系统科学与数学
J. Sys. Sci. & Math. Scis.
6 (4) (1986), 281—285

HAMILTONIAN PROPERTY IN THE SQUARE OF A CONNECTED GRAPH

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1. Introduction

In this article, all graphs are finite, undirected, without loops or multiple edges; most of the graph-theory terminology used here can be found in standard texts.

The subdivision graph of $K_{1,3}$ is denoted by $S(K_{1,3})$ (see Fig. 1).

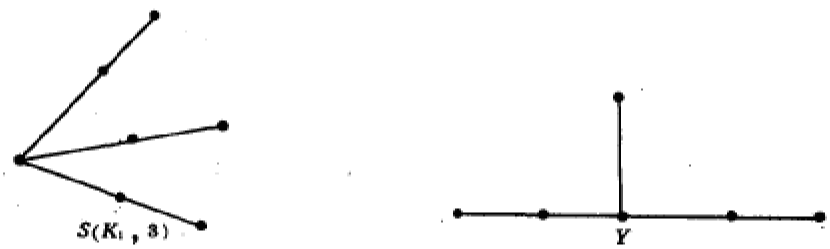

Fig. 1 The graphs $S(K_{1,3})$ and Y

A graph G is $(H_i, \cdots, H_k)$-free ($k \geqslant 1$), if G contains no induced subgraph isomorphic to H_i, for any $i = 1, \cdots, k$. If $k = 1$, we simply say G is H_1-free.

Various Hamiltonian-like properties are investigated in the square of graphs. The main results are the following.

Theorem A[1]. *If G is a graph, then G^2 is Hamiltonian if and only if G^2 is vertex pancyclic.*

Theorem B[2]. *If G is a 2-connected graph, then G^2 is Hamiltonian.*

Theorem C[3]. *For any tree T, T^2 is Hamiltonian if and only if T is $S(K_{1,3})$-free.*

Thus, it is of interest to study Hamiltonian properties in the square of a connected graph G, where G is not a tree and not 2-connected. New results are following on the

Received January 8, 1985.

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Theorem D[4]. *If G is a connected $K_{1,3}$-free graph, then G^2 is vertex pancyclic.*

Gould and Jacobson extended the result of Matthews. They obtained

Theorem E[5]. *If G is a connected Y-free graph, then G^2 is vertex pancyclic, where the forbidden subgraph Y is shown in Fig. 1.*

Theorem F[5]. *If G is a connected $(K_{1,4}, S(K_{1,3}), F, W)$-free graph of order $n \geqslant 3$, then G^2 is vertex pancyclic, where forbidden subgraphs F and W are shown in Fig. 2.*

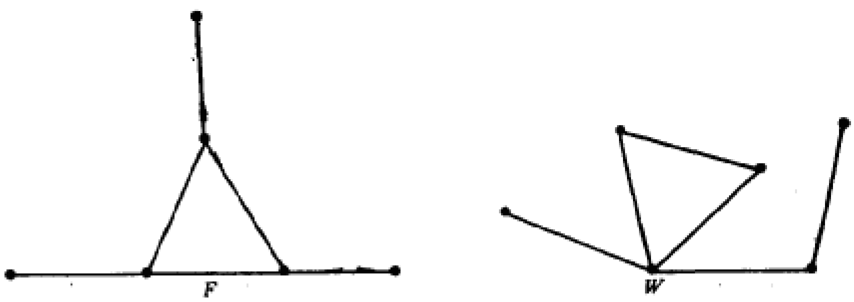

Fig. 2 The forbidden subgraphs F and W

The above results lead Gould and Jacobson to conclude the following conjectures[5].

Conjecture 1. If G is a connected $S(K_{1,3})$-free graph, then G^2 is vertex pancyclic.

Conjecture 2. If G is a connected $(S(K_{1,3}), K_{1,4})$-free graph, then G^2 is vertex pancyclic.

It is obvious that if Conjecture 1 is proved, then Conjecture 2 is true, too.

In this article we prove Conjecture 1; by Theorem C, it is best possible in general.

2. Main Results

We begin with a useful lemma.

Lemma. *Let G be a connected graph, but not a tree and not 2-connected. Then there exists a cutpoint v of G such that $d_G(v) \geqslant 3$.*

Proof. It is obvious.

Theorem. *If G is a connected $S(K_{1,3})$-free graph, then it is vertex pancyclic, where $|V(G)| \geqslant 3$.*

Proof. We proceed by induction on the order of the graph.

From Theorem C, our theorem is verified for any connected $S(K_{1,3})$-free graph G, $|V(G)| \leqslant 7$. Suppose our theorem is true for graphs whose order is less than n, $n > 7$ being a natural number, and let G be a connected $S(K_{1,3})$-free graph and $|V(G)| = n$. By Theorems B and C, we can suppose, without loss of generality, that G is not 2-connected and not a tree. From Theorem A, it suffices to show G^2 is Hamiltonian.

By the lemma, there exists a cutpoint v in graph G such that $d_G(v) \geqslant 3$. We denote $N(v) = \{v_1, v_2, \cdots, v_k\}$, $k \geqslant 3$.

Now consider a spanning tree T of G containing all edges from v to the vertices of $N(v)$. Let G_i be the subgraph of G induced by v and vertices of the branch of T containing $v_i (i = 1, 2, \cdots, k)$. By the choice of the cutpoint v, we have $|V(G_i)| \leqslant |V(G)| - 2 = n - 2 (i = 1, 2, \cdots, k)$.

Since v is a cutpoint, there exists at least one vertex $v_j \in N(v)$ for each $v_i \in N(v)$, $1 \leqslant i, j \leqslant k$, such that v_i and v_j belong to two components of $G - v$, respectively. First we consider the subgraph G_i^* of G induced by the set $V(G_i) \cup \{v_j\} (i = 1, 2, \cdots, k)$. It is clear that $|V(G_i^*)| = |V(G_i)| + 1 \leqslant n - 1$, and G_i^* is a connected $S(K_{1,3})$-free graph. So, by the induction hypothesis, the graph $(G_i^*)^2$ contains a Hamiltonian circuit C_i^*. Note that the degree of vertex v_j in $(G_i^*)^2$ is 2, and the vertex v_j is only adjacent to two vertices v and v_i. Hence there exists a Hamiltonian path P_i' from v to v_i in graph G_i^2. Let w_i be the vertex which is adjacent to v on the Hamiltonian path P_i' of G_i^2. If $|V(G_i)| > 2$, then $vw_i \in E(G^2)$ and $vw_i \notin E(G)$, namely $d_G(w_i, v) = 2$, $v_i w_i \in E(G)$. If $|V(G_i)| = 2$, then $v_i = w_i$, and G_i^2 is merely traceable.

Thus we prove that if $|V(G_i)| > 2$, then the graph $G_i^2 - v$ contains a Hamiltonian path P_i from the vertex v_i to a vertex w_i, where $d_G(v, w_i) = 2$; that is $w_i \in N(v_i) (i = 1, 2, \cdots, k)$. Denote $P_i = v_i, \cdots, w_i (i = 1, 2, \cdots, k)$. If $|V(G_i)| = 2$, then P_i is only a point, and $v_i = w_i$.

Denote $M = \{w_i : 1 \leqslant i \leqslant k\}$.

Now we consider the graph $G^2 - v$. Let A be a minimal set of paths in graph $G^2 - v$ which satisfies

(1) Each path P_i belongs to set $A (i = 1, 2, \cdots, k)$.

(2) If $w_i \in M$ is one end of a path $Q \in A$ and there exists a vertex $v_j \in N(v)$ such that v_j is adjacent to w_i in $G^2 - v$, then the path $Q w_i v_j P_j$ belongs to the set A, where P_j and Q are pairwise disjoint.

(3) If a path $Q \in A$ and $w_i \in M$ is one end of the path Q, and $w_j \in M$ is adjacent to w_i in the graph $G^2 - v$, then the path $Q w_i w_j \overline{P}_j$ belongs to the set A, where $\overline{P}_j$ is the reverse of the path P_j, and the path P_j and Q are pairwise disjoint.

(4) If a path $Q \in A$ and its two ends belong to $N(v)$, then the path $Q v_i v_j P_j$ belongs to the set A, where P_j and Q are pairwise disjoint.

From the definition of set A, we have the following properties:

Property (a). $A \neq \emptyset$.

Property (b). If a path $Q \in A$, x and y are two ends of Q, respectively, then $\{x, y\} \subseteq N(v) \cup M$.

Property (c). If a path $Q \in A$, x and y are two ends of Q, respectively, and suppose $y \in M$, then we claim that $x \in N(v)$.

Property (d). If $Q \in A$ and $v_i \notin V(Q)$, $v_i \in N(v)$, then $V(P_i) \cap V(Q) = \emptyset$; if $v_i \in V(Q)$, then $V(P_i) \subseteq V(Q)$.

Let Q_1 be a longest path of the set A.

It is clear that the path Q_1 is not a spanning path of $G^2 - v$; otherwise, the theorem is true.

By Property (d), there exists path $Q \in A$ in graph G^2-$(\{v\} \cup V(Q_1))$. Let A_1 be the subset of A in graph $G^2 - (\{v\} \cup V(Q_1))$; then $A_1 \neq \emptyset$. So we can say that Q_2 is the longest path of set A_1.

Now, if $V(Q_1) \cup V(Q_2) = V(G) \backslash \{v\}$. It is obvious that G^2 is Hamiltonian by Properties (b) and (c). Otherwise, there exists a longest path $Q_3 \in A$ in graph G^2-$(\{v\} \cup V(Q_1) \cup V(Q_2))$. We will prove this is impossible.

Note, by definition, that:

(1) The length of each path $Q_i (i = 1,2,3)$ is at least 1, namely, $|V(Q_i)| \geqslant 2$.

(2) There exists one end belonging to M in each path $Q_i (i = 1, 2, 3)$, since every path Q_i is maximal.

Without loss of generality, suppose w_i is an end of Q_i, and v_i is the vertex which is adjacent to w_i in graph $G_i (i = 1, 2, 3)$.

Now, we consider the subgraph H of G induced by set $\{v, v_1, w_1, v_2, w_2, v_3, w_3\}$. If $v_1 v_2 \in E(G)$, then the path $Q_1 w_1 v_2 P_2 \in A$ is longer than Q_1; this is impossible. So $v_1 v_2 \notin E(G)$. Similarly, $v_1 v_3 \notin E(G)$ and $v_2 v_3 \notin E(G)$.

If $v_1 w_2 \in E(G)$, then $w_1 w_2$ is an edge of graph $G^2 - v$, and the path $Q_1 w_1 w_2 \bar{P}_2 \in A$ is longer than Q_1; this is impossible. So $v_1 w_2$ is not an edge of graph G; similarly, $v_i w_j \notin E(G)$, $i \neq j$, $1 \leqslant i, j \leqslant 3$.

Finaly, $w_1 w_2 \in E(G)$ is likewise impossible; similarly, $w_1 w_3$ and $w_2 w_3$ are not edges of G.

From above, the induced subgraph H is isomorphic to $S(K_{1,3})$. This is impossible, and so G^2 is Hamiltonian.

The proof is complete.

3. Conclusion

Finally, we note that Gould and Jacobson made a mistake in the proof of Theorem F (namely, Theorem 3 in [5]). Suffice it to say that G_i is the graph in Fig. 3.

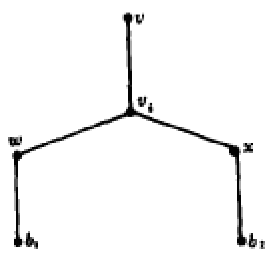

Fig. 3 The graph G_i

ACKNOWLEDGMENT The authors are very grateful to Professor Tian Feng for his guidance.

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平方图的汉米尔顿性

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摘　　要

一个图G的平方图(记作 G^2)，是在G中把所有距离为 2 的点对用边相邻接而形成的图. 本文主要结果是:

定理. 如果G是连通,无 $S(K_{1,3})$ 导出子图的图,则 G^2 是顶点泛圈图.

这样，Gould 和 Jacobson 提出的两个猜想得到证明. 结合这一方向上已有的工作，平方图的汉米尔顿问题基本上得到满意的解决.