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SMALLEST REGULAR GRAPHS WITH GIRTH PAIR (4,5)

Shi Ronghua
(Mathematics Department, Qinghai Teachers' College, Xining)

1. Introduction

Smallest k-regular graphs with given girth have been intensely studied [3], [4], and smallest k-regular graphs with girth pair (g, h) are studied also [1], [2].

Let G be a connected graph which is not a tree. The odd (even) girth of G is the length of a shortest odd (even) cycle in G. If there is no odd (even) cycle in G, then the odd (even) girth of G is taken as ∞. Let g be the girth of G which is the smaller of the odd girth and even girth, and let h be the bigger one. Then (g, h) is called the girth pair of G. A k-regular graph with girth pair (g, h) is called a $(k; g, h)$-graph. The minimum number of vertices of a $(k; g, h)$-graph is denoted by $f(k; g, h)$.

If three integers k, g, h satisfy the following standard restrictions: (1) $k \geqslant 3$, (2) $3 \leqslant g < h$, (3) $g + h$ odd, then there exists a $(k; g, h)$-graph [2].

Harary and Kovács [2] have given $f(2s; 4, 5) = 5s$ and an infinite family of smallest $(2s; 4, 5)$-graphs. But for $f(2s+1; 4, 5)$ they gave only $f(2s+1; 4, 5) \geqslant 5s+3$ and $(2s+1; 4, 5)$-graphs of order $6s+2$ for every $s \geqslant 1$. They believed that these graphs are smallest $(2s+1; 4, 5)$-graphs. In this paper, we give $f(2s+1; 4, 5)$ and construct an infinite family of smallest $(2s+1; 4, 5)$-graphs for $s \geqslant 1$.

2. The Function $f_H(2s+1; 4, 5)$

The minimum number of vertices of a $(2s+1; 4, 5)$-graph containing graph H is denoted by $f_H(2s+1; 4, 5)$, where H is a 3-regular graph of order 8 illustrated in Fig. 1.

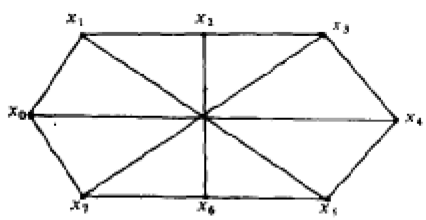

Fig. 1

Theorem 1. $f_H(2s+1; 4, 5) = p$, *where s is a natural number and p the smallest*

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even integer not less than $\frac{8}{3}(2s+1)$. *Graphs* H^0_{2s+1}, H^1_{2s+1} *and* H^2_{2s+1} *are smallest* $(2s+1; 4, 5)$*-graphs containing the graph* H *(see Fig. 2—4).*

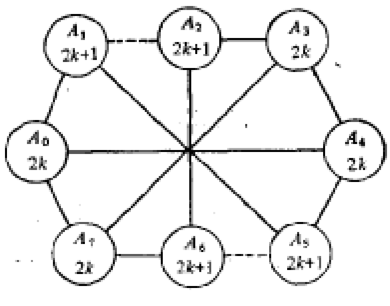

Fig. 2

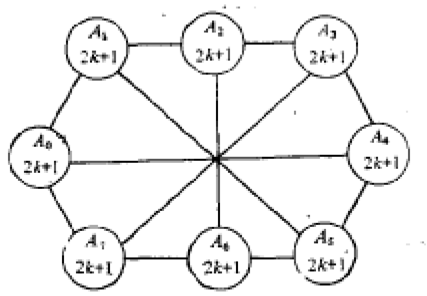

Fig. 3

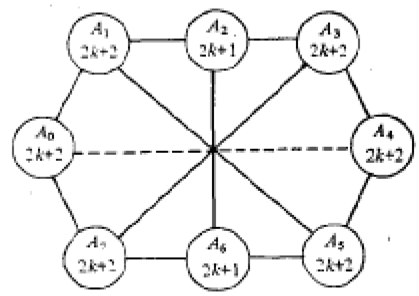

Fig. 4

Proof. First, we prove that $f_H(2s+1; 4, 5) \geqslant \frac{8}{3}(2s+1)$. Let G be a $(2s+1; 4, 5)$-graph containing the graph H. We define the following sets:

$$N_i = N(x_i) \backslash V(H), \quad i = 0, 1, \cdots, 7,$$

where $N(x_i) = \{y \in V(G); x_i y \in E(G)\}$, $V(G)$, $E(G)$ are the sets of vertices and edges of G, respectively. Then $|N_i| = 2s - 2$, $i = 0, 1, \cdots, 7$. Let $N_{ij} = N_i \cap N_j$, $i \neq j$, $i, j = 0, 1, \cdots, 7$. Since G does not contain a triangle, all N_{ij}, except the following sixteen sets, are empty:

$$N_{02}, N_{03}, N_{05}, N_{06}, N_{13}, N_{14}, N_{16}, N_{17},$$
$$N_{24}, N_{25}, N_{27}, N_{35}, N_{36}, N_{46}, N_{47}, N_{57}.$$

Let $N_{ijl} = N_i \cap N_j \cap N_l$, $i \neq j$, $j \neq l$, $i \neq l$, $i, j, l = 0, 1, \cdots, 7$. Similarly, all N_{ijl}, except the following eight sets, are empty:

$$N_{025}, N_{035}, N_{036}, N_{136}, N_{146}, N_{147}, N_{247}, N_{257}.$$

Moreover, there is no vertex in $V(G)$ adjacent to any four vertices of $V(H)$, since G does not contain a triangle. Thus, the intersection of any four distinct sets among $N_i (i = 0, 1, \cdots, 7)$ is empty.

For the sake of convenience we use the subscripts modulo 8.

We denote

$$|N| = \sum_{i=0}^{7} |N_i|, \qquad \alpha = \sum_{\substack{i \neq j \\ i,j \in \{0,1,\cdots,7\}}} |N_{ij}|,$$

and

$$\beta = \sum_{\substack{i \neq j,\, j \neq l,\, l \neq i \\ i,j,l \in \{0,1,\cdots,7\}}} |N_{ijl}|.$$

Thus we have

$$|N| = 8(2s-2) - \alpha + \beta. \tag{1}$$

On the other hand, from $|N_i| = 2s-2$ $(i = 0, 1, \cdots, 7)$, we derive that

$$2s - 2 - \sum_{j \in \{i+2, i+3, i+5, i+6\}} |N_{ij}| + (|N_{i(i+2)(i+5)}| + |N_{i(i+3)(i+6)}| + |N_{i(i+3)(i+5)}|) \geqslant 0 \qquad (i = 0, 1, \cdots, 7).$$

Thus we have

$$8(2s-2) - 2\alpha + 3\beta \geqslant 0. \tag{2}$$

Similarly,

$$2s - 2 - (|N_{i(i+2)(i+5)}| + |N_{i(i+3)(i+6)}| + |N_{i(i+3)(i+5)}|) \geqslant 0 \qquad (i = 0, 1, \cdots, 7).$$

Hence

$$8(2s-2) - 3\beta \geqslant 0. \tag{3}$$

From (1), (2) and (3), we derive that

$$|N| \geqslant \frac{8}{3}(2s-2).$$

Hence

$$|V(G)| \geqslant |V(H)| + |N| \geqslant 8 + \frac{8}{3}(2s-2) = \frac{8}{3}(2s+1). \tag{4}$$

Since G is a $(2s+1)$-regular graph, the order of G must be even. Therefore we get

$$f_H(2s+1; 4, 5) \geqslant p.$$

Now, let $s = 3k + i$ $(0 \leqslant i \leqslant 2)$. In order to prove

$$f_H(2s+1; 4, 5) = p,$$

we construct a $(2s+1; 4, 5)$-graph H^i_{2s+1} of order p which contains H as subgraph in the following way.

1. $i = 0$. So $2s + 1 = 6k + 1$, $p = 16k + 4$ $(k \geqslant 1)$. We construct H^0_{2s+1} as follows. Let $|V(H^0_{2s+1})| = 16k + 4$ and $\{A_j : j = 0, 1, \cdots, 7\}$ be a partition of $V(H^0_{2s+1})$, where each A_j is an independent set of H^0_{2s+1}, and $|A_0| = |A_3| = |A_4| = |A_7| = 2k$,

$|A_1| = |A_2| = |A_5| = |A_6| = 2k+1$ (see Fig. 2). If a set A_j has been joined to A_l by a real line in Fig. 2, then the subgraph of H^0_{2s+1} induced by $A_j \cup A_l$ is a complete bipartite graph. If a set A_j has been joined to A_l by a dotted line, then the subgraph induced by $A_j \cup A_l$ is a graph obtained from the complete bipartite graph by omitting a 1-factor.

2. $i = 1$. So $2s+1 = 6k+3$, $p = 16k+8$ ($k \geqslant 0$). We construct H^1_{2s+1} as follows. Let $|V(H^1_{2s+1})| = 16k+8$ and $\{A_j; j = 0, 1, \cdots, 7\}$ be a partition of $V(H^1_{2s+1})$, where each A_j is an independent set of H^1_{2s+1}, and $|A_j| = 2k+1$ for $j = 0, 1, \cdots, 7$ (see Fig. 3). The meaning of the real line in Fig. 3 is the same as above. Note that if $k = 0$ the graph H^1_3 is the graph H.

3. $i = 2$. So $2s+1 = 6k+5$, $p = 16k+14$ ($k \geqslant 0$). We construct H^2_{2s+1} as follows. Let $|V(H^2_{2s+1})| = 16k+14$ and $\{A_j; j = 0, 1, \cdots, 7\}$ be a partition of $V(H^2_{2s+1})$, where each A_j is an independent set of H^2_{2s+1}, and $|A_2| = |A_6| = 2k+1$, $|A_j| = 2k+2$ for $j \in \{0, 1, 3, 4, 5, 7\}$ (see Fig. 4). The meanings of real and dotted lines in Fig. 4 are the same as above.

It is obvious that the graphs H^0_{2s+1}, H^1_{2s+1} and H^2_{2s+1} are $(2s+1; 4,5)$-graphs of order p.

This completes the proof of Theorem 1.

We see that $s \leqslant 2$ or $s = 4$, and p is just equal to the smallest even integer not less than $5s+3$. It is shown that $f(3; 4, 5) = 8$, $f(5; 4, 5) = 14$ and $f(9; 4, 5) = 24$. It is interesting that $p = 6s+2$ for $s \leqslant 3$. But p is less than $6s+2$ for $s \geqslant 4$.

Thus, we assume below that $s \geqslant 3$ ($s \neq 4$).

3. Graphs G^0_{2s+1} and G^1_{2s+1}

If $s = 2k$, $k \geqslant 2$ being an integer, then $2s+1 = 4k+1$. Now we construct the graph G^0_{2s+1} as follows. Let $|V(G^0_{2s+1})| = 10k+6$ and $\{A_0, A_1, A_2, A_3, A_4, B_0, B_1, B_2, B_3, B_4\}$ be a partition of $V(G^0_{2s+1})$. Each A_i (or B_i) is an independent set of G^0_{2s+1}, and $|A_0| = |B_0| = |B_1| = |B_4| = k$, $|B_2| = |B_3| = |A_1| = \cdots = |A_4| = k+1$ (see Fig. 5). The meanings of real and dotted lines in Fig. 5 are the same as above.

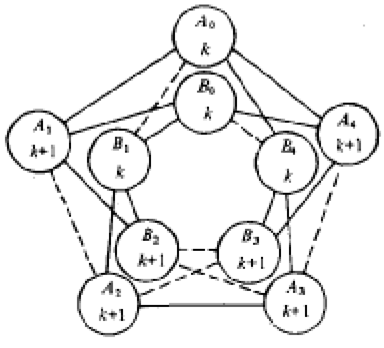

Fig. 5

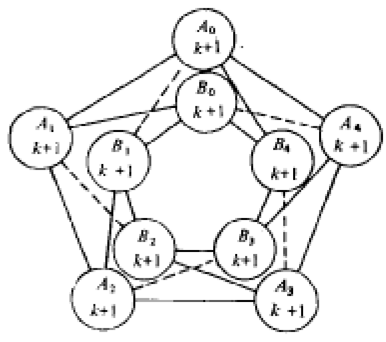

Fig. 6

If $s = 2k+1$, $k \geqslant 1$ being an integer, then $2s+1 = 4k+3$. We construct the graph G^1_{2s+1} as follows. Let $|V(G^1_{2s+1})| = 10k+10$ and $\{A_0, A_1, A_2, A_3, A_4, B_0, B_1, B_2, B_3, B_4\}$ be a partition of $V(G^1_{2s+1})$, where each A_i (or B_i) is an independent set of

G^1_{2s+1}, and $|A_i| = |B_i| = k + 1$ $(i = 0, 1, \cdots, 4)$ (see Fig. 6). The meanings of real and dotted lines in Fig. 6 are the same as above.

It is obvious that the graphs G^0_{2s+1}, G^1_{2s+1} are $(2s + 1;\ 4,\ 5)$-graphs. Therefore we get immediately the following lemma:

Lemma 1. *If $s \geqslant 3$ is odd, then*

$$f(2s + 1;\ 4,\ 5) \leqslant 5s + 5.$$

If $s \geqslant 3$ is even, then

$$f(2s + 1;\ 4,\ 5) \leqslant 5s + 6.$$

4. The Function $f(2s + 1;\ 4,\ 5)$

Let G be a $(2s + 1;\ 4,\ 5)$-graph and $C^5 = x_0x_1x_2x_3x_4x_0$ be a cycle of length 5 in G. We define the following sets:

$$V = V(G),\ C = \{x_0, x_1, x_2, x_3, x_4\},\ N_i = N(x_i)\backslash C,\ (i = 0, 1, \cdots, 4),$$

$$A_i = N_{i-1} \cap N_{i+1}\ (i = 0, 1, \cdots, 4),\ N = \bigcup_{i=0}^{4} N_i \text{ and } A = \bigcup_{i=0}^{4} A_i.$$

Note that the subscripts are reduced modulo 5.

Since G does not contain a triangle, each A_i is an independent set of G, and $A_i \cap A_j = \varnothing$ $(i \neq j,\ i, j = 0, 1, \cdots, 4)$ (see Fig. 7).

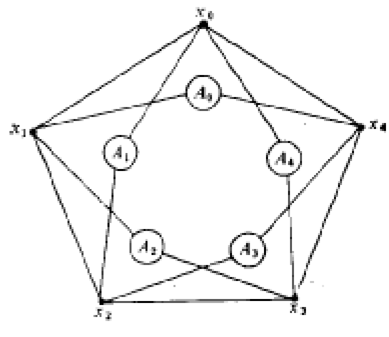

Fig. 7

Obviously, we have

$$|N| = 5(2s - 1) - |A|, \tag{5.1}$$
$$|N\backslash A| = 2(5s - |A|) - 5, \tag{5.2}$$
$$|N_0| = 2s - 1 \geqslant |A_4| + |A_1|, \tag{6.1}$$
$$|N_1| = 2s - 1 \geqslant |A_0| + |A_2|, \tag{6.2}$$
$$|N_2| = 2s - 1 \geqslant |A_1| + |A_3|, \tag{6.3}$$
$$|N_3| = 2s - 1 \geqslant |A_2| + |A_4|, \tag{6.4}$$
$$|N_4| = 2s - 1 \geqslant |A_3| + |A_0|. \tag{6.5}$$

Lemma 2. *Let G be a $(2s+1; 4, 5)$-graph and $|V(G)| < 5s+5$. Then $5s-3 \geqslant |A| \geqslant 5s-4$, $|V\backslash(A\cup C)| \leqslant 3$. Furthermore, if $|A| = 5s-3$, then $|N\backslash A| = 1$; if $|A| = 5s-4$, then $|N\backslash A| = 3$.*

Proof. From (6.1)—(6.5) we obtain that $10s-5 \geqslant 2|A|$. So $5s-3 \geqslant |A|$. On the other hand, if $|A| \leqslant 5s-5$, from (5.1) we have

$$|V| \geqslant |C| + |N| = 5 + \frac{1}{2}(10(2s-1) - 2|A|) \geqslant 5s+5,$$

which contradicts the assumption of the lemma. Hence

$$5s-3 \geqslant |A| \geqslant 5s-4.$$

The rest of the lemma follows by (5.2) immediately.

By Lemma 2 we get immediately that $f(2s+1; 4, 5) \geqslant 5s+3$.

Lemma 3. *Let G be a $(2s+1; 4, 5)$-graph with $|V(G)| < 5s+5$, and $u \in V(G)\backslash(A\cup C)$. Then there are at most two non-empty sets among the sets $N(u)\cap A_i$, $i = 0, 1, \cdots, 4$. Furthermore, $N(u)\cap A_i \neq \varnothing$ implies $N(u)\cap A_{i+1} = \varnothing$, $i = 0, 1, \cdots, 4$.*

Proof. If the lemma is not true, we distinguish the following two cases.

Case 1. There is an index l, $0 \leqslant l \leqslant 4$, such that $N(u)\cap A_l \neq \varnothing$, $N(u)\cap A_{l-2} \neq \varnothing$ and $N(u)\cap A_{l+2} \neq \varnothing$. Suppose the vertices u_l, u_{l-2}, and u_{l+2} belong to the sets $N(u)\cap A_l$, $N(u)\cap A_{l-2}$ and $N(u)\cap A_{l+2}$, respectively. We see that the subgraph of G induced by the set $\{u, u_{l+2}, x_{l-2}, x_{l-1}, u_l, x_{l+1}, x_{l+2}, u_{l-2}\}$ is the graph H, illustrated in Fig. 1. Form Theorem 1, we get $|V| \geqslant p$. But this is impossible, since $p \geqslant 5s+5$ for $s \geqslant 3$ $(s \neq 4)$.

Case 2. There is an index $l, 0 \leqslant l \leqslant 4$, such that $N(u)\cap A_{l-1} \neq \varnothing$, $N(u)\cap A_l \neq \varnothing$ and $N(u)\cap A_{l+1} \neq \varnothing$. By case 1 we see that other two sets $N(u)\cap A_i = \varnothing$, $i = l-2, l+2$. Since G does not contain a triangle, we have $N(u)\cap C = \varnothing$. From $|V\backslash(A\cup C)| \leqslant 3$, we can suppose that $|N(u)\cap(V\backslash(A\cup C))| = t$ $(0 \leqslant t \leqslant 2)$. Since $d(u) = 2s+1$, we have

$$|N(u)\cap A_l| + |N(u)\cap A_{l-1}| + |N(u)\cap A_{l+1}| = 2s+1-t.$$

We consider the following two subcases.

Subcase 1. $|N(u)\cap A_l| \geqslant \frac{1}{2}(2s+1-t)$.

Suppose $x \in N(u)\cap A_{l-1}$. Since G does not contain a triangle, $N(x)\cap(N(u)\cap A_l) = \varnothing$. We see that

$$\begin{aligned}|N(x)\cap(A\cup C)| &\leqslant 2 + |A_l| + |A_{l-2}| - |N(u)\cap A_l| \\ &\leqslant 2 + (2s-1) - \frac{1}{2}(2s+1-t) \\ &= s + \frac{1}{2}(1+t).\end{aligned}$$

Hence,

$$|N(x)\cap(V\backslash(A\cup C))|\geqslant 2s+1-s-\frac{1}{2}(1+t)$$
$$=s+\frac{1}{2}(1-t).$$

In the same way as above, since G does not contain a triangle, we get

$$|V\backslash(A\cup C)|\geqslant s+\frac{1}{2}(1-t)+t=s+\frac{1}{2}(1+t)\geqslant 4.$$

Therefore, $|V|\geqslant 5s+5$, which is a contradiction.

Subcase 2. $|N(u)\cap A_{l-1}|+|N(u)\cap A_{l+1}|\geqslant\frac{1}{2}(2s+1-t)$.

Now we suppose $x\in N(u)\cap A_l$, and consider $N(x)$. Then the result can be proved similarly as above.

Thus, we get immediately that there are at most two nonempty sets in the sets $N(u)\cap A_i$, $i=0,1,\cdots,4$.

Finally, if there is an index l, $0\leqslant l\leqslant 4$, such that $N(u)\cap A_l\neq\varnothing$ and $N(u)\cap A_{l+1}\neq\varnothing$, then we can also derive a contradiction in a similar way.

This completes the proof of Lemma 3.

Theorem 2. *If* $s\geqslant 3$ ($s\neq 4$) *is an integer, then* $f(2s+1;4,5)\geqslant 5s+5$.

Proof. We assume that $f(2s+1;4,5)<5s+5$. Let G be a $(2s+1;4,5)$-graph with $|V(G)|<5s+5$. From Lemma 2 we have either $|A|=5s-3$ or $|A|=5s-4$.

Case 1. $|A|=5s-3$.

From Lemma 2, $|N\backslash A|=1$. Let $x\in N\backslash A$, $xx_0\in E(G)$ without loss of generality. Now (6.1) becomes $2s-2=|A_4|+|A_1|$, and (6.2)—(6.5) become equalities. Hence we get that $|A_0|=|A_1|=|A_4|=s-1$, $|A_2|=|A_3|=s$. Since $|V(G)|<5s+5$, we have $|V\backslash(A\cup C)|\leqslant 2$.

If $|V\backslash(A\cup C)|=1$, then x is just the only vertex in $V\backslash(A\cup C)$. Since $xx_0\in E(G)$ and G does not contain a triangle, $N(x)\cap A_1=\varnothing$, $N(x)\cap A_4=\varnothing$. From Lemma 3, we have either $N(x)\cap A_2=\varnothing$ or $N(x)\cap A_3=\varnothing$. In either case, we always have $|N(x)|\leqslant 1+|A_1|+s=2s$. This contradicts that G is a $(2s+1)$-regular graph.

If $|V\backslash(A\cup C)|=2$, let $\{v,x\}=V\backslash(A\cup C)$. Then $v\notin N\cup C$, $N(v)\cap C=\varnothing$. Hence $|N(v)\cap A|\geqslant 2s$; otherwise, $d(v)<2s+1$. But by Lemma 3 we have $|N(v)\cap A|\leqslant 2s-1$, a contradiction.

Case 2. $|A|=5s-4$.

In this case, $|N\backslash A|=3$. Let $B=N\backslash A=\{u,v,w\}$. Since $|V|<5s+5$, there must be $|V|=5s+4$, and $\{A,B,C\}$ is a partition of V.

Similarly to Case 1, we can see that the induced subgraph $G(B)$ has not any isolated vertex; otherwise the degree of the isolated vertex of $G(B)$ would be less than $2s+1$. Moreover, $G(B)$ obviously is not a complete graph. Hence $G(B)$ is a path, and we let $G(B) = uvw$. Furthermore, without loss of generality, we can suppose that $v \in N(x_0)$, namely $x_0v \in E(G)$. Assume $ux_i \in E(G)$, $wx_j \in E(G)$, $1 \leqslant i \leqslant j \leqslant 4$, and consider the following two subcases respectively.

Subcase 1. $i = j$.

If $i = j = 1$, then (6.1) becomes $2s-2 = |A_4| + |A_1|$, (6.2) becomes $2s-3 = |A_0| + |A_2|$ and (6.3)—(6.5) become equalities. Hence, we have $|A_0| = |A_1| = s-2$, $|A_2| = s-1$, $|A_3| = s+1$ and $|A_4| = s$. From Lemma 3, the vertices u and w need be joined to each vertex of $A_1 \cup A_3$; otherwise their degrees would be less than $2s+1$. Thus, the vertex v can only be adjacent to the vertices of $A_0 \cup A_2$; otherwise G contains a triangle. Hence $|N(v)| \leqslant 3 + |A_1| + |A_3| = 2s$, which is impossible.

If $i = j = 2, 3$, or 4, we can prove it in a similar way.

Subcase 2. $1 \leqslant i < j \leqslant 4$.

We take $i = 1, j = 2$; other cases can be proved similarly. If $i = 1$, $j = 2$, then (6.1) becomes $2s-2 = |A_4| + |A_1|$, (6.2) becomes $2s-2 = |A_0| + |A_2|$, (6.3) becomes $2s-2 = |A_1| + |A_3|$, (6.4) and (6.5) become equalities. Hence we have $|A_0| = |A_3| = s-1$, $|A_2| = s-2$, $|A_3| = |A_4| = s$. From Lemma 3, we have either $N(u) \subseteq \{x_1, v\} \cup A_1 \cup A_3$ or $N(u) \subseteq \{x_1, v\} \cup A_1 \cup A_4$. In either case, we have the following result:

$$|N(u)| \leqslant 2 + s - 2 + s = 2s.$$

It contradicts that G is a $(2s+1)$-regular graph.

This completes the proof of Theorem 2.

From above, together with Harary-Kovács' result [2], we get immediately the following main result:

Theorem 3. (1) $f(2s; 4, 5) = 5s$, *where* $s > 1$ *is an integer, the graphs* G_{2s} *(Fig. 8) are smallest graphs.*

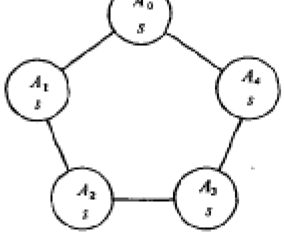

Fig. 8

(2) $f(3; 4, 5) = 8$, *the graph* H *is a smallest;*

$f(5; 4, 5) = 14$, *the graph* H_5^2 *is a smallest;*

$f(9; 4, 5) = 24$, *the graph* H_9^1 *is a smallest.*

(3) $f(2s+1;4,5)=5s+5$, *where* $s\geqslant 3$ *is an odd number, the graphs* G^1_{2s+1} *are smallest. Furthermore, when* $s\geqslant 9$, *each smallest* $(2s+1;4,5)$*-graph does not contain the graph* H.

(4) $f(2s+1;4,5)=5s+6$, *where* $s\geqslant 6$ *is an even number, the graphs* G^0_{2s+1} *are smallest. Furthermore, when* $s\geqslant 12$, *each smallest* $(2s+1;4,5)$*-graph does not contain the graph* H.

5. Acknowledgment

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围长对是(4,5)的最小正则图

施 容 华
（青海师范学院数学系，西宁）

摘 要

我们把围长对是 (g,h) 的 k-正则图称为 $(k;g,h)$-图；$(k;g,h)$-图的顶点的最少数目用 $f(k;g,h)$ 表示．本文证明了

$$f(2s+1;4,5)=\begin{cases}8, & s=1,\\ 14, & s=2,\\ 24, & s=4,\\ 5s+5, & s\text{ 是奇数},\ s\geqslant 3,\\ 5s+6, & s\text{ 是偶数},\ s\geqslant 6.\end{cases}$$

我们还构造了最小 $(2s+1;4,5)$-图，$s\geqslant 1$ 的无限族．这样，我们就完全解决了 Harary 和 Kovács[2] 提出的问题 1．