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DISCRETE PROBABILITY DISTRIBUTIONS GENERATED BY THE GENERALIZED STER SUMMATION

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Summary
A generalization of the STER summation is presented. Relations between probability generating functions and moments of the generating and generated distributions are analyzed. It is shown that the Yule distribution is invariant with respect to the considered summation.

Key words: Discrete probability distributions; partial sums distributions; the Yule distribution.

1 Introduction

The STER distribution, introduced in Bissinger (1965), represents Sums successively Truncated from the Expectation of the Reciprocal, that is

$$Q_i = A \sum_{j=i+1}^{\infty} \frac{1}{j} P_j, \quad i = 0, 1, 2, \ldots,$$

where $\{P_j\}_{j=0}^{\infty}$ is the original (parent) probability distribution, $\{Q_i\}_{i=0}^{\infty}$ is the generated (descendant) probability distribution and A is a norming constant. A slight generalization can be found in Wimmer and Altmann (2001), where several pairs of parent and descendant distributions from the generalized hypergeometric family were studied (see also Johnson et al., 1992, and Wimmer and Altmann, 1999). Several applications of STER distributions can be found in the literature. Consider the inventory decision model from Prichard and Eagle (1965): given the demand distribution $\{P_i\}$ and a prescribed administrative constant $\alpha \in (0, 1)$, it is shown that the critical stock level Y is such that $(Y - 1)$ represents the $(1 - \alpha)$-quantile of the descendant distribution $\{Q_i\}$. In the income underreporting model from Xekalaki (1983) it is pointed out that the true income distribution is identical to the reported one if and only if it has the Yule distribution. And the latter is related to the STER distribution (cf. Section 3).

In this note we study some properties of the following generalization of the STER distributions: for integers $k \geq 0$ and $l \geq 1$ define

$$Q_i(k,l) = A(k,l) \sum_{j=i}^{\infty} \frac{1}{j+l} \; P_{j+k}, \quad i = 0, 1, 2, \dots . \tag{1.1}$$

In Section 2 we present recursive formulas to compute the probabilities $\{Q_i(k,l)\}$ and its associated probability generating function and (descending) factorial moments.

In Section 3 we discuss the invariant distributions (k-displaced Yule distributions). And the latter is related to the STER summations.

2 Generalized STER summation

Let X be a random variable with distribution $\{P_i\}_{i\geq 0}$ and let $X_{k,l}$ be a random variable with distribution $\{Q_i(k,l)\}_{i\geq 0}$. We call X the $A(k,l)$-parent of $X_{k,l}$ and $X_{k,l}$ the $A(k,l)$-descendant of X. It is easy to see that if $k = l = 1$ in (1.1) we obtain the usual STER summation. Also, if the parent distribution of X is given then the $A(k,l)$-descendant distribution is uniquely determined by (1.1). The converse is not true. If $k > 0$ then

$$P_i = c\; (i - k + l)\; [Q_{i-k}(k,l) - Q_{i-k+1}(k,l)]$$

holds for some constant c. So we cannot uniquely identify $P_0, P_1, \dots$. Due to this fact, there are (for $k > 0$) many $A(k,l)$-parent distributions for a given $A(k,l)$-descendant distribution.

Lemma 2.1 *The $A(k,l)$-descendant distribution satisfies for $i = 0, 1, 2, \dots$*

$$Q_{i+1}(k,l) = Q_i(k,l) - \frac{A(k,l)}{i+l} \; P_{i+k} \tag{2.1}$$

with

$$A(k,l) = \left(1 - \sum_{i=0}^{k-1} P_i - (l-1) \sum_{i=k}^{\infty} \frac{1}{i-k+l} \; P_i \right)^{-1} . \tag{2.2}$$

Moreover, the probability generating functions $G(t) = \sum_{i=0}^{\infty} P_i t^i$ and $H(t) = \sum_{i=0}^{\infty} Q_i(k,l) t^i$ satisfy

$$H(t) = \frac{A(k,l)}{1-t} \int_t^1 \left(G(z) - \sum_{i=0}^{k-1} P_i \; z^i - (l-1) \sum_{i=k}^{\infty} \frac{1}{i-k+l} \; P_i \; z^i \right) z^{-k} \, \mathrm{d}z. \tag{2.3}$$

Proof. From the definition (1.1) we obtain (2.1). Now let $Q_i = Q_i(k,l)$ and we have

$$1 = \sum_{i=0}^{\infty} Q_i = A(k,l) \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \frac{P_{j+k}}{j+l} = A(k,l) \sum_{i=k}^{\infty} \left(\frac{i-k+1}{i+k+l} \right) P_i.$$

And (2.2) follows since

$$\sum_{i=k}^{\infty} \left(\frac{i-k+1}{i-k+l} \right) P_i = \sum_{i=k}^{\infty} \left(P_i - \frac{l-1}{i-k+l} \; P_i \right).$$

For $0 < t < 1$ write

$$(1-t) \; H(t) = Q_0 - \sum_{j=0}^{\infty} (Q_{j+1} - Q_j) \; t^{j+1}.$$

Using (1.1) and (2.1) we have

$$\begin{aligned}
(1-t)H(t) &= A(k,l) \sum_{j=0}^{\infty} \left[\frac{1}{j+l} \; P_{j+k} - \frac{1}{j+l} \; P_{j+k} \; t^{j+1} \right] \\
&= A(k,l) \sum_{j=0}^{\infty} \int_t^1 \frac{j+1}{j+l} \; P_{j+k} \; z^j \; \mathrm{d}z \\
&= A(k,l) \int_0^1 \left[\sum_{i=k}^{\infty} \left(P_i \; z^i - \frac{l-1}{i-k+l} \; P_i \; z^i \right) \right] z^{-k} \; \mathrm{d}z
\end{aligned}$$

and (2.3) follows. □

The special case $(k,l) = (1,1)$ of Lemma 2.1 can be found in Johnson et al. (1992) and is also treated in Wimmer and Altmann (2001).

For a real number x and an integer $r \geq 0$ let $x_{(r)} = x(x-1)\ldots(x-r+1)$ denote the r-th descending factorial of x and set $x_{(0)} = 1$. Let $\mu_{(r)}$ and $\nu_{(r)}$ denote the r-th descending factorial moments of X and $X_{k,l}$ respectively.

Lemma 2.2 *If $\mu_{(r)}$ and $\nu_{(r)}$ exist for all $r \geq 1$ then*

$$\begin{aligned}
\nu_{(r)} &= \frac{A(k,l)}{n+1} \left\{ \sum_{i=0}^{r} \binom{r}{i} (-k)_{(i)} \; \mu_{(r-i)} - \sum_{i=0}^{k-1} (i-k)_{(r)} \; P_i \right. \\
&\quad \left. -(l-1) \sum_{i=k}^{\infty} \frac{(i-k)_{(r)}}{i-k+l} \; P_i \right\}. \qquad (2.4)
\end{aligned}$$

Proof. Let $J(t) = \sum_{i=0}^{\infty} \mu_{(i)} t_i / i!$ and $L(t) = \sum_{i=0}^{\infty} \nu_{(i)} t^i / i!$ be the factorial moment generating function of X and $X_{k,l}$. Using the fact that $L(t) = H(1+t)$ and (2.3) we have

$$t\ L(t) = A(k,l) \int_1^{1+t} \left[\frac{G(z)}{z^k} - \sum_{i=0}^{k-1} P_i\ z^{i-k} - (l-1) \sum_{i=k}^{\infty} \frac{z^{i-k}}{i-k+l}\ P_i \right] \mathrm{d}z. \tag{2.5}$$

Since $\mu_{(r)} = \frac{\partial^r}{\partial t^r} J(t)\Big|_{t=0}$, $\nu_{(r)} = \frac{\partial^r}{\partial t^r} L(t)\Big|_{t=0}$ and $J(t) = G(1+t)$ we can differentiate (2.5) $r+1$ times and evaluate at $t=0$ to obtain (2.4). □

By taking $r = 1$ in (2.4) we get

$$E(X_{k,l}) = \frac{A(k,l)}{2} \left[E(X) - k - \sum_{i=0}^{k-1} (i-k)\ P_i - (l-1) \sum_{i=k}^{\infty} \frac{i-k}{i-k+l}\ P_i \right].$$

3 Invariant distributions

For an integer $r \geq 0$ let $x^{(r)} = x(x+1)\cdots(x+r-1)$ be the r-th factorial of x and set $x^{(0)} = 1$. We say that X has the r-displaced Yule distribution if

$$P_i = \frac{b\ (r+1)^{(i-r)}}{(b+r+1)^{(i-r+1)}},\ b > 0,\ i = r, r+1, \ldots. \tag{3.1}$$

Let $X_{k,l}^{(r)}$ denote a random variable with the r-displaced $A(k,l)$-descendant distribution of X, that is, $P(X_{k,l}^{(r)} = i) = Q_{i-r}$ if $i = r, r+1, \ldots$ and $P(X_{k,l}^{(r)} = i) = 0$, otherwise.

Theorem 3.1 *For $k \geq 1$ we have $X = X_{k,k}^{(k-1)}$ if and only if X has the $(k-1)$-displaced Yule distribution.*

Proof. The case $k = 1$ is trivial. Let $k \geq 2$. If $X = X_{k,k}^{(k-1)}$ then $P_i = 0$ for $i = 0, 1, \ldots, k-2$ and

$$P_{i+1} = P_i - \frac{A(k,k)}{i+1}\ P_{i+1} \text{ for } i = k-1, k, \ldots.$$

It is easy to see that for $i = 0, 1, 2, \ldots$

$$P_{k+i} = P_{k-1}\ \frac{k^{(i+1)}}{(k + A(k,k))^{(i+1)}}.$$

From Erdélyi (1953) we have the formula

$$\sum_{i=0}^{\infty} \frac{(r+1)^{(i)}}{(i+r+2)^{(i)}} = \frac{c+r+1}{c}, \ c, r > 0, \tag{3.2}$$

and we obtain

$$1 = \sum_{i=1}^{\infty} P_i = P_{k-1} \sum_{i=0}^{\infty} \frac{k^{(i)}}{(k+A(k,k))^{(i)}} = P_{k-1} \ \frac{k+A(k,k)-1}{A(k,k)-1}.$$

It follows that for $i = k-1, k, \dots$ we have

$$P_i = \frac{(A(k,k)-1) \ k^{(i-k+1)}}{(k+A(k,k)-1)^{(i-k+2)}},$$

which is the $(k-1)$-displaced Yule distribution.

Now let X with the $(k-1)$-displaced Yule distribution (3.1) with $r = k-1$. Using Lemma 2.1 and (3.2) we can determine the value of $A(k,k)$,

$$\begin{aligned} (A(k,k))^{-1} &= 1 - P_{k-1} - (k-1) \sum_{i=k}^{\infty} \frac{1}{i} \ P_i \\ &= 1 - \frac{b}{b+k} \sum_{i=0}^{\infty} \frac{(k-1)^{(i)}}{(b+k+1)^{(i)}} = \frac{1}{b+1}. \end{aligned}$$

It remains to show that for $i = k-1, k, \dots$

$$P(X_{k,k}^{(k-1)} = i) = R_i = \frac{b \ k^{(i-k+1)}}{(b+k)^{(i-k+2)}}.$$

We will prove it by induction. For $i = k-1$ we have

$$\begin{aligned} R_{k-1} &= Q_0 = A(k,k) \sum_{i=k}^{\infty} \frac{1}{i} \ P_i = b \ (b+1) \sum_{i=k}^{\infty} \frac{k^{(i-k+1)}}{i(b+k)^{(i-k+2)}} \\ &= \frac{b(b+1)}{(b+k)(b+k+1)} \sum_{i=0}^{\infty} \frac{k^{(i)}}{(b+k+2)^{(i)}} = \frac{b}{b+k}. \end{aligned}$$

Consequently,

$$\begin{aligned} R_{i+1} &= Q_{i-k+2} = R_i - \frac{A(k,k)}{i+1} \ P_{i+1} \\ &= \frac{b \ k^{(i-k+1)}}{(b+k)^{(i-k+2)}} - \frac{(b+1) \ b \ k^{(i-k+2)}}{(i+1)(b+k)^{(i-k+3)}} \end{aligned}$$

$$= \frac{b\ (b+i+2)\ k^{(i-k+1)} - b\ (b+1)\ k^{(i-k+1)}}{(b+k)^{(i-k+3)}}$$
$$= \frac{b\ k^{(i-k+2)}}{(b+k)^{(i-k+3)}},$$

which means that $X_{k,k}^{(k-1)}$ has the $(k-1)$-displaced Yule distribution. $\square$

The invariance of the Yule distribution with respect to the $A(1,1)$-summation and its extension to the r-displaced case can be found in Xekalaki (1983).

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