrak{g}$, (Derdzinksi 1992, p114-115); the Weinberg angle corresponds to a choice of metric in the lie algebra of the electroweak force, (ibid., p104-111); the values chosen for the masses of the elementary quarks and leptons correspond to the choice of a finite family of irreducible unitary representations of the local space-time symmetry group, from a continuous infinity of alternatives on offer (McCabe 2009); and the choice of a specific Kobayashi-Maskawa matrix corresponds to the selection of a specific orthogonal decomposition $\sigma_{d'} \oplus \sigma_{s'} \oplus \sigma_{b'}$ of the fibre bundle which represents a generalization of the $\{d, s, b\}$ quark flavours, (Derdzinski 1992, p160).

Nevertheless, Lee Smolin (2009) argues against the notion that there exists a multiverse of (timeless) universes. Smolin believes that the need to invoke a multiverse is rooted in the dichotomy between laws and initial conditions in existing theoretical physics, and suggests moving beyond this paradigm.

[2]Note that the standard ‘model’ is, in terms of mathematical logic, a theory and not a model.

A choice of initial conditions, however, is merely one of the means by which particular solutions to the laws of physics are identified. More generally, there are boundary conditions, and free parameters in the equations, which have no special relationship to the nature of time. To reiterate, each theory in physics represents a type of physical system by a species of mathematical structure, for which there are, generally, many possible non-isomorphic models; the laws associated with that theory select a particular sub-class of models. As Earman puts it, "a practitioner of mathematical physics is concerned with a certain mathematical structure and an associated set $\mathfrak{M}$ of models with this structure. The... laws L of physics pick out a distinguished sub-class of models $\mathfrak{M}_L := \text{Mod}(L) \subset \mathfrak{M}$, the models satisfying the laws L (or in more colorful, if misleading, language, the models that "obey" the laws L)," (p4, 2002). One might add that if those laws contain a set of free parameters $\{p_i : i = 1, ..., n\}$, then one has a different class of models $\mathfrak{M}_{L(p_i)}$ for each set of combined values of the parameters $\{p_i\}$. The application of a theory to explain or predict a particular empirical phenomenon, then requires the selection of a particular solution, i.e., a particular model. The choice of initial conditions and boundary conditions is then simply a way of picking out a particular model of a theory.

One point of nomenclature to note here is that, whilst mathematical logicians consider a theory to be the set of sentences which define a species of structure, physicists consider the laws which define a sub-class of mathematical models to define a theory. If one retains the same species of mathematical structure, but one changes the laws imposed upon it, then, as far as physicists are concerned, one obtains a different theory. Thus, for example, whilst general relativity represents space-time as a 4-dimensional Lorentzian manifold, if one changes the laws imposed by general relativity upon a Lorentzian manifold, (the Einstein field equations), then one obtains a different physical theory.

The crucial point, however, is that any theory whose domain extends to the entire universe, (i.e. any cosmological theory), potentially has a multiverse associated with it: namely, the class of all models of that theory. Irrespective of whether a future theory abolishes the dichotomy between laws and initial conditions, as Smolin prescribes, the application of that theory will require a means of identifying particular models of the species of mathematical structure selected by the theory. If there is only one physical universe, as Smolin claims, then the problem of contingency will remain: why does this particular model exist and not any one of the other possibilities? The invocation of a multiverse solves the problem of contingency by postulating that all the possible models physically exist.

3 Lagrangians and multiverses

At a classical level, the equations of a theory can be economically specified by its Lagrangian, hence physicists tend to identify a theory with its Lagrangian. In superstring theory, for example, there are five candidate theories precisely because there are five candidate Lagrangians. This point is particularly crucial because it also explains why physicists associate different (effective) theories with different 'vacua'.

The Lagrangians of particle physics typically contain scalar fields, such as the Higgs field postulated to exist by the unified electroweak theory. These scalar fields appear in certain terms of the Lagrangian. The scalar fields have certain values which constitute minima of their respective potential energy functions, and such minima are called vacuum states (or ground states). If one assumes that in the current universe such scalar fields reside in a vacuum state (as the consequence of a process called symmetry breaking), then the form of the Lagrangian changes to specify this special case. After symmetry breaking, the Lagrangian is not the Lagrangian of the fundamental theory, but an 'effective' Lagrangian. Hence, the selection of a vacuum state changes the form of the Lagrangian, and because a Lagrangian defines a theory, the selection of a vacuum state for a scalar field is seen to define the selection of a theory. Physicists therefore tend to talk, interchangeably, about the number of possible vacua, and the number of possible (effective) theories in string theory. The collection of different string theory vacua defines the so-called string theory 'landscape', and this landscape defines a type of multiverse.

However, it should be carefully noted that the string theory landscape defines a collection of different (effective) theories, not a collection of models of a fixed theory. Hence, even if one fixes a particular string theory, with a particular Lagrangian, there is still a multiverse consisting of the class of all models of that theory.

4 Mathematical logic, theories of everything, and multiverses

A final theory of everything, with no free parameters, has often been postulated as a superior alternative to the multiverse generated by our current suite of theories, with their various free parameters. The idea here is that the values of the free parameters in current theories, will follow by definition from the axioms of a final theory, in the same way that the value of pi follows from the axioms of classical Euclidean geometry. However, whilst there may be no free parameters in a final theory, the absence of free parameters is no guarantee that a theory will possess only one model. Hence, even if a final, parameter-free, theory of everything is obtainable, it may still generate a multiverse consisting of all its mutually non-isomorphic models.

However, before we proceed to consider the conditions under which a theory of everything will generate a multiverse, we first need to address the frequent question of whether Gödel's incompleteness theorem is inconsistent with the possibility of a theory of everything. It's a question which, curiously, has received scant attention in the foundations of physics literature.

To understand the question, first we'll need to introduce some concepts from mathematical logic. Here, a *theory* T is defined to be a set of sentences, in some language, which is closed under logical implication. In other words, any sentence which can be derived from a subset of the sentences in a theory, is itself a sentence in the theory. An *intepretation* of a language identifies: the domain over which the variables in the language range; the elements in the domain which correspond to the constants in the language; which elements in the domain possess the predicates in the language; which n-tuples of elements are related by the n-ary relations in the language; and which elements in the domain result from performing n-ary operations upon n-tuples in the domain. A *model* $\mathfrak{U}$ for a theory T is an interpretation of the langauge in which that theory is expressed, which renders each sentence in the theory as true. A *mathematical structure* is a set equipped with properties, relationships, operations and distinguished elements.

To reiterate, theories generally have many different models. For example, each different vector space is a model for the theory of vector spaces, and each different group is a model for the theory of groups. The class of groups and the class of vector spaces can be said to be species of mathematical structure. Conversely, given any structure or model $\mathfrak{U}$, there is a theory $\mathrm{Th}\,\mathfrak{U}$ which consists of the sentences which are true in the structure $\mathfrak{U}$.

Now, a theory T is defined to be *complete* if for any sentence σ, either σ or its negation $\neg\sigma$ belongs to T. A theory T is defined to be *decidable* if there is an effective procedure of deciding whether any given sentence σ belongs to T, (where an 'effective procedure' is generally defined to be a finitely-specifiable sequence of algorithmic steps). A theory is *axiomatizable* if there is a decidable set of sentences in the theory, whose closure under logical implication equals the entire theory.[3]

Gödel's incompleteness theorem revolves around the theory of Peano arithmetic (the theory of conventional additional and multiplicational arithmetic), and a particular model $\mathfrak{R} = (\mathbb{N}; \mathbf{0}, \mathbf{S}, <, +, \cdot, \mathbf{E})$ of Peano arithmetic, whose theory $\mathrm{Th}\,\mathfrak{R}$ can be referred to as 'number theory' (Enderton, p182).[4] It transpires that the theory of Peano arithmetic is both incomplete and undecidable. Moreover, whilst Peano arithmetic is axiomatizable, Gödel demonstrated that number theory is undecidable and non-axiomatizable. Gödel obtained sentences σ, which are true in the model, but which cannot be proven from the theory of the model. These sentences are of the self-referential form, $\sigma =$ 'I am not provable from A', where A is a subset of sentences in the theory.

It should be recognized that an incomplete theory is a highly generic occurrence in mathematics, and is not in itself a pathology of some kind. The axiomatic theory of groups, for example, is incomplete. Moreover, an incomplete theory can be turned into a complete theory by adding more axioms. For

[3]In this context, it should be noted that Tegmark (1998, 2008) draws a distinction between *formal systems* and mathematical structures, rather than a distinction between theories and models. The distinctions are almost equivalent, but the notion of a formal system is not equivalent to the notion of a theory. A formal system is an axiomatic theory, and because not all theories can be axiomatized, not all theories are formal systems.

[4]$\mathbb{N}$ is the set of natural numbers, $\mathbf{0}$ denotes the number zero as a distinguished element, $\mathbf{S}$ is the successor function, $S(n) = n + 1$, $<$ is the ordering relation on $\mathbb{N}$, and $+, \cdot, \mathbf{E}$ are addition, multiplication and exponentiation.

example, whilst the theory of fields is not complete, the theory of algebraically closed fields of characteristic zero *is* complete (Enderton p156). The undecidability of a theory can also be solved in some cases by adding more axioms, but the crucial point is that Gödel discovered a type of undecidability which could never be remedied by the addition of extra axioms.

Any theory which includes number theory will be undecidable, hence if a final theory of everything includes number theory, then the final theory will also be undecidable. The use of number theory is fairly pervasive in mathematical physics, hence, at first sight, this appears to be highly damaging to the prospects for a final theory of everything in physics.

In some mitigation, for the application of mathematics to the physical world, one's conscience may be fairly untroubled by the difficulties of self-referential sentences. However, undecidable sentences which are free from self-reference have been found in various branches of mathematics. It has, for example, been proven that there is no general means of proving whether or not a pair of 'triangulated' 4-dimensional manifolds are homeomorphic (topologically identical).

A final theory of everything might have no need of number theory, and might well be complete and decidable. However, even if a final theory of everything is incomplete and undecidable, it is the models $\mathfrak{U}$ of a theory which purport to represent physical reality, and whilst the theory of a model, $\text{Th}\,\mathfrak{U}$, may be undecidable, it is guaranteed to be complete. That is, every sentence in the language of the theory will either belong or not belong to $\text{Th}\,\mathfrak{U}$.

The potential undecidability of the theory of the structure of our universe, constitutes a potential epistemological limit; it is potentially a limit on what can be proven about the structure of our universe. However, the guaranteed completeness of the theory of the structure of our universe, entails that there is no ontological limit to the existence of such a theory.

The concepts of mathematical logic, introduced to explain Gödel's theorem, can also be exploited to shed further light on the question of multiverses in mathematical physics.

Recall that any physical theory whose domain extends to the entire universe, (i.e. any cosmological theory), potentially has a multiverse associated with it: namely, the class of all models of that theory. Both complete and incomplete theories are capable of generating such multiverses. The class of models of a complete theory will be mutually non-isomorphic, but they will nevertheless be *elementarily equivalent*.

Two models of a theory are defined to be elementarily equivalent if they share the same truth-values for all the sentences of the language. Whilst isomorphic models must be elementarily equivalent, there is no need for elementarily equivalent models to be isomorphic. For example, the structure $(\mathbb{R}, <_R)$ consisting of the real numbers, equipped with its conventional ordering relationship, is elementarily equivalent to $(\mathbb{Q}, <_Q)$, the set of rational numbers equipped with its conventional ordering relationship. However, whilst $\mathbb{Q}$ is a countable set, $\mathbb{R}$ is uncountable; there cannot be an isomorphic mapping between sets of different cardinality, hence these structures are non-isomorphic (Enderton p97-98).

Recalling that a complete theory T is one in which any sentence σ, or its negation $\neg\sigma$, belongs to the theory T, it follows that every model of a complete theory must be elementarily equivalent.

Alternatively, if a theory is such that there are sentences which are true in some models but not in others, then that theory must be incomplete. In this case, the models of the theory will be mutually non-isomorphic *and* elementarily inequivalent.

Hence, mathematical logic suggests that the application of mathematical physics to the universe as a whole can generate two different types of multiverse: classes of non-isomorphic but elementarily equivalent models; and classes of model which are both non-isomorphic and elementarily inequivalent.

The question then arises: are there any conditions under which a theory has only one model, up to isomorphism? In other words, are there conditions under which a theory doesn't generate a multiverse, and the problem of contingency ('Why this universe and not some other?') is eliminated?

A corollary of the upward Löwenheim-Skolem theorem provides an answer to this. The latter entails that if a theory has a model of any infinite cardinality, then it will have models of all infinite cardinalities.[5] Models of different cardinality obviously cannot be isomorphic, hence any theory, complete or incomplete, which has at least one model of infinite cardinality, will have a multiverse associated with. (In the case of a complete theory, the models of different cardinality will be elementarily equivalent, even if they are non-isomorphic). Needless to say, general relativity has models which employ the cardinality of the continuum, hence general relativity, for example, will possess models of every cardinality.

For a theory of mathematical physics to have only one possible model, it must have only a finite model. A theory of everything must have a unique finite model if the problem of contingency, and the potential existence of a multiverse is to be eliminated.

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